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On the distribution of the sequence $(n\alpha)$ with transcendental α

by

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1. Introduction. Let $\alpha \in \mathbb{R}$ be irrational with regular continued fraction expansion $\alpha = [a_0, a_1, a_2, \ldots]$ (i.e. $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for all $i \ge 1$) and convergents $p_n/q_n = [a_0, a_1, \ldots, a_n]$. (Sometimes we write $a_n(\alpha)$ and $p_n(\alpha)/q_n(\alpha)$ to stress the dependence on α.) It is a classic result of P. Bohl [5], W. Sierpiński [15], [16] and H. Weyl [17], [18] that the sequence $(n\alpha)_{n\ge1}$ is uniformly distributed modulo 1. This property is studied from a quantitative viewpoint by means of the speed of convergence in the limit relation $\lim_{N\to\infty} D_N^*(\alpha) = 0$ where the quantity

$$D_N^*(\alpha) = \sup_{0\le x\le 1}\left|\frac{1}{N}\sum_{n=1}^{N} c_{[0,x)}(\{n\alpha\}) - x\right|$$

is called *discrepancy*. According to a theorem of W. M. Schmidt [11] the convergence is best possible if $D_N^*(\alpha) = O((\log N)/N)$. It was first observed by H. Behnke [4] that this estimate is satisfied if and only if α is *of bounded density*, i.e. $\sum_{i=1}^{m} a_i = O(m)$ as $m \to \infty$. For α of bounded density the map $\alpha \mapsto \nu^*(\alpha) = \limsup_{N\to\infty} ND_N^*(\alpha)/\log N$ is used to obtain more detailed information. It was proved by Y. Dupain and V. T. Sós [6] that $\inf_{\alpha\in B} \nu^*(\alpha) = \nu^*([\overline{2}])$ where B denotes the set of numbers of bounded density and $[\overline{2}] = [2,2,2,\ldots] = 1+\sqrt{2}$ is used as a convenient shorthand notation. J. Schoißengeier [14] expressed $\nu^*(\alpha)$ in terms of the continued fraction expansion of α after he had obtained partial results in [13]. Employing these results C. Baxa [3] showed the following:

(1) Let $B^q := \{\alpha \in B \mid \alpha \text{ is a quadratic irrationality}\}$. Then we have $\nu^*(B) = \overline{\nu^*(B^q)} = [\nu^*([\overline{2}]), \infty)$.

(2) Let $b \ge 4$ be an even integer, $B_b := \{\alpha = [a_0, a_1, a_2, \ldots] \in B \mid a_i \ge b$ for all $i \ge 1\}$ and $B_b^q := \{\alpha \in B_b \mid \alpha \text{ is a quadratic irrationality}\}$. Then $\nu^*(B_b) = \overline{\nu^*(B_b^q)} = [\nu^*([\overline{b}]), \infty)$.

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It is the purpose of the present paper to strengthen these results and to prove:

Theorem 1. *Let* $B^t := \{\alpha \in B \mid \alpha$ *is transcendental*$\}$ *and* $B^u := \{\alpha \in B \mid \alpha$ *is a* U_2*-number*$\}$. *Then*

$$\nu^*(B^t) = \nu^*(B^u) = [\nu^*([\overline{2}]), \infty).$$

Theorem 2. *Let* $B_b^t := \{\alpha \in B_b \mid \alpha$ *is transcendental*$\}$ *and* $B_b^u := \{\alpha \in B_b \mid \alpha$ *is a* U_2*-number*$\}$ (*where again* $b \geq 4$ *is assumed to be an even integer*). *Then*

$$\nu^*(B_b^t) = \nu^*(B_b^u) = [\nu^*([\overline{b}]), \infty).$$

Remarks. (1) For a more detailed and leisurely exposition of the problem and its history the reader is referred to [3].

(2) In contrast to Theorem 1 we see that $\nu^*(B^q) \subsetneqq [\nu^*([\overline{2}]), \infty)$ since $\nu^*(\alpha)$ is transcendental if α is a quadratic irrationality. This follows from Theorem 1 in §4 of [14] as the logarithm of an algebraic number $\neq 1$ is always transcendental.

2. Criteria for transcendence. Our criteria are a variant of a method used by E. Maillet [7, Chapter 7] and A. Baker [1], [2] (see also [8, §36]). We will follow rather closely parts of [1] and [2] with two major differences:

(1) We will use a theorem by W. M. Schmidt which became available only a few years later [9] and was generalized in [10] (compare also with [12]).

(2) We do not aim at criteria of great generality but at specific ones which are well suited for our purpose. This explains the special shape of Corollary 6 below.

Definition. If β is algebraic then $H(\beta)$ denotes the classical *absolute height*. This means, if $p(X) = \sum_{i=0}^{m} a_i X^i \in \mathbb{Z}[X] \setminus \{0\}$ with $\gcd(a_0, \ldots, a_m) = 1$ and $p(\beta) = 0$ (and $\deg p$ minimal with this property) then

$$H(\beta) = \max_{0 \leq i \leq m} |a_i|.$$

Theorem 3 (W. M. Schmidt). *Let* $\alpha \in \mathbb{R}$ *be algebraic but neither rational nor a quadratic irrationality and* $\delta > 0$. *Then there exist only finitely many* $\beta \in \mathbb{R}$ *which are rational or quadratic irrationalities such that* $|\alpha - \beta| < H(\beta)^{-3-\delta}$.

Corollary 4. *Let* $\alpha \in \mathbb{R}$ *have "quasiperiodic" but not periodic continued fraction expansion*

$$\alpha = [0, a_1, \ldots, a_{\nu_1 - 1}, \overline{a_{\nu_1}, \ldots, a_{\nu_1 + k_1 - 1}}^{\lambda_1}, \overline{a_{\nu_2}, \ldots, a_{\nu_2 + k_2 - 1}}^{\lambda_2}, \ldots]$$

(*i.e.* $\nu_n = \nu_1 + \sum_{i=1}^{n-1} \lambda_i k_i$). *If* α *is algebraic then* $\limsup_{i \to \infty} q_{\nu_{i+1} - 1} q_{\nu_i + k_i - 1}^{-3-\delta} < \infty$. (*Here* $\overline{a_\nu, \ldots, a_{\nu+k}}^{\lambda}$ *indicates that the partial quotients* $a_\nu, \ldots, a_{\nu+k}$

should be repeated λ *times. For example* $[0,\overline{1,2,3}^2,\overline{5}^3,7,\ldots]=[0,1,2,3,1,2,3,5,5,5,7,\ldots]$.)

Proof. For $i\ge 1$ we define the quadratic irrationality

$$\beta_i := [0,a_1,\ldots,a_{\nu_1-1},\overline{a_{\nu_1},\ldots,a_{\nu_1+k_1-1}}^{\lambda_1},\ldots \\ \ldots,\overline{a_{\nu_{i-1}},\ldots,a_{\nu_{i-1}+k_{i-1}-1}}^{\lambda_{i-1}},\overline{a_{\nu_i},\ldots,a_{\nu_i+k_i-1}}].$$

For $k\le \nu_{i+1}-1$ we have $a_k(\alpha)=a_k(\beta_i)$ and we may write p_k/q_k for $p_k(\alpha)/q_k(\alpha)=p_k(\beta_i)/q_k(\beta_i)$. Now

$$L_i\beta_i^2+M_i\beta_i+N_i=0$$

with

$$\begin{aligned} L_i &= q_{\nu_i-2}q_{\nu_i+k_i-1}-q_{\nu_i-1}q_{\nu_i+k_i-2},\\ M_i &= q_{\nu_i-1}p_{\nu_i+k_i-2}+p_{\nu_i-1}q_{\nu_i+k_i-2}-p_{\nu_i-2}q_{\nu_i+k_i-1}-q_{\nu_i-2}p_{\nu_i+k_i-1},\\ N_i &= p_{\nu_i-2}p_{\nu_i+k_i-1}-p_{\nu_i-1}p_{\nu_i+k_i-2}, \end{aligned}$$

and therefore

$$H(\beta_i)\le \max\{|L_i|,|M_i|,|N_i|\}<2q_{\nu_i+k_i-1}^2.$$

Theorem 3 implies

$$q_{\nu_{i+1}-1}^{-2}>|\alpha-\beta_i|>C(\alpha,\delta)H(\beta_i)^{-3-\delta}>C(\alpha,\delta)2^{-3-\delta}q_{\nu_i+k_i-1}^{-6-2\delta}$$

for a certain $C(\alpha,\delta)>0$. The corollary follows immediately.

LEMMA 5. *Keeping all notations of Corollary* 4 *we have*

$$0<|L_i\alpha^2+M_i\alpha+N_i|<8q_{\nu_i+k_i+1}^4q_{\nu_{i+1}-1}^{-2}.$$

Proof. Let $\overline{\beta}_i$ denote the conjugate of β_i. If $|\overline{\beta}_i|\ge 1$ it follows from $L_i\overline{\beta}_i^2+M_i\overline{\beta}_i+N_i=0$ that

$$\begin{aligned} |\overline{\beta}_i|^2\le |L_i\overline{\beta}_i^2| &= |M_i\overline{\beta}_i+N_i|<2q_{\nu_i+k_i-1}^2(|\overline{\beta}_i|+1)\\ &\le 4q_{\nu_i+k_i-1}^2|\overline{\beta}_i| \end{aligned}$$

and therefore $|\overline{\beta}_i|<4q_{\nu_i+k_i-1}^2$, which remains true even if $|\overline{\beta}_i|<1$. This implies $|\alpha-\overline{\beta}_i|\le 1+|\overline{\beta}_i|<1+4q_{\nu_i+k_i-1}^2<8q_{\nu_i+k_i-1}^2$ and thus

$$\begin{aligned} |L_i\alpha^2+M_i\alpha+N_i| &= |L_i|\cdot|\alpha-\beta_i|\cdot|\alpha-\overline{\beta}_i|\\ &< q_{\nu_i+k_i-1}^2\cdot q_{\nu_{i+1}-1}^{-2}\cdot 8q_{\nu_i+k_i-1}^2=8q_{\nu_i+k_i-1}^4q_{\nu_{i+1}-1}^{-2}. \end{aligned}$$

COROLLARY 6. (1) *Let* $b>a>1$ *be integers and* $\alpha=[0,\overline{a}^{\lambda_1},\overline{b}^{\lambda_2},\overline{a}^{\lambda_3},\overline{b}^{\lambda_4},\ldots]$. *If*

$$\limsup_{n\to\infty}\left(\lambda_{n+1}-13\frac{\log b}{\log a}(\lambda_1+\ldots+\lambda_n)\right)=\infty$$

then α *is transcendental.*

(2) *If even* $\limsup_{n\to\infty}\lambda_{n+1}/(\lambda_1+\ldots+\lambda_n)=\infty$ *then* α *is a* U_2*-number.*

Proof. If $i > 1$ then

$$q_{\nu_i+k_i-1} = q_{\nu_i} \le (b+1)^{1+\lambda_1+\ldots+\lambda_{i-1}} \le (b^2)^{2(\lambda_1+\ldots+\lambda_{i-1})} = a^{4\frac{\log b}{\log a}(\lambda_1+\ldots+\lambda_{i-1})}$$

and therefore

$$q_{\nu_{i+1}-1} q_{\nu_i+k_i-1}^{-13/4} \ge a^{\lambda_i - 13\frac{\log b}{\log a}(\lambda_1+\ldots+\lambda_{i-1})}$$

and (1) follows immediately from Corollary 4.

We have

$$H(L_iX^2 + M_iX + N_i) = \max\{|L_i|, |M_i|, |N_i|\} < 2q_{\nu_i+k_i-1}^2 \le 2b^{4(\nu_i+k_i-1)}$$

where H denotes the height of a polynomial just for once. Now estimating $q_{\nu_i+k_i-1} \le b^{2(\nu_i+k_i-1)}$ and $q_{\nu_{i+1}-1} \ge a^{\nu_{i+1}-1}$ we deduce from Lemma 5 that

$$0 < |L_i\alpha^2 + M_i\alpha + N_i| < b^{-(2(\nu_{i+1}-1)\log a - 8(\nu_i+k_i-1)\log b - 3\log 2)/\log b} = (2b^{4(\nu_i+k_i-1)})^{-\Psi_i}$$

with

$$\Psi_i = \frac{2(\nu_{i+1}-1)\log a - 8(\nu_i+k_i-1)\log b - 3\log 2}{4(\nu_i+k_i-1)\log b + \log 2}.$$

Obviously $\limsup_{i\to\infty} \Psi_i = \infty$ is equivalent to $\limsup_{i\to\infty} \nu_{i+1}/\nu_i = \infty$ and therefore to $\limsup_{i\to\infty} \lambda_i/(\lambda_1 + \ldots + \lambda_{i-1}) = \infty$.

3. Values of $\nu^*(\alpha)$ for transcendental α

LEMMA 7. *Let $a < b$ be even positive integers and $\nu^*([\overline{a}]) < \mu < \nu^*([\overline{b}])$. Then there exists a transcendental $\alpha = [0, a_1, a_2, \ldots]$ (and even a U_2-number α) such that $a_i \in \{a, b\}$ for all $i \ge 1$ and $\nu^*(\alpha) = \mu$.*

Proof. The function

$$f_{ab}(x) = \frac{1}{8} \cdot \frac{a + xb}{\log([\overline{a}]) + x\log([\overline{b}])}$$

increases for positive x, $f_{ab}(0) = \nu^*([\overline{a}])$ and $\lim_{x\to\infty} f_{ab}(x) = \nu^*([\overline{b}])$. Therefore there is a unique $Q \in (0, \infty)$ such that $\mu = f_{ab}(Q)$. Let $(\sigma_n)_{n\ge1}$ be a strictly increasing sequence of integers such that $\sigma_1 Q \ge 1$ and

$$\limsup_{n\to\infty}\Big(\sigma_{n+1} - 13(Q+1)\frac{\log b}{\log a}(\sigma_1 + \ldots + \sigma_n)\Big) = \infty \tag{1}$$

or even

$$\limsup_{n\to\infty} \frac{\sigma_{n+1}}{\sigma_1 + \ldots + \sigma_n} = \infty \tag{2}$$

are satisfied. Let $\lambda_{2n-1} = 2\sigma_n$ and $\lambda_{2n} = 2[\sigma_n Q]$ for $n \ge 1$. Furthermore, let $\alpha = [0, \overline{a}^{\lambda_1}, \overline{b}^{\lambda_2}, \overline{a}^{\lambda_3}, \overline{b}^{\lambda_4}, \ldots]$. Using Corollary 6 it is easy to check that α is transcendental if (1) and a U_2-number if (2) is satisfied. Employing a special case of Theorem 1 in §3 of [14] which was already stated as Theorem 1 in §4 of [13] we see that

$$\nu^*(\alpha) = \frac{1}{4} \limsup_{m\to\infty} \frac{1}{\log q_m} \max\Big(\sum_{1\le i\le m, 2|i} a_i, \sum_{1\le i\le m, 2\nmid i} a_i \Big) = \frac{1}{8} \limsup_{m\to\infty} \frac{1}{\log q_m} \sum_{i=1}^{m} a_i$$

where we used the fact that $\lim_{m\to\infty} \log q_{m+1}/\log q_m = 1$ for numbers of bounded density and that

$$\max\Big(\sum_{1\le i\le m, 2|i} a_i, \sum_{1\le i\le m, 2\nmid i} a_i \Big) = \frac{1}{2} \sum_{i=1}^{m} a_i + \Delta \quad \text{with } |\Delta| \le b/2.$$

If $\lambda_1 + \ldots + \lambda_{2k-1} < m \le \lambda_1 + \ldots + \lambda_{2k+1}$ then

$$\log q_m = (\lambda_1 + \lambda_3 + \ldots + \lambda_{2k-1} + r_{2k+1}) \log([\overline{a}]) + (\lambda_2 + \lambda_4 + \ldots + \lambda_{2k-2} + r_{2k}) \log([\overline{b}]) + O(k)$$

with an implicit constant that depends on a and b only. Here

$$1 \le r_{2k} = m - (\lambda_1 + \ldots + \lambda_{2k-1}) \le \lambda_{2k}, \quad r_{2k+1} = 0 \quad \text{if } m \le \lambda_1 + \ldots + \lambda_{2k},$$

$$r_{2k} = \lambda_{2k}, \quad 1 \le r_{2k+1} = m - (\lambda_1 + \ldots + \lambda_{2k}) \le \lambda_{2k+1} \quad \text{if } m > \lambda_1 + \ldots + \lambda_{2k}.$$

(If the reader considers this step to be too sketchy he or she may want to consult the proof of Theorem 4.3 in [3].) Therefore $\nu^*(\alpha) = \frac{1}{8} \limsup_{m\to\infty} h(m)$ where

$$h(m) = \frac{(\lambda_1 + \lambda_3 + \ldots + \lambda_{2k-1} + r_{2k+1})a + (\lambda_2 + \lambda_4 + \ldots + \lambda_{2k-2} + r_{2k})b}{(\lambda_1 + \lambda_3 + \ldots + \lambda_{2k-1} + r_{2k+1}) \log([\overline{a}]) + (\lambda_2 + \lambda_4 + \ldots + \lambda_{2k-2} + r_{2k}) \log([\overline{b}])}.$$

Obviously $\max\{h(m) \mid \lambda_1 + \ldots + \lambda_{2k-1} < m \le \lambda_1 + \ldots + \lambda_{2k+1}\} = h(\lambda_1 + \ldots + \lambda_{2k})$ and thus

$$\nu^*(\alpha) = \tfrac{1}{8} \lim_{k\to\infty} \sup_{m\ge k} h(m) = \tfrac{1}{8} \lim_{k\to\infty} \sup_{m>\lambda_1+\ldots+\lambda_{2k-1}} h(m) = \tfrac{1}{8} \lim_{k\to\infty} \sup_{m\ge k} h(\lambda_1 + \ldots + \lambda_{2m}) = \tfrac{1}{8} \limsup_{k\to\infty} h(\lambda_1 + \ldots + \lambda_{2k}) = \mu$$

since $\lim_{k\to\infty} (\lambda_2 + \lambda_4 + \ldots + \lambda_{2k})/(\lambda_1 + \lambda_3 + \ldots + \lambda_{2k-1}) = Q$.

LEMMA 8. *Let a be an even positive integer. Then there exists a transcendental $\alpha = [0, a_1, a_2, \ldots]$ (and even a U_2-number α) such that $a_i \in \{a, a+2\}$ for all $i \geq 1$ and $\nu^*(\alpha) = \nu^*([\overline{a}])$.*

Proof. Let $\lambda_1 = 1$ and $\lambda_{2n+1} = n(\lambda_1 + \lambda_3 + \ldots + \lambda_{2n-1})$ for $n \geq 1$. Finally, put $\alpha = [0, \overline{a}^{\lambda_1}, a+2, \overline{a}^{\lambda_3}, a+2, \overline{a}^{\lambda_5}, \ldots]$. Then α is a U_2-number according to Corollary 6 and $\nu^*(\alpha) = \nu^*([\overline{a}])$ by Theorem 5.1 in [3].

Proof of Theorems 1 and 2. Let b be a positive even integer. Then

$$[\nu^*([\overline{b}]), \infty) = \{\nu^*([\overline{b}])\} \cup \bigcup_{k=1}^{\infty} (\nu^*([\overline{b}]), \nu^*([\overline{b+2k}]))$$

and both theorems follow from Lemmata 7 and 8, the theorem of Y. Dupain and V. T. Sós [6] and Theorem 3.1 of [3].

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