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Normality of numbers generated by the values of polynomials at primes

by

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To the memory of Norikata Nakagoshi

1. Introduction. Let $r \geq 2$ be a fixed integer and let $\theta = 0.a_1 a_2 \ldots$ be the r-adic expansion of a real number θ with $0 < \theta < 1$. Then θ is said to be *normal* to base r if, for any block $b_1 \ldots b_l \in \{0, 1, \ldots, r-1\}^l$,

$$n^{-1} N(\theta; b_1 \ldots b_l; n) = r^{-l} + o(1)$$

as $n \to \infty$, where $N(\theta, b_1 \ldots b_l; n)$ is the number of indices $i \leq n - l + 1$ such that $a_i = b_1$, $a_{i+1} = b_2, \ldots, a_{i+l-1} = b_l$. Let $(m)_r$ denote the r-adic expansion of an integer $m \geq 1$. For any infinite sequence $\{m_1, m_2, \ldots\}$ of positive integers, we consider the number $0.(m_1)_r(m_2)_r \ldots$ whose r-adic expansion is obtained by the concatenation of the strings $(m_1)_r, (m_2)_r, \ldots$ of r-adic digits, which will be written simply as $0.m_1 m_2 \ldots \ (r)$.

Copeland and Erdős [1] proved that the number $0.m_1 m_2 \ldots (r)$ is normal to base r for any increasing sequence $\{m_1, m_2, \ldots\}$ of positive integers such that, for every positive $\varrho < 1$, the number of m_i's up to x exceeds x^ϱ provided x is sufficiently large. In particular, the normality of the number

$$0.23571113 \ldots \ (r)$$

defined by the primes was established. Davenport and Erdős [2] proved that the number

$$0.f(1)f(2) \ldots f(n) \ldots \ (r)$$

is normal to base r, where $f(x)$ is any nonconstant polynomial taking positive integral values at all positive integers.

In this paper, we prove the following

Theorem. *Let $f(x)$ be as above. Then the number*

$$\alpha(f) = 0.f(2)f(3)f(5)f(7)f(11)f(13) \ldots \ (r)$$

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defined by the values of $f(x)$ *at primes is normal to base* r. *More precisely, for any block* $b_1 \ldots b_l \in \{0, 1, \ldots, r-1\}^l$, *we have*

$$n^{-1} N(\alpha(f); b_1 \ldots b_l; n) = r^{-l} + O\left(\frac{1}{\log n}\right) \tag{1}$$

as $n \to \infty$, *where the implied constant depends possibly on* r, f, *and* l.

2. Preliminary of the proof of the Theorem. Let $\alpha(f) = 0.a_1 a_2 \ldots a_n \ldots$ be the r-adic expansion of the number $\alpha(f)$ given in the Theorem. Then each a_n belongs to the corresponding string $(f(p_\nu))_r$, where p_ν is the νth prime and $\nu = \nu(n)$ is defined by

$$\sum_{i=1}^{\nu-1} ([\log_r f(p_i)] + 1) < n \le \sum_{i=1}^{\nu} ([\log_r f(p_i)] + 1).$$

Here $[t]$ denotes the greatest integer not exceeding the real number t. We put $x = x(n) = p_{\nu(n)}$, so that

$$\begin{aligned} n &= \sum_{p \le x} \log_r f(p) + O(\pi(x)) + O(\log_r f(x)) \\ &= \frac{dx}{\log r} + O\left(\frac{x}{\log x}\right), \end{aligned} \tag{2}$$

where $d \ge 1$ is the degree of the polynomial $f(t)$, p runs through prime numbers, and $\pi(x)$ is the number of primes not exceeding x. We used here the prime number theorem:

$$\pi(x) = \operatorname{Li} x + O\left(\frac{x}{(\log x)^G}\right),$$

where G is a positive constant given arbitrarily and

$$\operatorname{Li} x = \int_2^x \frac{dt}{\log t}.$$

Then we have

$$\begin{aligned} N(\alpha(f); b_1 \ldots b_l; n) &= \sum_{p \le x} N(f(p); b_1 \ldots b_l) + O(\pi(x)) + O(\log_r f(x)) \\ &= \sum_{p \le x} N(f(p); b_1 \ldots b_l) + O\left(\frac{n}{\log n}\right) \end{aligned}$$

with $x = x(n) = p_{\nu(n)}$.

Let j_0 be a large constant. Then for each integer $j \ge j_0$, there is an integer n_j such that

$$r^{j-2} \le f(n_j) < r^{j-1} \le f(n_j + 1) < r^j.$$

We note that

$$n_j \gg\ll r^{j/d}$$

and that $n_j < n \le n_{j+1}$ if and only if the r-adic expansion of $f(n)$ is of length j; namely,

$$(f(n))_r = c_{j-1} \dots c_1 c_0 \in \{0, 1, \dots, r-1\}^j, \quad c_{j-1} \ne 0. \tag{3}$$

For any $x > r^{j_0}$, we define an integer $J = J(x)$ by

$$n_J < x \le n_{J+1},$$

so that

$$J = \log_r f(x) + O(1) \gg\ll \log x. \tag{4}$$

Let n be an integer with $n_j < n \le n_{j+1}$ and $j_0 < j \le J$, so that $(f(n))_r$ can be written as in (3). We denote by $N^*(f(n); b_1 \dots b_l)$ the number of occurrences of the block $b_1 \dots b_l$ appearing in the string $\underbrace{0 \dots 0}_{J-j} c_{j-1} \dots c_1 c_0$ of length J. Then we have

$$\begin{aligned} 0 &\le \sum_{p \le x} N^*(f(p); b_1 \dots b_l) - \sum_{p \le x} N(f(p); b_1 \dots b_l) \\ &\le \sum_{j=j_0+1}^{J-1} (J-j)(\pi(n_{j+1}) - \pi(n_j)) + O(1) \\ &\le \sum_{j=j_0+1}^{J-1} \pi(n_{j+1}) + O(1) \ll \sum_{j=1}^{J-1} \frac{r^{j/d}}{J} \ll \frac{x}{\log x} \end{aligned}$$

and so

$$N(\alpha(f); b_1 \dots b_l; n) = \sum_{p \le x} N^*(f(p); b_1 \dots b_l) + O\left(\frac{n}{\log n}\right) \tag{5}$$

with $x = x(n) = p_{\nu(n)}$.

We shall prove in Sections 4 and 5 that

$$\sum_{p \le x} N^*(f(p); b_1 \dots b_l) = r^{-l} \pi(x) \log_r f(x) + O\left(\frac{x}{\log x}\right) \tag{6}$$

which, combined with (5) and (2), yields (1).

3. Lemmas

Lemma 1 ([9; 4.19]). *Let $F(x)$ be a real function, k times differentiable, and satisfying $|F^{(k)}(x)| \geq \lambda > 0$ throughout the interval $[a,b]$. Then*

$$\Big|\int_a^b e(F(x))\,dx\Big| \leq c(k)\lambda^{-1/k}.$$

Lemma 2 ([3; p. 66, Theorem 10]). *Let*

$$F(t) = \frac{h}{q}t^d + \alpha_1 t^{d-1} + \ldots + \alpha_k,$$

where h, q are coprime integers and α_i's are real. Suppose that

$$(\log x)^\sigma \leq q \leq x^d(\log x)^{-\sigma},$$

where $\sigma > 2^{6d}(\sigma_0 + 1)$ with $\sigma_0 > 0$. Then

$$\Big|\sum_{p\leq x} e(F(p))\Big| \leq c(d)x(\log x)^{-\sigma_0}$$

as $x \to \infty$, where p runs through the primes.

Lemma 3 ([3; p. 2, Lemma 1.3 and p. 5, Lemma 1.6]). *Let*

$$F(x) = b_0x^d + b_1x^{d-1} + \ldots + b_{d-1}x + b_d$$

be a polynomial with integral coefficients and let q be a positive integer. Let D be the greatest common divisor of q, b_0, $b_1, \ldots,$ and b_{d-1}. Then

$$\Big|\sum_{n=1}^{q} e\Big(\frac{F(n)}{q}\Big)\Big| \leq d^{3\omega(q/D)}D^{1/d}q^{1-1/d}$$

as $q \to \infty$, where $\omega(n)$ is the number of distinct prime divisors of n.

Lemma 4 ([6; Corollary of Lemma]). *Let $F(x)$ be a polynomial with real coefficients with leading term Ax^d, where $A \neq 0$ and $d \geq 2$. Let a/q be a rational number with $(a,q) = 1$ such that $|A - a/q| < q^{-2}$. Assume that*

$$(\log Q)^H \leq q \leq Q^d/(\log Q)^H,$$

where $H > d^2 + 2^dG$ with $G \geq 0$. Then

$$\Big|\sum_{1\leq n\leq Q} e(F(n))\Big| \ll Q(\log Q)^{-G}.$$

Lemma 5 ([7; Theorem], cf. [8; Theorem 1]). *Let $f(t)$ and $b_1 \ldots b_l$ be as in Theorem. Then*

$$\sum_{n\leq y} N(f(n); b_1 \ldots b_l) = r^{-l}y\log_r f(y) + O(y)$$

as $y \to \infty$, where the implied constant depends possibly on r, f, and l.

4. Proof of the Theorem. We have to prove the inequality (6). We write

$$\sum_{p\le x} N^*(f(p); b_1 \ldots b_l) = \sum_{p\le x}\sum_{m=l}^{J} I\biggl(\frac{f(p)}{r^m}\biggr),$$

where

$$I(t) = \begin{cases} 1 & \text{if } \sum_{k=1}^{l} b_k r^{-k} \le t-[t] < \sum_{k=1}^{l} b_k r^{-k} + r^{-l},\\ 0 & \text{otherwise.}\end{cases}$$

There are functions $I_-(t)$ and $I_+(t)$ such that $I_-(t) \le I(t) \le I_+(t)$, having Fourier expansion of the form

$$I_\pm(t) = r^{-l} \pm J^{-1} + \sum_{\substack{\nu=-\infty\\ \nu\ne 0}}^{\infty} A_\pm(\nu)e(\nu t)$$

with

$$|A_\pm(\nu)| \ll \min(|\nu|^{-1}, J\nu^{-2}),$$

where $e(x) = e^{2\pi i x}$ ([10; Chap. 2, Lemma 2]). We choose a large constant c_0 and put

$$M = [c_0 \log_r J]. \tag{7}$$

Then it follows that

$$\begin{aligned} &\sum_{p\le x} N^*(f(p); b_1 \ldots b_l) \qquad (8)\\ &\lesseqgtr \Bigl(\sum_{l\le m\le dM} + \sum_{dM<m\le J-M} + \sum_{J-M<m\le J}\Bigr)\sum_{p\le x} I_\pm\biggl(\frac{f(p)}{r^m}\biggr)\\ &= \sum\nolimits_1 + \frac{\pi(x)}{r^l}(J-dM) + \sum\nolimits_2 + \sum\nolimits_3 + O(\pi(x)), \end{aligned}$$

where d is the degree of the polynomial $f(x)$,

$$\sum\nolimits_1 = \sum\nolimits_{1(\pm)} = \sum_{l\le m\le dM}\sum_{p\le x} I_\pm\biggl(\frac{f(p)}{r^m}\biggr),$$

$$\sum\nolimits_2 = \sum\nolimits_{2(\pm)} = \sum_{dM<m\le J-M}\sum_{1\le|\nu|\le J^2} A_\pm(\nu)\sum_{p\le x} e\biggl(\frac{\nu}{r^m}f(p)\biggr),$$

$$\sum\nolimits_3 = \sum\nolimits_{3(\pm)} = \sum_{J-M<m\le J}\sum_{1\le|\nu|\le J^2} A_\pm(\nu)\sum_{p\le x} e\biggl(\frac{\nu}{r^m}f(p)\biggr).$$

We first estimate $\sum_2$. Suppose that $dM \le m \le J-M$. Then, writing the leading coefficient of the polynomial $\nu r^{-m} f(t)$ as a/q with $(a,q)=1$, we have

$$(\log x)^{\sigma} \le q \le x^{d}(\log x)^{-\sigma}$$

with a large constant σ, so that by Lemma 2,

$$\sum_{p\le x} e\Big(\frac{\nu}{r^m} f(p)\Big) \ll x(\log x)^{-\sigma_0},$$

where $\sigma_0 > 3$ is a constant. Therefore we obtain

$$\textstyle\sum_2 \ll x(\log x)^{2-\sigma_0} \ll \dfrac{x}{\log x}. \tag{9}$$

Next we estimate $\sum_3$. We appeal to the prime number theorem of the form referred to in Section 2. Then it follows that

$$\begin{aligned}\sum_{p\le x} e\Big(\frac{\nu}{r^m} f(p)\Big) &= \int_2^x e\Big(\frac{\nu}{r^m} f(t)\Big)\, d\pi(t) + O(1)\\ &= \int_2^x e\Big(\frac{\nu}{r^m} f(t)\Big)\frac{dt}{\log t} + O\Big(\frac{x}{(\log x)^G}\Big)\\ &= \int_{x(\log x)^{-G}}^{x} e\Big(\frac{\nu}{r^m} f(t)\Big)\frac{dt}{\log t} + O\Big(\frac{x}{(\log x)^G}\Big)\\ &\ll \frac{1}{\log x}\sup_{\xi}\Big|\int_{x(\log x)^{-G}}^{\xi} e\Big(\frac{\nu}{r^m} f(t)\Big)\, dt\Big| + O\Big(\frac{x}{(\log x)^G}\Big)\\ &\ll \frac{1}{\log x}\Big(\frac{|\nu|}{r^m}\Big)^{-1/d} + O\Big(\frac{x}{(\log x)^G}\Big),\end{aligned}$$

using the second mean-value theorem and Lemma 1 with $|\nu r^{-m} f^{(d)}(t)| \gg |\nu| r^{-m}$. Therefore we have

$$\begin{aligned}\textstyle\sum_3 &\ll \sum_{1\le|\nu|\le J^2} |\nu|^{-1} \sum_{J-M\le m\le J}\Big(\frac{1}{\log x}\Big(\frac{|\nu|}{r^m}\Big)^{-1/d} + O\Big(\frac{x}{(\log x)^G}\Big)\Big)\\ &\ll \frac{1}{\log x}\sum_{1\le|\nu|\le J^2}\frac{1}{|\nu|^{1+1/d}}\sum_{m\le J} r^{-m/d} + O\Big(\frac{x}{(\log x)^{G-2}}\Big)\\ &\ll \frac{x}{\log x}.\end{aligned} \tag{10}$$

To prove the Theorem, it remains to show that

$$\textstyle\sum_1 = \dfrac{\pi(x)}{r^l}\, dM + O\Big(\dfrac{x}{\log x}\Big), \tag{11}$$

since this together with (4), (8), (9), and (10) implies

$$\sum_{p\le x} N^*(f(p); b_1 \dots b_l) = \frac{\pi(x)}{r^l} J + O(\pi(x))$$
$$= \frac{\pi(x)}{r^l}\log_r f(x) + O\left(\frac{x}{\log x}\right),$$

which is the inequality (6).

5. Proof of Theorem (continued). We shall prove the inequality (11) in three steps.

First step. Suppose that $l \le m \le dM$, where M is given by (7) with (4). We appeal to the prime number theorem for arithmetic progressions of the following form ([4; Sect. 17]): Let $\pi(x; q, a)$ be the number of primes $p \le x$ in an arithmetic progression $p \equiv a \pmod q$ with $(a, q) = 1$ and let $\varphi(n)$ be the Euler function. Then

$$\pi(x; q, a) = \frac{1}{\varphi(q)} \operatorname{Li} x + O(xe^{-c\sqrt{\log x}})$$

uniformly in $1 \le q \le (\log x)^H$, where $c > 0$ is a constant which depends on a constant $H > 0$ given arbitrary. (A weaker result $O(x(\log x)^{-G})$ is enough for our purpose.) Let B denote the least common multiple of all denominators of the coefficients, other than the constant term, of $f(t)$. Then

$$\begin{aligned}\sum_{p\le x} I_\pm\left(\frac{f(p)}{r^m}\right) &= \sum_{\substack{p\le x\\(p,Br)=1}} I_\pm\left(\frac{f(p)}{r^m}\right) + O(1)\\
&= \sum_{\substack{a \bmod Br^m\\(a,Br)=1}} I_\pm\left(\frac{f(a)}{r^m}\right)\pi(x; Br^m, a) + O(1)\\
&= \sum_{\substack{a \bmod Br^m\\(a,Br)=1}} I_\pm\left(\frac{f(a)}{r^m}\right)\left(\frac{1}{\varphi(Br^m)}\operatorname{Li} x + O\left(\frac{x}{(\log x)^G}\right)\right)\\
&\quad + O(1)\\
&= \frac{\pi(x)}{\varphi(Br^m)} \sum_{\substack{a \bmod Br^m\\(a,Br)=1}} I_\pm\left(\frac{f(a)}{r^m}\right) + O\left(r^m \frac{x}{(\log x)^G}\right).\end{aligned}$$

Hence we have

$$\sum\nolimits_1 \lesseqgtr \sum_{l\le m\le dM} \frac{\pi(x)}{\varphi(Br^m)} \sum_{\substack{a \bmod Br^m\\(a,Br)=1}} I_\pm\left(\frac{f(a)}{r^m}\right) + O\left(Mr^{dM}\frac{x}{(\log x)^G}\right) \tag{12}$$

$$= \sum_{l\le m\le dM} \frac{\pi(x)}{\varphi(Br^m)} \sum_{a \bmod Br^m} I_\pm\left(\frac{f(a)}{r^m}\right) \sum_{b|(a,Br)} \mu(b) + O\left(\frac{x}{\log x}\right)$$

$$= \sum_{b|Br} \mu(b) \sum_{l\le m\le dM} \frac{\pi(x)}{\varphi(Br^m)} \sum_{\substack{a \bmod Br^m \\ b|a}} I_\pm\left(\frac{f(a)}{r^m}\right) + O\left(\frac{x}{\log x}\right)$$

$$= \pi(x)\frac{Br}{\varphi(Br)} \sum_{b|Br} \mu(b) \sum_{l\le m\le dM} \frac{1}{Br^m} \sum_{1\le n\le Br^m/b} I_\pm\left(\frac{f(bn)}{r^m}\right)$$

$$+ O\left(\frac{x}{\log x}\right),$$

where $\mu(n)$ is the Möbius function. Note that $Br = O(1)$.

Second step. We shall prove that, for each $b\,|\,Br$,

$$\sum_{l\le m\le dM} \frac{1}{Br^m} \sum_{1\le n\le Br^m/b} I_\pm\left(\frac{f(bn)}{r^m}\right) = \sum_{l\le m\le dM} \frac{1}{Br^M} \sum_{1\le n\le Br^M/b} I_\pm\left(\frac{f(bn)}{r^m}\right) + O(1). \tag{13}$$

If $l \le m \le M$, then we have

$$\frac{1}{Br^m} \sum_{1\le n\le Br^m/b} I_\pm\left(\frac{f(bn)}{r^m}\right) = \frac{1}{Br^M} \sum_{1\le n\le Br^M/b} I_\pm\left(\frac{f(bn)}{r^m}\right),$$

so that

$$\sum_{l\le m< M} \frac{1}{Br^m} \sum_{1\le n\le Br^m/b} I_\pm\left(\frac{f(bn)}{r^m}\right) = \sum_{l\le m\le M} \frac{1}{Br^M} \sum_{1\le n< Br^M/b} I_\pm\left(\frac{f(bn)}{r^m}\right). \tag{14}$$

If $d = 1$, (14) implies (13). So in what follows we assume $d \ge 2$ and $M \le m \le dM$. We have

$$\sum_{1\le n\le Br^m/b} I_\pm\left(\frac{f(bn)}{r^m}\right)$$

$$\lesseqgtr \frac{Br^m}{b}\cdot\frac{1}{r^l} + O\left(\frac{r^m}{J}\right) + O\left(\sum_{1\le|\nu|\le J^2} \frac{1}{|\nu|}\left|\sum_{1\le n\le Br^m/b} e\left(\frac{\nu}{r^m} f(bn)\right)\right|\right)$$

$$= \frac{Br^m}{b}\cdot\frac{1}{r^l} + O\left(\frac{r^m}{J}\right) + O(r^{m(1-1/d)}J^{2/d}\log J),$$

since, by Lemma 3,

$$\Bigg|\sum_{1\le n\le Br^m/b} e\bigg(\frac{\nu}{r^m}f(bn)\bigg)\Bigg| \ll (r^m,\nu)^{1/d}r^{m(1-1/d)}.$$

Hence we get

$$\sum_{M\le m\le dM}\frac{1}{Br^m}\sum_{1\le n\le Br^m/b} I_\pm\bigg(\frac{f(bn)}{r^m}\bigg) = \frac{(d-1)M}{br^l} + O(1). \tag{15}$$

In the rest of this step, we shall prove the inequality

$$\sum_{M\le m\le dM}\frac{1}{Br^M}\sum_{1\le n\le Br^M/b} I_\pm\bigg(\frac{f(bn)}{r^m}\bigg) = \frac{(d-1)M}{br^l} + O(1), \tag{16}$$

which together with (15) and (14) yields (13).

P r o o f o f (16). It is easily seen that

$$\begin{aligned}(17)\quad &\sum_{M\le m\le dM}\frac{1}{Br^M}\sum_{1\le n\le Br^M/b} I_\pm\bigg(\frac{f(bn)}{r^m}\bigg)\\ &\lesseqgtr \frac{1}{Br^M}\sum_{M\le m\le dM}\ \sum_{1\le n\le Br^M/b}\bigg(\frac{1}{r^l}+O\bigg(\frac{1}{J}\bigg)\\ &\qquad\qquad + \sum_{1\le|\nu|\le J^2} A_\pm(\nu)e\bigg(\frac{\nu}{r^m}f(bn)\bigg)\bigg)\\ &= \frac{(d-1)M}{br^l} + O(1)\\ &\quad + O\bigg(\sum_{1\le|\nu|\le J^2}\frac{1}{|\nu|}\cdot\frac{1}{Br^M}\sum_{M\le m\le dM}\bigg|\sum_{1\le n\le Br^M/b} e\bigg(\frac{\nu}{r^m}f(bn)\bigg)\bigg|\bigg).\end{aligned}$$

We estimate the last sum. Let H be a large constant. For any ν, m, b, we can choose, by Dirichlet's theorem, coprime integers a and $q = q(\nu, m, b)$ such that

$$1 \le q \le Q^d/(\log Q)^H, \qquad Q = Br^M/b$$

and

$$\bigg|\frac{\nu}{r^m}b^d - \frac{a}{q}\bigg| < \frac{(\log Q)^H}{qQ^d} \qquad (\le 1/q^2).$$

If

$$(\log Q)^H \le q \le Q^d/(\log Q)^H,$$

then by Lemma 4,

$$\Bigg|\sum_{1\le n\le Br^M/b} e\left(\frac{\nu}{r^m}f(bn)\right)\Bigg| \ll \frac{Q}{(\log Q)^G} \ll \frac{r^M}{(\log J)^2}.$$

Hence the contribution of these sums in the last term in (17) is

$$\ll \frac{1}{Br^M}(d-1)M\log J\cdot\frac{r^M}{(\log J)^2} = O(1).$$

Otherwise, we have

$$1 \le q \le (\log Q)^H \qquad (\gg\ll M^H).$$

In particular, $(\nu/r^m)b^d \neq a/q$, since $m \ge M$. Hence

$$\frac{1}{qr^m} \le \left|\frac{\nu}{r^m}b^d - \frac{a}{q}\right| \ll \frac{M^H}{qr^{dM}},$$

so that

$$(dM \ge)\ m \ge dM - H_1\log M,$$

with a large constant H_1. From this it follows that

$$\frac{d}{dt}\cdot\frac{\nu}{r^m}f(bt) \gg\ll \frac{\nu}{r^m}t^{d-1} \ll J^2 r^{-M+H_1\log M} = o(1)$$

throughout the interval $[1, Br^M/b]$. Thus by a van der Corput's lemma ([9; Lemma 4.8]) we have

$$\begin{aligned}\sum_{1\le n\le Br^M/b} e\left(\frac{\nu}{r^m}f(bn)\right) &= \int_1^{Br^M/b} e\left(\frac{\nu}{r^m}f(bt)\right)dt + O(1)\\ &\ll \left|\frac{\nu}{r^m}f^{(d)}(t)\right|^{-1/d} + O(1) \ll \left(\frac{|\nu|}{r^m}\right)^{-1/d},\end{aligned}$$

using again Lemma 1. Hence the contribution of these sums to the last term in (17) is

$$\ll \frac{1}{Br^M}\sum_{M\le m\le dM}\ \sum_{1\le|\nu|\le J^2}\frac{1}{|\nu|}\left(\frac{|\nu|}{r^m}\right)^{-1/d} = O(1).$$

Combining these results, we obtain (16).

Third step. It follows from (12) with (13) that

$$\sum\nolimits_1 \lesseqgtr \pi(x)\frac{Br}{\varphi(Br)}\sum_{b|Br}\mu(b)\frac{1}{Br^M}\sum_{l\le m\le dM}\ \sum_{1\le n\le Br^M/b} I_{\pm}\biggl(\frac{f(bn)}{r^m}\biggr) + O\biggl(\frac{x}{\log x}\biggr)$$

$$\lesseqgtr \pi(x)\frac{Br}{\varphi(Br)}\sum_{b|Br}\mu(b)\frac{1}{Br^M}\sum_{l\le m\le dM}\ \sum_{1\le n\le Br^M/b} I\biggl(\frac{f(bn)}{r^m}\biggr) + O\biggl(\frac{x}{\log x}\biggr).$$

We put, in Lemma 5, $y = Br^M/b$, so that $\log_r f(by) = dM + O(1)$. Then we have

$$\sum_{l\le m\le dM}\ \sum_{1\le n\le Br^M/b} I\biggl(\frac{f(bn)}{r^m}\biggr) = \sum_{n\le y} N(f(bn); b_1\ldots b_l) + O(r^M)$$
$$= r^{-l} y \log_r f(by) + O(r^M)$$
$$= r^{-l}\frac{Br^M}{b}\, dM + O(r^M).$$

Therefore we obtain

$$\sum\nolimits_1 \gtreqless \frac{Br}{\varphi(Br)}\sum_{b|Br}\frac{\mu(b)}{b}\cdot\frac{dM}{r^l}\pi(x) + O\biggl(\frac{x}{\log x}\biggr)$$
$$= r^{-l} dM\pi(x) + O\biggl(\frac{x}{\log x}\biggr),$$

which is (11). The proof of the Theorem is now complete.

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