ACTA ARITHMETICA
LXXXI.3 (1997)

Cyclic coverings of an elliptic curve with two branch points and the gap sequences at the ramification points

by

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1. Introduction. Let C be a complete non-singular irreducible algebraic curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic 0, which is called a *curve* in this paper. Let P be its point. A positive integer γ is called a *gap* at P if there exists a regular 1-form ω on C such that $\mathrm{ord}_P(\omega) = \gamma - 1$. We denote by $G(P)$ the set of gaps at P. Then the cardinality of $G(P)$ is equal to g. Now the sequence $\{\gamma_1, \ldots, \gamma_g\} = G(P)$ with $\gamma_i < \gamma_j$ for $i < j$ is called the *gap sequence* at P.

Let $\pi : C \to C'$ be a cyclic covering of curves of degree d with total ramification points P. It is well known that in the case of $C' = \mathbb{P}^1$ and $d = 2$ we have $G(P) = \{1, 3, \ldots, 2g - 1\}$. In the case of $C' = \mathbb{P}^1$ and $d = 3$ (resp. 4) the gap sequences $G(P)$ are known (see [1], [2], [3] (resp. [4], Prop. 4.5)). If $C' = \mathbb{P}^1$ and d is a prime number ≥ 5, we can also determine the gap sequences $G(P)$ (for example, see [5], Prop. 1). In this paper we shall consider the case $C' = E$ where E is an elliptic curve. If $d = 2$, then $G(P)$ are known ([4], Prop. 2.9, 3.10). However, for $d \geq 3$ there are only a few results on the gap sequences $G(P)$. For example, I. Kuribayashi and K. Komiya ([8], Th. 5) showed the following: If $\pi : C \to E$ is a cyclic covering of an elliptic curve of degree 6 which is branched over three points P'_i $(i = 1, 2, 3)$ such that $\sharp\pi^{-1}(P'_i) = i$, then the gap sequence $G(P_1)$ can be determined, where P_1 denotes the point of C over P'_1. Moreover, the author ([6], Lemma 4.6) showed the following: Let E be an elliptic curve with the origin Q'. Let P'_1 (resp. P'_2) be a point of E such that $P'_1 \neq Q'$ and $2[P'_1] = [Q']$ (resp. $P'_2 \neq Q'$ and $3[P'_2] = [Q']$), where for any positive integer m and any point P' of the elliptic curve E the multiplication of P' by m is denoted by $m[P']$. Then there is an element z of $\mathbf{K}(E)$ such

1991 *Mathematics Subject Classification*: Primary 14H55; Secondary 14H30, 14H52, 11A05.

that $\operatorname{div}(z) = 4P_1' + 3P_2' - 7Q'$ where $\mathbf{K}(E)$ denotes the function field of E. Let $\pi : C \to E$ be the surjective morphism of curves corresponding to the inclusion $\mathbf{K}(E) \subset \mathbf{K}(E)(z^{1/7}) = \mathbf{K}(C)$. If P_2 denotes the point of C over P_2', then the gap sequence $G(P_2)$ is equal to $\{1, 2, 3, 4, 5, 7, 13\}$. In this paper we shall prove the generalization of the above statement for the degree of the covering $\pi : C \to E$, which is the following:

Main Theorem. *Let $g \geq 7$. We can construct cyclic coverings $\pi : C \to E$ of an elliptic curve E of degree g which have only two ramification points P_1 and P_2, which are totally ramified, such that*

$$G(P_1) = G(P_2) = \{1, \ldots, g-2, g, 2g-1\}.$$

Now we consider the following situation. Let G be a finite subset of the set $\mathbb{N}$ of positive integers such that the complement $\mathbb{N}_0 \setminus G$ of G in the additive semigroup $\mathbb{N}_0$ of non-negative integers forms its subsemigroup. If the cardinality of G is g, then $\{\gamma_1, \ldots, \gamma_g\} = G$ with $\gamma_i < \gamma_j$ for $i < j$ is called a *gap sequence of genus g*. We say that a gap sequence G is *Weierstrass* if there exists a pointed curve (C, P) such that $G = G(P)$. Let $a(G) = \min\{h \in \mathbb{N}_0 \setminus G \mid h > 0\}$. Then $a(G) \leq g + 1$. If $a(G) = g + 1$, then $G = \{1, \ldots, g\}$. In this case G is Weierstrass, because for any point P of a curve of genus g except finitely many points we have $G(P) = \{1, \ldots, g\}$. If $a(G) = g$, then there is a positive integer $k \leq g-1$ such that $G = \{1, \ldots, g-1, g+k\}$. These $g-1$ kinds of gap sequences are Weierstrass (cf. [9], Th. 14.5). If l is a fixed integer ≥ 2, then for any sufficiently large g there exists a non-Weierstrass gap sequence G of genus g such that $a(G) = g - l$ (cf. [7], Th. 3.5 and 4.5). Hence we pose the following problem: *Is any gap sequence G of genus g with $a(G) = g - 1$ Weierstrass?*

Now we say that G is *primitive* if $2a(G) > \gamma_g$. Since any gap sequence of genus $g \leq 7$ is Weierstrass (cf. [6], Th. 4.7), combining the Main Theorem with Lemma 1 we get the following:

Any non-primitive gap sequence G of genus g with $a(G) = g - 1$ is Weierstrass.

In Sections 2, 3 and 4 we construct our desired cyclic coverings $\pi : C \to E$ of an elliptic curve in the cases when $g \equiv 3$, 1 and 0 mod 4 respectively. In Section 5 the case when $g \equiv 2 \bmod 4$ is treated. In this case we need an arithmetic lemma (Key Lemma 4) which is important for the constructions of the coverings $\pi : C \to E$.

2. The case $g \equiv 3 \bmod 4$**.** First we prove the following:

Lemma 1. *Let G be a non-primitive gap sequence of genus $g \geq 3$ with $a(G) = g - 1$. Then $G = \{1, \ldots, g-2, g, 2g-1\}$.*

Proof. Let $G = \{\gamma_1, \ldots, \gamma_g\}$ with $\gamma_i < \gamma_j$ for $i < j$. In view of $a(G) = g-1$ we must have $\gamma_i = i$ for $i \le g-2$ and $\gamma_{g-1} \ge g$. Since G is non-primitive, we have $\gamma_g > 2a(G) = 2g-2$. It is a well-known fact that $\gamma_g \le 2g-1$ (for example, see [4], Lemma 2.1), which implies that $\gamma_g = 2g-1$. Suppose that $\gamma_{g-1} \ge g+1$. Then $\mathbb{N}_0 \backslash G$ contains $g-1$ and g. Since $\mathbb{N}_0 \backslash G$ is a subsemigroup of $\mathbb{N}_0$, we must have $\gamma_g = 2g-1 \in \mathbb{N}_0 \setminus G$, which is a contradiction. Hence we obtain $\gamma_{g-1} = g$. ∎

In the remainder of this section we will prove the Main Theorem in the case $g \equiv 3 \bmod 4$ with $g \ge 7$.

Let $g = 4h+3 = 2n+1$ with $h \in \mathbb{N}$ and $n = 2h+1$. Let E be an elliptic curve over k with the origin Q'. Let P_1' be a point of E such that $P_1' \neq Q'$ and $2[P_1'] = [Q']$. Moreover, P_2' denotes a point of E such that $n[P_2'] = [Q']$ and $m[P_2'] \neq [Q']$ for any positive integer $m < n$. Hence in view of $g \ge 7$ we have $P_2' \neq Q'$. Moreover, $P_1' \neq P_2'$, because $2hP_2' + P_2' = nP_2' \sim nQ' = (2h+1)Q' \sim 2hP_1' + Q'$. Now we have

$$(n+1)P_1' + nP_2' \sim 2(h+1)P_1' + nQ' \sim 2(h+1)Q' + nQ' = (2n+1)Q'.$$

Hence we may take $z \in \mathbf{K}(E)$ such that $\mathrm{div}(z) = (n+1)P_1' + nP_2' - (2n+1)Q'$.

Let C be the curve whose function field $\mathbf{K}(C)$ is $\mathbf{K}(E)(z^{1/(2n+1)})$. Moreover, $\pi : C \to E$ denotes the surjective morphism of curves corresponding to the inclusion $\mathbf{K}(E) \subset \mathbf{K}(C)$. Then we may take $y \in \mathbf{K}(C)$ and $\sigma \in \mathrm{Aut}(\mathbf{K}(C)/\mathbf{K}(E))$ such that

$$\sigma(y) = \zeta_{2n+1} y \quad \text{and} \quad \mathrm{div}_E(y^{2n+1}) = (n+1)P_1' + nP_2' - (2n+1)Q',$$

where ζ_{2n+1} is a primitive $(2n+1)$th root of unity. Then there are only two branch points P_1' and P_2' of π. Moreover, $\pi^{-1}(P_i')$ consists of only one point P_i for $i = 1, 2$. Hence the ramification index of P_i is $2n+1$ for $i = 1, 2$. Therefore

$$\mathrm{div}(y) = (n+1)P_1 + nP_2 - \pi^*(Q'),$$

where π^* denotes the pull-back of π. If we denote by g the genus of C, then by the Riemann–Hurwitz formula we have $g = 2n+1$. Hence

$$\mathrm{div}(dy) = nP_1 + (n-1)P_2 - 2\pi^*(Q') + \sum_{i=1}^{3} \pi^*(R_i'),$$

where R_i''s are points of E which are distinct from P_1', P_2' and Q', because $\mathrm{div}(dy)$ is invariant under $\mathrm{Aut}(\mathbf{K}(C)/\mathbf{K}(E))$.

We set

$$D_0' = -P_1' - P_2' - Q' + \sum_{i=1}^{3} R_i',$$

$$D'_{2l+1} = -(2l+2)Q' + lP'_1 + lP'_2 + \sum_{i=1}^{3} R'_i \quad \text{for } 0 \le l \le n-1$$

and

$$D'_{2l} = -(2l+1)Q' + lP'_1 + (l-1)P'_2 + \sum_{i=1}^{3} R'_i \quad \text{for } 1 \le l \le n.$$

First we show that $l(D'_0) = 1$, i.e., D'_0 is linearly equivalent to 0, where for any divisor D' on E the number $l(D')$ denotes the dimension of the k-vector space

$$L(D') = \{f \in \mathbf{K}(E) \mid \mathrm{div}_E(f) \ge -D'\}.$$

Since

$$\sigma\left(\frac{dy}{y}\right) = \frac{d(\sigma y)}{\sigma y} = \frac{d(\zeta_{2n+1} y)}{\zeta_{2n+1} y} = \frac{dy}{y},$$

the 1-form dy/y on C is regarded as the one on E. Hence there exists an element f of $\mathbf{K}(E)$ such that $f dy/y$ is regular. Then

$$\mathrm{div}_E(f) = P'_1 + P'_2 + Q' - \sum_{i=1}^{3} R'_i$$

because

$$\begin{aligned} 0 \le \mathrm{div}_C\left(\frac{f dy}{y}\right) &= \mathrm{div}_C(f) + \mathrm{div}_C\left(\frac{dy}{y}\right) \\ &= \mathrm{div}_C(f) - P_1 - P_2 - \pi^*(Q') + \sum_{i=1}^{3} \pi^*(R'_i). \end{aligned}$$

Hence

$$D'_0 = -P'_1 - P'_2 - Q' + \sum_{i=1}^{3} R'_i \sim 0.$$

Moreover, $l(D'_r) = 1$ for any r with $1 \le r \le 2n$, because $\deg(D'_r) = 1$.

To compute the numbers $l(D'_r - P'_1)$ and $l(D'_r - P'_2)$ we show that $mP'_1 \not\sim mP'_2$ for any positive integer m with $m \le n$. In fact, suppose that there exists a positive integer $m \le n$ such that $mP'_1 \sim mP'_2$. If m is even, then

$$mP'_2 \sim \frac{m}{2} 2P'_1 \sim \frac{m}{2} 2Q' = mQ',$$

which is a contradiction. Let m be odd. Then $2mP'_2 \sim 2mP'_1 \sim 2mQ'$. If $m < n/2$, then

$$(n-2m)P'_2 = nP'_2 - 2mP'_2 \sim nQ' - 2mQ' = (n-2m)Q',$$

a contradiction. If $n/2 < m < n$, then $(2m-n)P_2' \sim (2m-n)Q'$, a contradiction. If $m = n$, then

$$(n-1)Q' + P_1' \sim (n-1)P_1' + P_1' \sim nP_2' \sim nQ',$$

which implies that $P_1' \sim Q'$. This is a contradiction. Hence we have shown that for any m with $0 < m \leq n$, $mP_1' \not\sim mP_2'$.

Now for any l with $0 \leq l \leq n-2$ we have $l(D_{2l+1}' - P_1') = 0$. In fact, suppose that $l(D_{2l+1}' - P_1') = 1$. Then

$$\begin{aligned} 0 &\sim D_{2l+1}' - P_1' - D_0' \sim (n-2l-1)Q' + lP_1' + (l+1-n)P_2' \\ &\sim (n-l-1)P_1' - (n-l-1)P_2', \end{aligned}$$

because $nQ' \sim nP_2'$ and $2P_1' \sim 2Q'$. Hence

$$1 \leq n-l-1 \leq n-1 \quad \text{and} \quad (n-l-1)P_1' \sim (n-l-1)P_2',$$

which is a contradiction.

Now in view of $2P_1' \sim 2Q'$ and $nP_2' \sim nQ'$ we have

$$D_{2n-1}' - P_1' - D_0' \sim -(2n-1)Q' + (n-1)Q' + nQ' = 0,$$

which implies that $D_{2n-1}' - P_1' \sim 0$. Hence

$$l(D_{2n-1}') = l(D_{2n-1}' - P_1') = 1 \quad \text{and} \quad l(D_{2n-1}' - 2P_1') = 0.$$

Suppose that $l(D_{2l}' - P_1') = 1$. Then in view of $2P_1' \sim 2Q'$ we have

$$0 \sim D_{2l}' - P_1' - D_0' \sim -2lP_1' + lP_1' + lP_2' = -lP_1' + lP_2',$$

a contradiction. Hence $l(D_{2l}' - P_1') = 0$ for any l with $1 \leq l \leq n$.

Next we show that $l(D_1' - P_2') = 0$. If $l(D_1' - P_2') = 1$, then

$$-2Q' + \sum_{i=1}^{3} R_i' - P_2' = D_1' - P_2' \sim 0 \sim D_0' \sim -P_1' - P_2' - Q' + \sum_{i=1}^{3} R_i',$$

which implies that $P_1' \sim Q'$. This is a contradiction. Now in view of $2P_1' \sim 2Q'$ we obtain $D_2' - P_2' \sim D_0' \sim 0$, which implies that

$$l(D_2') = l(D_2' - P_2') = 1 \quad \text{and} \quad l(D_2' - 2P_2') = 0.$$

Let $1 \leq l \leq n-1$. Suppose that $l(D_{2l+1}' - P_2') = 1$. Then

$$-(2l+2)Q' + lP_1' + (l-1)P_2' + \sum_{i=1}^{3} R_i' \sim D_0' \sim -P_1' - P_2' - Q' + \sum_{i=1}^{3} R_i',$$

which implies that $-(l+1)P_1' \sim -(2l+1)Q' + lP_2'$. Since $nP_2' \sim nQ'$ and n is odd, we have

$$nP_2' - (l+1)P_1' \sim (n-(2l+1))Q' + lP_2' \sim (n-(2l+1))P_1' + lP_2',$$

which implies that $(n-l)P_2' \sim (n-l)P_1'$. This contradicts $mP_1' \not\sim mP_2'$ for any $0 < m < n$. Hence $l(D_{2l+1}' - P_2') = 0$ for any $1 \leq l \leq n-1$.

Let $2 \le l \le n$. Suppose that $l(D'_{2l} - P'_2) = 1$. Then

$$-(2l+1)Q' + lP'_1 + (l-2)P'_2 + \sum_{i=1}^{3} R'_i \sim -P'_1 - P'_2 - Q' + \sum_{i=1}^{3} R'_i,$$

which implies that $(l+1)P'_1 + (l-1)P'_2 \sim 2lQ' \sim 2lP'_1$. Hence $(l-1)P'_2 \sim (l-1)P'_1$, a contradiction. Therefore $l(D'_{2l} - P'_2) = 0$ for any $2 \le l \le n$.

Now let f be an element of $\mathbf{K}(E)$ and set

$$\mathrm{div}_E(f) = \sum_{P' \in E} m(P')P'.$$

Then for any non-negative integer r we obtain

$$\begin{aligned}\mathrm{div}_C\left(\frac{f\,dy}{y^{1-r}}\right) &= ((2n+1)m(P'_1) + n + (n+1)(r-1))P_1 \\ &\quad + ((2n+1)m(P'_2) + n - 1 + n(r-1))P_2 \\ &\quad + (m(Q') - r - 1)\pi^*(Q') \\ &\quad + \sum_{i=1}^{3}(m(R'_i)+1)\pi^*(R'_i) + \sum_{P' \in S} m(P')\pi^*(P'),\end{aligned}$$

where we set $S = E \setminus \{P'_1, P'_2, Q', R'_1, R'_2, R'_3\}$. We note that if $R'_1 \ne R'_2$ and $R'_2 = R'_3$ (resp. $R'_1 = R'_2 = R'_3$), then

$$\sum_{i=1}^{3}(m(R'_i)+1)\pi^*(R'_i)$$

is replaced by

$$(m(R'_1)+1)\pi^*(R'_1) + (m(R'_2)+2)\pi^*(R'_2) \quad (\text{resp. } (m(R'_1)+3)\pi^*(R'_1)).$$

For each $r = 0, 1, \ldots, 2n$, we take a non-zero element $f_r \in L(D'_r)$ and set $\phi_r = f_r\,dy/y^{1-r}$. Then by the above,

$$\mathrm{ord}_{P_i}(\phi_0) = 2n + 1 - 1 = g - 1 \quad \text{for } i = 1, 2.$$

For any l with $0 \le l \le n-2$ we have

$$\mathrm{ord}_{P_1}(\phi_{2l+1}) = n + l + 1 - 1 \quad \text{and} \quad \mathrm{ord}_{P_2}(\phi_{2l+1}) = n - l - 1.$$

Let $l = n-1$, i.e., $2l+1 = 2n-1$. Since $L(D'_{2n-1}) = L(D'_{2n-1} - P'_1)$ and $L(D'_{2n-1}) \supset L(D'_{2n-1} - P'_2) = (0)$, we obtain

$$\mathrm{ord}_{P_1}(\phi_{2n-1}) = 4n + 1 - 1 = 2g - 1 - 1 \quad \text{and} \quad \mathrm{ord}_{P_2}(\phi_{2n-1}) = 1 - 1.$$

Let $l = 1$, i.e., $2l = 2$. Since $L(D'_2) \supset L(D'_2 - P'_1) = (0)$ and $L(D'_2) = L(D'_2 - P'_2)$, we obtain

$$\mathrm{ord}_{P_1}(\phi_2) = 1 - 1 \quad \text{and} \quad \mathrm{ord}_{P_2}(\phi_2) = 2g - 1 - 1.$$

For any l with $2 \le l \le n$ we have

$$\mathrm{ord}_{P_1}(\phi_{2l}) = l - 1 \quad \text{and} \quad \mathrm{ord}_{P_2}(\phi_{2l}) = 2n - l + 1 - 1.$$

Hence for each $r = 0, 1, \ldots, 2n$, ϕ_r is a regular 1-form on C. Therefore $G(P_1) = G(P_2) = \{1, \ldots, g-2, g, 2g-1\}$.

3. The case $g \equiv 1 \bmod 4$**.** In this section we prove the Main Theorem in the case $g \equiv 1 \bmod 4$ with $g \ge 9$.

Let $g = 4h+1 = 2n+1$ with $h \in \mathbb{N}$, $h \ge 2$ and $n = 2h$. Let E be an elliptic curve over k with the origin Q'. Let P_1' be a point of E such that $P_1' \ne Q'$ and $2[P_1'] = [Q']$. Moreover, P_2' denotes a point of E such that $n[P_2'] = -[P_1']$ and $m[P_2'] \ne -[P_1']$ for any positive integer $m < n$, where $-[P_1']$ denotes the inverse of P_1' under the addition on the elliptic curve E. Then $P_2' \ne Q'$ and $P_1' \ne P_2'$. Moreover, $(n+1)P_1' + nP_2' \sim nQ' + P_1' + (n+1)Q' - P_1' = (2n+1)Q'$. Hence we may take $z \in \mathbf{K}(E)$ such that $\mathrm{div}(z) = (n+1)P_1' + nP_2' - (2n+1)Q'$.

Let C, $\pi : C \to E$, $y \in \mathbf{K}(C)$, P_1, P_2, R_i', D_0', D_{2l+1}' and D_{2l}' be as in Section 2. Then, in the same way as in Section 2, D_0' is linearly equivalent to zero. Moreover, $l(D_r') = 1$ for any r with $1 \le r \le 2n$.

To compute the numbers $l(D_r' - P_1')$ and $l(D_r' - P_2')$ we show that for any positive integer m with $m \le n$, $mP_1' \not\sim mP_2'$. In fact, suppose that there exists a positive integer $m \le n$ such that $mP_1' \sim mP_2'$. If m is odd, then $mP_2' + P_1' \sim (m+1)P_1' \sim (m+1)Q'$. This contradicts $m[P_2'] \ne -[P_1']$ for any positive integer $m < n$. If m is even, then

$$\begin{aligned}(n+1)Q' &\sim nP_2' + P_1' = (n-m)P_2' + P_1' + mP_2' \\ &\sim (n-m)P_2' + P_1' + mP_1' \sim (n-m)P_2' + P_1' + mQ',\end{aligned}$$

which implies that $(n-m)P_2' + P_1' \sim (n+1-m)Q'$. This is a contradiction.

For any l with $0 \le l \le n-2$ we have $l(D_{2l+1}' - P_1') = 0$. In fact, suppose that $l(D_{2l+1}' - P_1') = 1$. Then $0 \sim D_{2l+1}' - P_1' - D_0' = -(2l+1)Q' + lP_1' + (l+1)P_2'$. Since $nP_2' + P_1' \sim (n+1)Q'$ and n is even, we have

$$\begin{aligned}nP_2' - lP_1' &\sim -P_1' + (n+1)Q' - (2l+1)Q' + (l+1)P_2' \\ &= -P_1' + (l+1)P_2' + (n-2l)Q' \sim -P_1' + (l+1)P_2' + (n-2l)P_1',\end{aligned}$$

which implies that $(n-l-1)P_2' \sim (n-l-1)P_1'$. This contradicts $mP_1' \not\sim mP_2'$ for $1 \le m \le n$. Since $nP_2' + P_1' \sim (n+1)Q'$ and n is even, we have

$$\begin{aligned}D_{2n-1}' - P_1' - D_0' &\sim -(2n-1)Q' + (n-1)P_1' + nP_2' \\ &= -(n-2)Q' + (n-2)P_1' \sim -(n-2)Q' + (n-2)Q' = 0,\end{aligned}$$

which implies that $l(D_{2n-1}') = 1 = l(D_{2n-1}' - P_1')$. Moreover, in the same way as in Section 2, we obtain $l(D_{2l}' - P_1') = 0$ for any l with $1 \le l \le n$.

Next, as in Section 2, we have

$$l(D_1' - P_2') = 0 \quad \text{and} \quad l(D_2') = l(D_2' - P_2') = 1.$$

Let $1 \le l \le n-1$. Suppose $l(D'_{2l+1} - P'_2) = 1$. Then $D'_{2l+1} - P'_2 \sim 0 \sim D'_0$, which implies that $-(l+1)P'_1 \sim -(2l+1)Q' + lP'_2$. Since $nP'_2 + P'_1 \sim (n+1)Q'$ and n is even, we have $nP'_2 - lP'_1 \sim (n+1)Q' - (2l+1)Q' + lP'_2 \sim (n-2l)P'_1 + lP'_2$, which implies that $(n-l)P'_2 \sim (n-l)P'_1$. This is a contradiction. Hence $l(D'_{2l+1} - P'_2) = 0$ for any $1 \le l \le n-1$.

As in Section 2 we have $l(D'_{2l} - P'_2) = 0$ for any $2 \le l \le n$. Therefore $G(P_1) = G(P_2) = \{1, \ldots, g-2, g, 2g-1\}$.

4. The case $g \equiv 0 \bmod 4$**.** First we show the following lemma, which is useful to construct the desired coverings of an elliptic curve in the even genus cases.

LEMMA 2. *Let $\pi_0 : C \to C_0$ be a finite morphism of curves of degree 2. Let $P \in C$ be a ramification point of π_0. Then $n \in \mathbb{N}_0 \setminus G(\pi_0(P))$ if and only if $2n \in \mathbb{N}_0 \setminus G(P)$.*

Proof. Suppose that $n \in \mathbb{N}_0 \setminus G(\pi_0(P))$, i.e., there exists $f_0 \in \mathbf{K}(C_0)$ such that $(f_0)_\infty = n\pi_0(P)$, where $(f_0)_\infty$ denotes the polar divisor of f_0. Since P is a ramification point of π_0, we have $(\pi_0^* f_0)_\infty = 2nP$, where π_0^* denotes the inclusion map $\mathbf{K}(C_0) \subset \mathbf{K}(C)$ corresponding to the surjective morphism $\pi_0 : C \to C_0$. Hence $2n \in \mathbb{N}_0 \setminus G(P)$.

Conversely, suppose that $2n \in \mathbb{N}_0 \setminus G(P)$, i.e., there exists $f \in \mathbf{K}(C)$ such that $(f)_\infty = 2nP$. Let σ be an involution of C such that $C/\langle\sigma\rangle \cong C_0$. Then we may take a local parameter t at P such that $\sigma^* t = -t$. Since we can write

$$f = c_{-2n} t^{-2n} + c_{-2n+1} t^{-2n+1} + \ldots$$

where c_{-2n} is a non-zero constant and c_i's ($i \ge -2n+1$) are constants, we obtain

$$\sigma^* f = c_{-2n} t^{-2n} - c_{-2n+1} t^{-2n+1} + \ldots$$

Hence

$$f + \sigma^* f = 2c_{-2n} t^{-2n} + 2c_{-2n+2} t^{-2n+2} + \ldots,$$

which implies that $(f + \sigma^* f)_\infty = 2nP$. Now

$$\sigma^*(f + \sigma^* f) = \sigma^* f + (\sigma^2)^* f = f + \sigma^* f,$$

which implies that $f + \sigma^* f \in \mathbf{K}(C_0)$. Therefore $(f + \sigma^* f)_\infty = n\pi_0(P)$ on C_0, which implies that $n \in \mathbb{N}_0 \setminus G(\pi_0(P))$. ∎

Using the above lemma we get the following:

PROPOSITION 3. *Let $\pi_0 : C \to C_0$ be a finite morphism of curves of degree 2. Suppose that the genus g of C is even and that the genus of C_0 is equal to $g/2$. Let $P \in C$ be a ramification point of π_0. If $G(P)$ contains $\{2, 4, \ldots, g-2, g, 2g-1\}$, then $G(P) = \{1, 2, \ldots, g-2, g, 2g-1\}$.*

P r o o f. Suppose that $G(P) \supset \{2, 4, \ldots, g-2, g, 2g-1\}$. Then by Lemma 2 we obtain

$$G(\pi_0(P)) = \{1, 2, \ldots, g/2\}.$$

If h is an even integer $> g$, then by the above we have $h/2 \in \mathbb{N}_0 \setminus G(\pi_0(P))$. Hence by Lemma 2 we get $h \in \mathbb{N}_0 \setminus G(P)$. On the other hand, if h is an even integer with $g+2 \le h \le 2g-2$, then $2g-1-h \in G(P)$. In fact, if $2g-1-h \in \mathbb{N}_0 \setminus G(P)$, then $2g-1 = h + (2g-1-h) \in \mathbb{N}_0 \setminus G(P)$, a contradiction. Hence $G(P)$ contains the set

$$\{2, 4, \ldots, g-2, g, 2g-1\} \cup \{2g-1-h \mid h \text{ is even with } g+2 \le h \le 2g-2\} = \{1, 2, 3, 4, \ldots, g-3, g-2, g, 2g-1\}.$$

Since the cardinality of $G(P)$ is g, we get the desired result. ■

Using this result we show the Main Theorem in the case $g \equiv 0 \bmod 4$ with $g \ge 8$.

Let $g = 4h = 2n$ with $h \in \mathbb{N}$, $h \ge 2$ and $n = 2h$. Let E be an elliptic curve over k with the origin Q'. Let P_1' be a point of E such that $(2n-1)[P_1'] = [Q']$ and $m[P_1'] \ne [Q']$ for any positive integer $m < 2n-1$. Moreover, P_2' denotes the point of E such that $[P_2'] = 3[P_1']$. Then $P_2' \ne Q'$ and $P_1' \ne P_2'$ because $g \ge 8$. Now we have

$$(n+1)P_1' + (n-1)P_2' \sim (n+1)P_1' + (n-1)(3P_1' - 2Q') \sim 2(2n-1)P_1' - (2n-2)Q' \sim 2nQ'.$$

Hence we may take $z \in \mathbf{K}(E)$ such that $\mathrm{div}(z) = (n+1)P_1' + (n-1)P_2' - 2nQ'$.

Let C be the curve whose function field $\mathbf{K}(C)$ is $\mathbf{K}(E)(z^{1/(2n)})$. Moreover, $\pi : C \to E$ denotes the surjective morphism of curves corresponding to the inclusion $\mathbf{K}(E) \subset \mathbf{K}(C)$. Then we may take $y \in \mathbf{K}(C)$ and $\sigma \in \mathrm{Aut}(\mathbf{K}(C)/\mathbf{K}(E))$ such that

$$\sigma(y) = \zeta_{2n} y \quad \text{and} \quad \mathrm{div}_E(y^{2n}) = (n+1)P_1' + (n-1)P_2' - 2nQ'.$$

Since n is even, we get $(2n, n+1) = (2n, n-1) = 1$. Therefore the branch points of π are P_1' and P_2' whose ramification indices are $2n$. Therefore

$$\mathrm{div}(y) = (n+1)P_1 + (n-1)P_2 - \pi^*(Q').$$

Moreover, by the Riemann–Hurwitz formula we have $g(C) = 2n = g$. Hence

$$\mathrm{div}(dy) = nP_1 + (n-2)P_2 - 2\pi^*(Q') + \sum_{i=1}^{3} \pi^*(R_i'),$$

where R_i''s are points of E which are distinct from P_1', P_2' and Q'.

Let D_0' and D_{2l}' $(1 \le l \le n-1)$ be as in Section 2. Moreover, we set

$$D_{n-1}' = D_{2(n/2-1)+1}' = -nQ' + \left(\frac{n}{2} - 1\right)P_1' + \left(\frac{n}{2} - 1\right)P_2' + \sum_{i=1}^{3} R_i'$$

and

$$D'_{n+1} = D'_{2\cdot n/2+1} = -(n+2)Q' + \left(\frac{n}{2}+1\right)P'_1 + \left(\frac{n}{2}-1\right)P'_2 + \sum_{i=1}^{3} R'_i.$$

Then $D'_0 \sim 0$. Moreover, for any l with $1 \le l \le n-1$ we have $l(D'_{2l}) = 1$ and $l(D'_{2l} - P'_1) = l(D'_{2l} - P'_2) = 0$. In fact, first assume $l(D'_{2l} - P'_1) = 1$. Then $0 \sim D'_{2l} - P'_1 - D'_0 \sim 4lP'_1 - 4lQ'$, which implies that $2n-1$ divides $4l$. In view of $1 \le l \le n-1$ we must have $4l = 2n-1$, which is a contradiction. Secondly, assume $l(D'_{2l} - P'_2) = 1$. Then $0 \sim D'_{2l} - P'_2 - D'_0 \sim -(4l-2)Q' + (4l-2)P'_1$, which implies that $2n-1$ divides $4l-2$. This is a contradiction. Now we have

$$D'_{n-1} - P'_1 - D'_0 \sim (2n-1)P'_1 - (2n-1)Q' \sim 0,$$

which implies that $l(D'_{n-1}) = l(D'_{n-1} - P'_1) = 1$ and $l(D'_{n-1} - 2P'_1) = 0$. Moreover, $D'_{n+1} - P'_2 - D'_0 \sim -(2n-1)Q' + (2n-1)P'_1 \sim 0$, which implies that $l(D'_{n+1}) = l(D'_{n+1} - P'_2) = 1$ and $l(D'_{n+1} - 2P'_2) = 0$.

Let $f \in \mathbf{K}(E)$ and set

$$\mathrm{div}_E(f) = \sum_{P' \in E} m(P')P'.$$

Then for any non-negative integer r we obtain

$$\begin{aligned}\mathrm{div}_C\left(\frac{f\,dy}{y^{1-r}}\right) &= (2nm(P'_1) + n + (n+1)(r-1))P_1 \\ &\quad + (2nm(P'_2) + n - 2 + (n-1)(r-1))P_2 \\ &\quad + (m(Q') - r - 1)\pi^*(Q') \\ &\quad + \sum_{i=1}^{3}(m(R'_i)+1)\pi^*(R'_i) + \sum_{P' \in S} m(P')\pi^*(P'),\end{aligned}$$

where we set $S = E \setminus \{P'_1, P'_2, Q', R'_1, R'_2, R'_3\}$.

For each $r \in \{0, 2, \ldots, 2n-2\} \cup \{n-1\} \cup \{n+1\}$ we take a non-zero element $f_r \in L(D'_r)$ and set $\phi_r = f_r dy/y^{1-r}$. Then, by the above, $\mathrm{ord}_{P_i}(\phi_0) = 2n-1 = g-1$ for $i = 1, 2$. For any l with $1 \le l \le n-1$ we have $\mathrm{ord}_{P_1}(\phi_{2l}) = 2l-1$ and $\mathrm{ord}_{P_2}(\phi_{2l}) = 2(n-l)-1$. Moreover,

$$\mathrm{ord}_{P_1}(\phi_{n-1}) = 4n-1-1 = 2g-1-1,$$

$$\mathrm{ord}_{P_2}(\phi_{n-1}) \ge -2n\left(\frac{n}{2}-1\right) + n - 2 + (n-1)(n-2) = 0,$$

$$\mathrm{ord}_{P_1}(\phi_{n+1}) \ge -2n\left(\frac{n}{2}+1\right) + n + (n+1)n = 0 \quad \text{and} \quad \mathrm{ord}_{P_2}(\phi_{n+1}) = 2g-1-1.$$

Hence $\phi_0, \phi_2, \ldots, \phi_{2n-2}, \phi_{n-1}, \phi_{n+1}$ are regular 1-forms on C. Therefore we get $G(P_i) \supset \{2, 4, \ldots, g-2, g, 2g-1\}$ for $i = 1, 2$.

Now let C_0 be the curve whose function field $\mathbf{K}(C_0)$ is $\mathbf{K}(E)(z^{1/n})$. Moreover, $\eta : C_0 \to E$ denotes the surjective morphism of curves corresponding to the inclusion $\mathbf{K}(E) \subset \mathbf{K}(C_0)$. Let $\pi_0 : C \to C_0$ be the double covering corresponding to the inclusion $\mathbf{K}(C_0) \subset \mathbf{K}(C)$. Since $\pi = \eta \circ \pi_0 : C \to E$ has only two ramification points P_1 and P_2, which are totally ramified, by the Riemann–Hurwitz formula we get $g(C_0) = g/2$. Moreover, P_1 and P_2 are ramification points of π_0. Therefore by Proposition 3 we obtain $G(P_1) = G(P_2) = \{1, 2, \ldots, g-2, g, 2g-1\}$.

5. The case $g \equiv 2 \bmod 4$**.** First we show the following arithmetic lemma which is the key to proving the next Proposition 5.

Key Lemma 4. *Let $l \geq 2$ be an integer and let $p_1, \ldots, p_l$ be distinct prime numbers. Then there is a partition*

$$\{i_1, \ldots, i_t\} \cup \{i_{t+1}, \ldots, i_l\} = \{1, \ldots, l\}$$

with $1 \leq t \leq l-1$ *such that* $(4p_{i_1} \ldots p_{i_t} + 1, p_{i_{t+1}} \ldots p_{i_l}) = 1$.

Proof. We may assume that $p_1, \ldots, p_l$ are odd. In fact, if $p_1 = 2$, then $(4p_2 \ldots p_l + 1, p_1) = 1$. We prove the lemma by induction on $l \geq 2$.

Let $l = 2$. We may assume that $p_1 < p_2$. Suppose that

$$(4p_1 + 1, p_2) \neq 1 \quad \text{and} \quad (4p_2 + 1, p_1) \neq 1,$$

which implies that $p_2 \mid (4p_1 + 1)$ and $p_1 \mid (4p_2 + 1)$. Let $4p_1 + 1 = mp_2$. Then m must be 1 or 3. Moreover, p_1 divides $(4p_2 + 1)m = 16p_1 + 4 + m$, which implies that $p_1 \mid (4 + m)$. Let $m = 1$. Then $p_1 \mid 5$, which implies that $p_1 = 5$. Hence $p_2 = 4p_1 + 1 = 21$ is not prime, a contradiction. Let $m = 3$. Then $p_1 \mid 7$, which implies that $p_1 = 7$. Hence $3p_2 = 4p_1 + 1 = 29$, a contradiction.

Let $l \geq 3$. We may assume that $p_l > p_j$ for all $j \neq l$. Suppose that

$$(4p_1 \ldots p_{i-1}p_{i+1} \ldots p_l + 1, p_i) \neq 1, \quad \text{i.e.,} \quad p_i \mid (4p_1 \ldots p_{i-1}p_{i+1} \ldots p_l + 1)$$

for all $i = 1, \ldots, l$. Then $p_l \nmid (4p_1 \ldots p_{i-1}p_{i+1} \ldots p_{l-1} + 1)$ for all $i = 1, \ldots, l-1$. In fact, suppose that $p_l \mid (4p_1 \ldots p_{i-1}p_{i+1} \ldots p_{l-1} + 1)$ for some i. In view of $p_l \mid (4p_1 \ldots p_{l-1} + 1)$ we get

$$p_l \mid 4p_1 \ldots p_{i-1}p_{i+1} \ldots p_{l-1}(p_i - 1),$$

which implies that $p_l \mid (p_i - 1)$. This contradicts $p_l > p_j$ for all $j \neq l$.

Moreover, we may assume that $p_i \mid (4p_1 \ldots p_{i-1}p_{i+1} \ldots p_{l-1} + 1)$ for each $i = 1, \ldots, l-1$. In fact, suppose that $p_i \nmid (4p_1 \ldots p_{i-1}p_{i+1} \ldots p_{l-1} + 1)$ for some i. In view of $p_l \nmid (4p_1 \ldots p_{i-1}p_{i+1} \ldots p_{l-1} + 1)$ we obtain a partition

$$\{1, \ldots, i-1, i+1, \ldots, l-1\} \cup \{i, l\} = \{1, \ldots, l\}$$

such that $(p_i p_l, 4p_1 \ldots p_{i-1}p_{i+1} \ldots p_{l-1} + 1) = 1$. Hence

$$p_i \mid 4p_1 \ldots p_{i-1}p_{i+1} \ldots p_{l-1}(p_l - 1)$$

for each $i = 1, \ldots, l-1$. Therefore $p_i \mid (p_l - 1)$ for all $i = 1, \ldots, l-1$, which implies that $p_l - 1 = mp_1 \ldots p_{l-1}$ for some integer m. If $m \geq 5$, then $p_l \geq 5p_1 \ldots p_{l-1} + 1$, which contradicts $p_l \mid (4p_1 \ldots p_{l-1} + 1)$. If $m \leq 3$, then $(mp_1 \ldots p_{l-1} + 1) \mid (4p_1 \ldots p_{l-1} + 1)$, a contradiction.

Hence $m = 4$. By the induction hypothesis there is a partition

$$\{i_1, \ldots, i_t\} \cup \{i_{t+1}, \ldots, i_{l-1}\} = \{1, \ldots, l-1\}$$

with $1 \leq t \leq l-2$ such that $(4p_{i_1} \ldots p_{i_t} + 1, p_{i_{t+1}} \ldots p_{i_{l-1}}) = 1$. In view of $p_l = 4p_1 \ldots p_{l-1} + 1 > 4p_{i_1} \ldots p_{i_t} + 1$ we get $p_l \nmid (4p_{i_1} \ldots p_{i_t} + 1)$. Hence we obtain $(4p_{i_1} \ldots p_{i_t} + 1, p_{i_{t+1}} \ldots p_{i_{l-1}} p_l) = 1$. ■

Using the Key Lemma we show the following proposition, which is crucial to the proof of the remaining case of the Main Theorem.

PROPOSITION 5. *Let $n = 10t + 3$ with an integer $t \geq 1$. Then there exists an integer s with $3 \leq s \leq (n-3)/2$ such that $s \mid (2n-1)$ and $(2n-1, n+2s) = 1$.*

Proof. First, we consider the case $2n - 1 = p_1^e p_2 \ldots p_r$ with $e \geq 2$ if $p_1 \geq 5$ or $e \geq 3$ if $p_1 = 3$, where $p_2, \ldots, p_r$ may not be distinct. Let $s = p_1 p_2 \ldots p_r$ and $q = p_1^{e-1}$. Then $s \mid (2n-1)$ and

$$\begin{aligned}(2n-1, n+2s) &= (2n-1, 2n+4s) = (2n-1, 4s+1) \\ &= (sq, 4s+1) = (q, 4s+1) = (p_1^{e-1}, 4p_1 p_2 \ldots p_r + 1) = 1.\end{aligned}$$

Moreover,

$$s = p_1 p_2 \ldots p_r = \frac{2n-1}{q} \leq \frac{2n-1}{5} \leq \frac{n-3}{2}$$

because $q = p_1^{e-1} \geq 5$ and $n \geq 13$.

Secondly, we consider the case $2n - 1 = p_1^2 p_2 \ldots p_r$ with $p_1 = 3$ where $p_1, \ldots, p_r$ are distinct. In view of $2n - 1 = 5(4t+1)$ we have $r \geq 2$. By Lemma 4 we have a partition

$$\{i_1, \ldots, i_t\} \cup \{i_{t+1}, \ldots, i_r\} = \{1, \ldots, r\}$$

with $1 \leq t \leq r-1$ such that $(4p_{i_1} \ldots p_{i_t} + 1, p_{i_{t+1}} \ldots p_{i_r}) = 1$. Hence we get $(4p_{i_1} \ldots p_{i_t} + 1, p_1 p_{i_{t+1}} \ldots p_{i_r}) = 1$. Let $s = p_{i_1} \ldots p_{i_t}$ and $q = p_1 p_{i_{t+1}} \ldots p_{i_r}$. Then $s \mid (2n-1)$ and

$$(2n-1, n+2s) = (q, 4s+1) = (p_1 p_{i_{t+1}} \ldots p_{i_r}, 4p_{i_1} \ldots p_{i_t} + 1) = 1.$$

Moreover,

$$s = \frac{2n-1}{q} \leq \frac{2n-1}{9} < \frac{n-3}{2}$$

because $q = p_1 p_{i_{t+1}} \ldots p_{i_r} \geq 9$.

Lastly, we consider the case $2n-1 = p_1p_2\dots p_r$ where $p_1,\dots,p_r$ are distinct. By Lemma 4 we have a partition $\{i_1,\dots,i_t\}\cup\{i_{t+1},\dots,i_r\} = \{1,\dots,r\}$ with $1\le t\le r-1$ such that $(4p_{i_1}\dots p_{i_t}+1, p_{i_{t+1}}\dots p_{i_r}) = 1$.

Let $t\le r-2$ or $p_i>3$ for all i. We set $s = p_{i_1}\dots p_{i_t}$ and $q = p_{i_{t+1}}\dots p_{i_r}$. Then $s\,|\,(2n-1)$ and $(2n-1, n+2s) = 1$. Moreover,

$$s = \frac{2n-1}{q} = \frac{2n-1}{p_{i_{t+1}}\dots p_{i_r}} \le \frac{2n-1}{5} \le \frac{n-3}{2}$$

because $n\ge 13$.

Let $t = r-1$ and $p_i = 3$ for some i. In this case $r\ge 3$, because $2n-1 = 5(4t+1)$ with $4t+1\ge 5$. Then we may assume that $p_1 = 3$. Let $p_r > p_j$ for all $j\ne r$. Moreover, we may assume either

(1) $(p_i, 4p_1\dots p_{i-1}p_{i+1}\dots p_r+1) = 1$ for some $i = 2,\dots,r$, or
(2) there exists a partition

$$\{i_1,\dots,i_t\}\cup\{i_{t+1},\dots,i_{r-1}\} = \{1,\dots,r-1\}$$

with $1\le t\le r-2$ such that $(p_{i_{t+1}}\dots p_{i_{r-1}}p_r, 4p_{i_1}\dots p_{i_t}+1) = 1$.

In fact, suppose that (1) does not hold, i.e.,

$$p_i\,|\,(4p_1\dots p_{i-1}p_{i+1}\dots p_r+1) \quad \text{for all } i = 2,\dots,r.$$

Then

$$p_r\nmid(4p_1\dots p_{i-1}p_{i+1}\dots p_{r-1}+1) \quad \text{for all } i = 2,\dots,r-1.$$

In fact, suppose that

$$p_r\,|\,(4p_1\dots p_{i-1}p_{i+1}\dots p_{r-1}+1) \quad \text{for some } i = 2,\dots,r-1.$$

In view of $p_r\,|\,(4p_1\dots p_{r-1}+1)$ we obtain $p_r\,|\,4p_1\dots p_{i-1}p_{i+1}\dots p_{r-1}(p_i-1)$, which implies that $p_r\,|\,(p_i-1)$. This contradicts $p_r>p_i$.

Moreover, we may assume that

$$p_i\,|\,(4p_1\dots p_{i-1}p_{i+1}\dots p_{r-1}+1) \quad \text{for all } i = 2,\dots,r-1.$$

In fact, suppose that

$$p_i\nmid(4p_1\dots p_{i-1}p_{i+1}\dots p_{r-1}+1) \quad \text{for some } i = 2,\dots,r-1.$$

In view of $p_r\nmid(4p_1\dots p_{i-1}p_{i+1}\dots p_{r-1}+1)$ we have a partition

$$\{1,\dots,i-1,i+1,\dots,r-1\}\cup\{i,r\} = \{1,\dots,r\}$$

such that $(p_ip_r, 4p_1\dots p_{i-1}p_{i+1}\dots p_{r-1}+1) = 1$. This case reduces to the case $t\le r-2$ in which we have already proven the statement. Hence in view of

$$p_i\,|\,(4p_1\dots p_{i-1}p_{i+1}\dots p_r+1) \quad \text{for all } i = 2,\dots,r-1$$

we have $p_i\,|\,4p_1\dots p_{i-1}p_{i+1}\dots p_{r-1}(p_r-1)$ for all $i = 1,\dots,r-1$, which implies $p_i\,|\,(p_r-1)$ for all $i = 2,\dots,r-1$. Therefore $p_2\dots p_{r-1}\,|\,(p_r-1)$, which in turn implies that $p_r - 1 = mp_2 \dots p_{r-1}$ where m is even. In view of $p_r \mid (4p_1p_2 \dots p_{r-1} + 1)$ with $p_1 = 3$ we have

$$12p_2 \dots p_{r-1} + 1 = m'p_r = m'(mp_2 \dots p_{r-1} + 1) = m'mp_2 \dots p_{r-1} + m'$$

with a positive integer m'. Then we must have $m' = 1$, i.e., $m = 12$. In fact, suppose that $m' \geq 2$. Then $12 - m'm > 0$, which implies that $12 > m'm \geq 2m'$. Hence $m' \leq 5$, which implies that

$$4 \geq m' - 1 = (12 - m'm)p_2 \dots p_{r-1} \geq p_2 \dots p_{r-1} \geq 5p_3 \dots p_{r-1}.$$

This is a contradiction. Hence $m' = 1$.

Therefore we obtain

$$p_r = 12p_2 \dots p_{r-1} + 1 = 4p_1p_2 \dots p_{r-1} + 1.$$

Since $p_1, p_2, \dots, p_{r-1}$ are distinct primes and $r - 1 \geq 2$, by Lemma 4 there exists a partition $\{i_1, \dots, i_t\} \cup \{i_{t+1}, \dots, i_{r-1}\} = \{1, \dots, r-1\}$ with $1 \leq t \leq r-2$ such that $(4p_{i_1} \dots p_{i_t} + 1, p_{i_{t+1}} \dots p_{i_{r-1}}) = 1$. In view of $p_r = 4p_1p_2 \dots p_{r-1} + 1 > 4p_{i_1} \dots p_{i_t} + 1$ we have $p_r \nmid (4p_{i_1} \dots p_{i_t} + 1)$. Hence $(4p_{i_1} \dots p_{i_t} + 1, p_{i_{t+1}} \dots p_{i_{r-1}}p_r) = 1$. Thus we have proven that if $t = r-1$ and $p_1 = 3$, then we may assume that either (1) or (2) holds.

In case (1) (resp. (2)) we set $s = p_1 \dots p_{i-1}p_{i+1} \dots p_r$ (resp. $s = p_{i_1} \dots p_{i_t}$) and $q = p_i \geq 5$ (resp. $q = p_{i_{t+1}} \dots p_{i_{r-1}}p_r \geq 15$). Then we have $s \mid (2n-1)$ and $(2n-1, n+2s) = (q, 4s+1) = 1$. Moreover,

$$s = \frac{2n-1}{q} \leq \frac{2n-1}{5} \leq \frac{n-3}{2}$$

because $n \geq 13$. ∎

Now we prove the Main Theorem in the case $g \equiv 2 \bmod 4$ with $g \geq 10$.

Let $g = 2n$ where n is an odd integer ≥ 5. First we show that there exists an odd integer s with $1 \leq s \leq (n-3)/2$ such that

$$s \mid (2n-1) \quad \text{and} \quad (2n-1, n+2s) = 1.$$

In fact, let $g \not\equiv 1 \bmod 5$, which implies that $n + 2 \not\equiv 0 \bmod 5$. Then

$$(2n-1, n+2) = (2n-1, 2n+4) = (2n-1, 5) = 1.$$

Hence in this case we may take $s = 1$. Let $g \equiv 1 \bmod 5$. Then we can write $n = 10t + 3$ with $t \geq 1$. By Proposition 5 we may take an integer s with $3 \leq s \leq (n-3)/2$ such that $s \mid (2n-1)$ and $(2n-1, n+2s) = 1$.

Now there exists a unique integer m with $0 < m \leq 2n-3$ such that

$$(m+1)(n+2s) \equiv 1 \bmod 2n-1.$$

In fact, in view of $(2n-1, n+2s) = 1$ there exists a unique integer $0 \leq m \leq 2n-3$ such that $(m+1)(n+2s) \equiv 1 \bmod 2n-1$. If $m = 0$, then

$n+2s \equiv 1 \bmod 2n-1$. Since

$$n+2s-1 \geq n+1 > 0 \quad \text{and} \quad n+2s-1 \leq n+2\cdot\frac{n-3}{2}-1 = 2n-4,$$

this contradicts $(2n-1) \mid (n+2s-1)$.

Let E be an elliptic curve over k with the origin Q'. Let P_1' be a point of E such that $(2n-1)[P_1'] = [Q']$ and $h[P_1'] \neq [Q']$ for any positive integer $h < 2n-1$. Moreover, P_2' denotes the point of E such that $[P_2'] = -m[P_1']$, i.e., $P_2' \sim -mP_1' + (m+1)Q'$. Then P_1', P_2' and Q' are distinct because $0 < m \leq 2n-3$. Now we obtain

$$(n-2s)P_1' + (n+2s)P_2' \sim 2nQ'.$$

In fact,

$$(n-2s)P_1' + (n+2s)P_2' \sim (-m(n+2s)+n-2s)P_1' + (n+2s)(m+1)Q'.$$

Then $-m(n+2s)+n-2s \equiv -1+2n \equiv 0 \bmod 2n-1$ because $(m+1)(n+2s) \equiv 1 \bmod 2n-1$. Hence

$$\begin{aligned}(n-2s)&P_1' + (n+2s)P_2' \\ &\sim \frac{-m(n+2s)+n-2s}{2n-1}(2n-1)P_1' + (n+2s)(m+1)Q' \sim 2nQ'.\end{aligned}$$

Hence we may take $z \in \mathbf{K}(E)$ such that

$$\operatorname{div}(z) = (n-2s)P_1' + (n+2s)P_2' - 2nQ'.$$

Let C be the curve whose function field $\mathbf{K}(C)$ is $\mathbf{K}(E)(z^{1/(2n)})$. Moreover, $\pi : C \to E$ denotes the surjective morphism of curves corresponding to the inclusion $\mathbf{K}(E) \subset \mathbf{K}(C)$. Then we may take $y \in \mathbf{K}(C)$ and $\sigma \in \operatorname{Aut}(\mathbf{K}(C)/\mathbf{K}(E))$ such that

$$\sigma(y) = \zeta_{2n} y \quad \text{and} \quad \operatorname{div}_E(y^{2n}) = (n-2s)P_1' + (n+2s)P_2' - 2nQ'.$$

Now we have $(n,s)=1$. In fact, $(n,s) \mid (2n-1, n+2s)$ because $s \mid (2n-1)$, which implies that $(n,s)=1$. Therefore $(2n, n+2s) = (s,n) = 1$ and $(2n, n-2s) = 1$, because n is odd. Therefore the branch points of π are P_1' and P_2' whose ramification indices are $2n$. Thus

$$\operatorname{div}(y) = (n-2s)P_1 + (n+2s)P_2 - \pi^*(Q').$$

Moreover, by the Riemann–Hurwitz formula we have $g(C) = 2n = g$. Hence

$$\operatorname{div}(dy) = (n-2s-1)P_1 + (n+2s-1)P_2 - 2\pi^*(Q') + \sum_{i=1}^{3} \pi^*(R_i'),$$

where R_i''s are points of E which are distinct from P_1', P_2' and Q'.

We set

$$D_0' = -P_1' - P_2' - Q' + \sum_{i=1}^{3} R_i',$$

which is linearly equivalent to zero. Let $l \in \{0, 1, \ldots, 2s-1\}$ be fixed. Then for any even $r > 0$ with

$$\frac{2ln-1}{2s} < r \leq \frac{2(l+1)n-1}{2s}$$

we set

$$D'_r = -(r+1)Q' + \left(\frac{r}{2} - l - 1\right)P'_1 + \left(\frac{r}{2} + l\right)P'_2 + \sum_{i=1}^{3} R'_i.$$

Next we show that for any r, $l(D'_r - P'_1) = 0$ and $l(D'_r - P'_2) = 0$, i.e., $D'_r - P'_1 \not\sim 0$ and $D'_r - P'_2 \not\sim 0$. Suppose that $D'_r - P'_1 \sim 0$. Then $0 \sim D'_r - P'_1 - D'_0$, which implies that

$$\left(\left(\frac{r}{2} + l + 1\right)(m+1) - r\right)Q' \sim \left(\left(\frac{r}{2} + l + 1\right)(m+1) - r\right)P'_1.$$

Hence

$$\left(\frac{r}{2} + l + 1\right)(m+1) - r \equiv 0 \bmod 2n-1.$$

In view of $s \mid (2n-1)$, we get

$$\left(\frac{r}{2} + l + 1\right)(m+1) - r \equiv 0 \bmod s.$$

Moreover, since $(m+1)(n+2s) \equiv 1 \bmod 2n-1$ we have $(m+1)n \equiv 1 \bmod s$. Hence

$$0 \equiv 2\left(\frac{r}{2} + l + 1\right)(m+1)n - 2rn \equiv 2(l+1) \bmod s,$$

which implies that $l+1 \equiv 0 \bmod s$. In view of $0 \leq l \leq 2s-1$ we have $l = s-1$ or $2s-1$.

Let $l = s-1$. Then r satisfies

$$\frac{2(s-1)n-1}{2s} < r \leq \frac{2sn-1}{2s}.$$

Moreover,

$$\left(\frac{r}{2} + s\right)(m+1) \equiv r \bmod 2n-1.$$

In view of $(m+1)(n+2s) \equiv 1 \bmod 2n-1$ we have

$$\begin{aligned}\frac{r}{2} + s &\equiv \left(\frac{r}{2} + s\right)(m+1)(n+2s) \equiv r(n+2s) \\ &\equiv \frac{r}{2}(1+4s) \bmod 2n-1,\end{aligned}$$

which implies that $s(2r-1) \equiv 0 \bmod 2n-1$. Hence we may set

$$2r - 1 = \frac{2n-1}{s} \cdot k \quad \text{with a positive odd integer } k.$$

Then
$$\frac{2(s-1)n-1}{2s} < r = \frac{(2n-1)k+s}{2s} \le \frac{2sn-1}{2s},$$
which implies that $2(k-s)n \le k-s-1 < 2(k-s+1)n$. If $k > s$, then
$$2n \le \frac{k-s-1}{k-s} = 1 - \frac{1}{k-s} < 1,$$
a contradiction. If $k = s$, then $0 \le -1$, a contradiction. Let $k - s = -1$. Since k and s are odd, this is a contradiction. If $k - s < -1$, then
$$2n < \frac{k-s-1}{k-s+1} = 1 + \frac{2}{-k+s-1} \le 3,$$
a contradiction.

Let $l = 2s - 1$. Then r satisfies
$$\frac{2(2s-1)n-1}{2s} < r \le \frac{4sn-1}{2s}.$$
Moreover,
$$\left(\frac{r}{2} + 2s\right)(m+1) \equiv r \bmod 2n-1.$$
Hence
$$\frac{r}{2} + 2s \equiv \left(\frac{r}{2} + 2s\right)(m+1)(n+2s) \equiv \frac{r}{2}(1+4s) \bmod 2n-1,$$
which implies that $2s(r-1) \equiv 0 \bmod 2n-1$. Therefore we may set
$$r - 1 = \frac{2n-1}{s} \cdot k \quad \text{with a positive odd integer } k.$$
Then
$$\frac{2(2s-1)n-1}{2s} < r = \frac{(4n-2)k+2s}{2s} \le \frac{4sn-1}{2s},$$
which implies that $4(k-s)n \le 2k-2s-1 < 2(2k-2s+1)n$. This is a contradiction.

Moreover, we prove that $D'_r - P'_2 \not\sim 0$. Suppose that $D'_r - P'_2 \sim 0$. Then $0 \sim D'_r - P'_2 - D'_0$, which implies that
$$\left(\left(\frac{r}{2} + l\right)(m+1) - r\right)Q' \sim \left(\left(\frac{r}{2} + l\right)(m+1) - r\right)P'_1.$$
Hence
$$\left(\frac{r}{2} + l\right)(m+1) - r \equiv 0 \bmod 2n-1.$$
In view of $s \,|\, (2n-1)$, we get
$$\left(\frac{r}{2} + l\right)(m+1) - r \equiv 0 \bmod s.$$

Since $(m+1)n \equiv 1 \bmod s$, we obtain

$$0 \equiv \left(\frac{r}{2}+l\right)(m+1)n - rn \equiv r/2 + l - nr \bmod s,$$

which implies that $0 \equiv r + 2l - 2nr \equiv 2l \bmod s$. Since s is odd, we have $l \equiv 0 \bmod s$, which implies that $l = 0$ or $l = s$.

Let $l = 0$. Then $2 \le r \le (2n-1)/(2s)$. Moreover,

$$\frac{r}{2}(m+1) \equiv r \bmod 2n-1.$$

Hence

$$\frac{r}{2} \equiv \frac{r}{2}(m+1)(n+2s) \equiv 2sr + \frac{r}{2} \bmod 2n-1,$$

which implies that $0 \equiv 2sr \bmod 2n-1$. Therefore $r \equiv 0 \bmod (2n-1)/s$, which contradicts $2 \le r \le (2n-1)/(2s)$.

Let $l = s$. Then

$$\frac{2sn-1}{2s} < r \le \frac{2(s+1)n-1}{2s}.$$

Moreover,

$$\left(\frac{r}{2}+s\right)(m+1) \equiv r \bmod 2n-1.$$

Hence

$$\frac{r}{2}+s \equiv \left(\frac{r}{2}+s\right)(m+1)(n+2s) \equiv \frac{r}{2}(4s+1) \bmod 2n-1,$$

which implies that $s \equiv 2sr \bmod 2n-1$. Hence we may set

$$2r-1 = \frac{2n-1}{s}\cdot k,$$

where k is an odd positive integer. If $k \ge s+2$, then

$$\begin{aligned} 2r-1 &\ge \frac{2n-1}{s}(s+2) > 2n-1+\frac{2n-1}{s} \\ &= \frac{2(s+1)n-1}{s} - 1 = 2\cdot\frac{2(s+1)n-1}{2s} - 1 \ge 2r-1, \end{aligned}$$

a contradiction. Now we have

$$2r-1 > 2\cdot\frac{2sn-1}{2s} - 1 = 2n - \frac{1}{s} - 1,$$

which implies that $2r-1 \ge 2n-1$. If $k \le s-2$, then

$$2n-1 \le 2r-1 \le \frac{2n-1}{s}(s-2) = 2n-1-\frac{2(2n-1)}{s} < 2n-1,$$

a contradiction. Hence $k = s$, which implies that

$$2r - 1 = \frac{2n-1}{s} \cdot s = 2n - 1.$$

Therefore $r = n$. Since r is even and n is odd, this is a contradiction. Hence $D'_r - P'_2 \not\sim 0$. Thus we obtain the following: Let $l \in \{0, 1, \ldots, 2s-1\}$ be fixed. Then for any even $r > 0$ with $(2ln-1)/(2s) < r \leq (2(l+1)n-1)/(2s)$ we get

$$l(D'_r) = 1 \quad \text{and} \quad l(D'_r - P'_1) = l(D'_r - P'_2) = 0.$$

Now in view of $(n, s) = 1$ there is a unique non-negative integer $q \leq 2s-1$ such that $(2q+1)n \equiv 2s+1 \bmod 4s$. Then we set

$$\begin{aligned} r_1 &= 2 \cdot \frac{2s + 1 - (2q+1)n + 4s(n-1)}{4s} + 1 \\ &= 2 \cdot \frac{(4s - 2q - 1)n - (2s-1)}{4s} + 1. \end{aligned}$$

Note that r_1 is an odd integer ≥ 3. In fact,

$$4s - 2q - 1 \geq 4s - 2(2s-1) - 1 = 1.$$

Hence in view of $s \leq (n-3)/2$ we get

$$(4s - 2q - 1)n - (2s-1) \geq n - (2s-1) \geq 2s + 3 - (2s-1) = 4 > 0,$$

which implies that $r_1 \geq 3$. Then we define

$$\begin{aligned} D'_{r_1} = &- (r_1+1)Q' + \frac{(4s-2q-1)n - (2s-1) - 4s(2s-q)}{4s} P'_1 \\ &+ \frac{(4s-2q-1)n - (2s-1) + 4s(2s-q)}{4s} P'_2 + \sum_{i=1}^{3} R'_i. \end{aligned}$$

Note that $\deg D'_{r_1} = 1$. We prove that $D'_{r_1} - P'_1 \sim 0$. In fact, in view of $P'_2 \sim (m+1)Q' - mP'_1$ we have

$$\begin{aligned} &D'_{r_1} - P'_1 - D'_0 \\ &\quad \sim \frac{(4s-2q-1)n(m-1) + (4s(2s-q) + 2s + 1)(m+1) - 2}{4s} (Q' - P'_1). \end{aligned}$$

Then

$$\begin{aligned} &(4s - 2q - 1)n(m-1) + (4s(2s-q) + 2s + 1)(m+1) - 2 \\ &\qquad = 4s(n(m-1) + 2s(m+1)) \\ &\qquad\quad - ((2q+1)n(m-1) - (2s+1)(m-1) + 4sq(m+1) - 4s). \end{aligned}$$

Let

$$u = \frac{(n+2s)(m+1) - 1}{2n-1},$$

which is a positive integer because $(m+1)(n+2s) \equiv 1 \bmod 2n-1$. We have

$$\begin{aligned}
&(2q+1)n(m-1)-(2s+1)(m-1)+4sq(m+1)-4s\\
&\quad = 2q((n+2s)(m+1)-2n)+(n+2s)(m+1)-2n-4sm-m+1-4s\\
&\quad = 2q((2n-1)u+1-2n)+(2n-1)u+1-2n\\
&\qquad -2((n+2s)(m+1)-n(m+1))-m+1\\
&\quad = (2n-1)((2q-1)u-2q+m).
\end{aligned}$$

Now $(2q-1)n = (2q+1)n-2n \equiv 2s+1-2n \equiv 0 \bmod s$, which implies that $s \mid (2q-1)$ because $(n,s)=1$. Moreover,

$$\begin{aligned}
(-2q+m)n &= -2qn+mn \equiv n-2s-1+mn\\
&= (n+2s)(m+1)-1-2s-2sm-2s \equiv 0 \bmod s
\end{aligned}$$

because $s \mid (2n-1)$. In view of $(n,s)=1$ we get $s \mid (-2q+m)$. Therefore $4s \mid ((2q-1)u-2q+m)$ because $(4,2n-1)=1$, which implies that

$$(2n-1)4s \mid ((2q+1)n(m-1)-(2s+1)(m-1)+4sq(m+1)-4s).$$

Moreover,

$$4s(n(m-1)+2s(m+1)) = 4s((m+1)(n+2s)-1-(2n-1)),$$

which implies that $4s(2n-1) \mid 4s(n(m-1)+2s(m+1))$. Hence the integer

$$\frac{(4s-2q-1)n(m-1)+(4s(2s-q)+2s+1)(m+1)-2}{4s}$$

is divisible by $2n-1$, which implies that $D'_{r_1}-P'_1 \sim 0$.

Next we set

$$r_2 = \frac{(2q+1)n-1}{2s} = 2\cdot\frac{(2q+1)n-(2s+1)}{4s}+1,$$

which is an odd integer because $(2q+1)n \equiv 2s+1 \bmod 4s$. Moreover, $3 \le r_2 \le 2n-3$. In fact, $1 \le 2q+1 \le 4s-1$ because $0 \le q \le 2s-1$. Hence

$$\frac{n-1}{2s} \le r_2 = \frac{(2q+1)n-1}{2s} \le \frac{(4s-1)n-1}{2s} = 2n-\frac{n+1}{2s}.$$

In view of $0 < 1 \le s \le (n-3)/2$ we have

$$1 < \frac{n-1}{n-3} \le \frac{n-1}{2s} \quad \text{and} \quad 2n-\frac{n+1}{2s} \le 2n-\frac{n+1}{n-3} < 2n-1.$$

Now we set

$$\begin{aligned}
D'_{r_2} = &-(r_2+1)Q' + \frac{(2q+1)n-(2s+1)-4sq}{4s}P'_1\\
&+\frac{(2q+1)n-(2s+1)+4sq}{4s}P'_2+\sum_{i=1}^{3}R'_i,
\end{aligned}$$

which is of degree 1. We prove that $D'_{r_2} - P'_2 \sim 0$. We have

$$D'_{r_2} - P'_2 - D'_0 \sim \frac{(2q+1)n(m-1) - (2s+1)(m-1) + 4sq(m+1) - 4s}{4s}(Q' - P'_1).$$

By the argument in the proof of $D'_{r_1} - P'_1 \sim 0$ we show that

$$\frac{(2q+1)n(m-1) - (2s+1)(m-1) + 4sq(m+1) - 4s}{4s}$$

is divisible by $2n-1$, which implies that $D'_{r_2} - P'_2 \sim D'_0 \sim 0$.

Now we are in a position to prove that $\{1, \ldots, g-2, g, 2g-1\}$ is the gap sequence at P_1 and P_2. Let $f \in \mathbf{K}(E)$ and set

$$\mathrm{div}_E(f) = \sum_{P' \in E} m(P')P'.$$

Then for any non-negative integer r we obtain

$$\begin{aligned}\mathrm{div}_C\left(\frac{f\,dy}{y^{1-r}}\right) &= (2nm(P'_1) + r(n-2s) - 1)P_1 \\ &\quad + (2nm(P'_2) + r(n+2s) - 1)P_2 + (m(Q') - r - 1)\pi^*(Q') \\ &\quad + \sum_{i=1}^{3}(m(R'_i) + 1)\pi^*(R'_i) + \sum_{P' \in S} m(P')\pi^*(P'),\end{aligned}$$

where we set $S = E \setminus \{P'_1, P'_2, Q', R'_1, R'_2, R'_3\}$. Fix $l \in \{0, 1, \ldots, 2s-1\}$, and let r be a positive even integer with $(2ln-1)/(2s) < r \leq (2(l+1)n-1)/(2s)$. If $f_r \in L(D'_r)$, then

$$\mathrm{ord}_{P_1}\left(\frac{f_r dy}{y^{1-r}}\right) = 2(l+1)n - 1 - 2sr \geq 0$$

and

$$\mathrm{ord}_{P_2}\left(\frac{f_r dy}{y^{1-r}}\right) = 2sr - (2ln-1) - 2 \geq 0.$$

In fact, suppose that $2sr - (2ln-1) = 1$, which implies that $r = ln/s$. We know that $(n,s) = 1$, n is odd and r is even. Hence l/s must be even, which implies that $l = 2us$ with a non-negative integer u. In view of $0 \leq l \leq 2s-1$ we must have $l = 0$, which implies that $r = 0$. This is a contradiction. Hence $2sr - (2ln-1) - 2 \geq 0$. Therefore $f_r dy/y^{1-r}$ is a regular 1-form on C, which implies that $2n - (2sr - 2nl)$ (resp. $2sr - 2nl$) is a gap at P_1 (resp. P_2).

Now we show that

$$\left\{2sr - 2nl \;\middle|\; l = 0, 1, \ldots, 2s-1,\ r \text{ is even} > 0 \text{ with } \frac{2ln-1}{2s} < r \leq \frac{2(l+1)n-1}{2s}\right\} = \{2, 4, \ldots, g-2\}.$$

First we show that the above elements $2sr-2nl$ are distinct. Let $l' \in \{0,1,\ldots,2s-1\}$ with $l' \geq l$ and let r' be even with

$$\frac{2l'n-1}{2s} < r' \leq \frac{2(l'+1)n-1}{2s} \quad \text{such that} \quad 2sr-2nl = 2sr'-2nl'.$$

Then $n(l'-l) = s(r'-r)$. In view of $(n,s)=1$ we obtain $s \mid (l'-l)$, which implies that $l'=l$ or $l'=l+s$. Hence we may assume that $l'=l+s$, which implies that $r'-r=n$. Since $r'-r$ is even and n is odd, this is a contradiction. Hence the elements $2sr-2nl$ are distinct.

Next if $l=0$ (resp. $l=2s-1$), then

$$\frac{2ln-1}{2s} = \frac{-1}{2s} < 0 \quad \left(\text{resp. } 2n-1 \leq \frac{2(l+1)n-1}{2s} = 2n - \frac{1}{2s} < 2n = g\right).$$

In view of $r>0$ the cardinality of the set of the elements $2sr-2nl$ is equal to that of $\{2,4,\ldots,g-2\}$. Moreover, $1 \leq 2sr-2nl$. In view of $r \leq (2(l+1)n-1)/(2s)$ we have $2sr-2nl \leq g-1$. Hence we obtain the desired result. Therefore $2,4,\ldots,g-2$ are gaps at P_1 and P_2.

Now if $f_0 \in L(D_0')$, then

$$\operatorname{ord}_{P_i}\left(\frac{f_0\,dy}{y}\right) = 2n-1 = g-1 \quad \text{for } i=1,2,$$

which implies that g is also a gap at P_1 and P_2. Let $f_{r_1} \in L(D'_{r_1} - P'_1) \neq \{0\}$. Then

$$\operatorname{ord}_{P_1}\left(\frac{f_{r_1}\,dy}{y^{1-r_1}}\right) = 4n-2 = (2g-1)-1$$

and

$$\operatorname{ord}_{P_2}\left(\frac{f_{r_1}\,dy}{y^{1-r_1}}\right) \geq -2n \cdot \frac{(4s-2q-1)n-(2s-1)+4s(2s-q)}{4s} + r_1(n+2s) - 1 = 0.$$

Therefore $f_{r_1}\,dy/y^{1-r_1}$ is a regular 1-form on C, which implies that $2g-1$ is a gap at P_1. Moreover, let $f_{r_2} \in L(D'_{r_2} - P'_2) \neq \{0\}$. Then

$$\operatorname{ord}_{P_1}\left(\frac{f_{r_2}\,dy}{y^{1-r_2}}\right) \geq -2n \cdot \frac{(2q+1)n-(2s+1)-4sq}{4s} + r_2(n-2s) - 1 = 0$$

and

$$\operatorname{ord}_{P_2}\left(\frac{f_{r_2}\,dy}{y^{1-r_2}}\right) = (2g-1)-1.$$

Therefore $2g-1$ is a gap at P_2. In the same way as in Section 4 we get $G(P_1) = G(P_2) = \{1,2,\ldots,g-2,g,2g-1\}$.

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Received on 26.11.1996 (3084)