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On explicit construction of Hilbert–Siegel modular forms of degree two

by

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Introduction. Several authors have developed the theory of lifting from the space of modular forms of one variable to that of modular forms on the orthogonal groups attached to quadratic forms over $\mathbb{Q}$ (cf. [1, 4–6, 8]). Shimura [9], [10] dealt with the problem of construction of arithmetic modular forms on orthogonal groups over totally real algebraic number fields. However, he did not take up the explicit calculation of the Fourier coefficients of lifted modular forms.

On the other hand, in [3], [4] we have established a correspondence $\Psi_k^{M,\chi}$ between the space $S_{(2k-1)/2}(M,\chi)$ of modular cusp forms of half integral weight $(2k-1)/2$ of level M to the space $M_k^{(2)}(M,\chi)$ of Maass forms of Siegel modular cusp forms of degree two of weight k of level M in such a way that it commutes with the actions of Hecke operators. We evaluated explicitly the Fourier coefficients of $\Psi_k^{M,\chi}(f)$ with a form f in $S_{(2k-1)/2}(M,\chi)$, and made clear a coincidence with Shimura's zeta functions attached to f and Andrianov's zeta functions attached to $\Psi_k^{M,\chi}(f)$. We note that these results are closely related to Saito–Kurokawa's conjecture concerning Siegel modular forms of degree two. Using the technique in the theory of group representation of Jacquet and Langlands, Piatetski-Shapiro [7] discussed Saito–Kurokawa's conjecture in the case of Siegel modular forms on $\mathrm{GpSp}(2,A_F)$ where A_F is the adele ring of an arbitrary number field F. Unfortunately, it seems that his approach is difficult to use for an explicit calculation of the Fourier coefficients of the lifted forms.

The first purpose of the present note is to show the existence of a correspondence $\Psi_{N'}$ between Hilbert modular forms f of half integral weight with respect to the principal congruence group and Hilbert–Siegel modular forms $\Psi_{N'}(f)$ of degree two attached to totally real number fields. The second one

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is to determine an explicit relation between the Fourier coefficients of f and $\Psi_{N'}(f)$.

Section 1 is a preliminary section. In Section 2, using theta series associated with Weil representations of quadratic forms, we shall construct Hilbert–Siegel modular forms of degree two of integral weight from Hilbert modular forms of half integral weight. In Section 3, we shall derive relations between the Fourier coefficients of those modular forms. Our results can be regarded as a development of those of [3]. We use theta function methods similar to those of Friedberg [2] and Kojima [3] (cf. [1], [5]).

1. Notation and preliminaries. We denote, as usual, by $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field. For a commutative ring A with the unity 1, we denote by A_n^m the set of $m \times n$ matrices with entries in A. Furthermore, we denote by $\mathrm{SL}_n(A)$ (resp. $\mathrm{GL}_n(A)$) the group of all matrices M with $\det(M) = 1$ (resp. $\det(M) \in A^\times$), where $A^\times$ is the group of all invertible elements in A, and set $A^n = A_1^n$ and $M_n(A) = A_n^n$ for simplicity. Let E_m be the unity of $\mathrm{GL}_m(A)$.

Throughout the paper, we denote by F a totally real algebraic number field of degree l of class number one. We denote by J_F the set of all embeddings of F into $\mathbb{C}$, and by $\tau_1, \ldots, \tau_l$ the elements of J_F. For $a \in F$, we set $a^{(i)} = \tau_i(a)$ $(1 \le i \le l)$. Let $\mathfrak{r}$ be the ring of all integers in F. Now, we denote by $\mathfrak{H}_n$ the complex upper half space of degree n, i.e.,

$$\mathfrak{H}_n = \{Z \in M_n(\mathbb{C}) \mid {}^tZ = Z, \Im(Z) > 0\}.$$

We set $\mathfrak{H} = \mathfrak{H}_1$ for simplicity. Let $\mathrm{Sp}(n, \mathbb{R})$ (resp. $\mathrm{Sp}(n, F)$) be the real symplectic group of degree n (resp. the symplectic group of degree n over F), i.e.,

$$\mathrm{Sp}(n, \mathbb{R}) = \{g \in \mathrm{GL}_{2n}(\mathbb{R}) \mid {}^tgJ_ng = J_n\}$$
$$(\text{resp. } \mathrm{Sp}(n, F) = \{g \in \mathrm{Sp}(n, \mathbb{R}) \cap M_{2n}(F)\})$$

with

$$J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

The group $\mathrm{Sp}(n, \mathbb{R})$ acts on $\mathfrak{H}_n$ by

$$Z \to g(Z) = (AZ+B)(CZ+D)^{-1} \quad \left(Z \in \mathfrak{H}_n, g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{R})\right).$$

To define Hilbert–Siegel modular forms, we take an arithmetic congruence subgroup $\Gamma_1^{(n)}(N)$ of $\mathrm{Sp}(n, F)$ in the form

$$\Gamma_1^{(n)}(N) = \{\gamma \in \mathrm{Sp}(n, F) \cap M_{2n}(\mathfrak{r}) \mid \gamma \equiv E_{2n}(N)\}$$

with a positive integer N. The group $\Gamma_1^{(n)}(N)$ can be embedded into $\mathrm{Sp}(n,\mathbb{R})^l$ $(=\mathrm{Sp}(n,\mathbb{R})\times\overset{(l)}{\cdots}\times\mathrm{Sp}(n,\mathbb{R}))$ by the mapping $\gamma\to(\gamma^{(1)},\dots,\gamma^{(l)})$ with $\gamma^{(\alpha)}=(\gamma_{i,j}^{(\alpha)})\in\Gamma_1^{(n)}(N)$. It is well known that $\Gamma_1^{(n)}(N)$ acts properly discontinuously on $\mathfrak{H}_n^l$ by

$$\gamma(Z)=(\gamma^{(1)}(Z_1),\dots,\gamma^{(l)}(Z_l))$$

for every $\gamma\in\Gamma_1^{(n)}(N)$ and $Z=(Z_1,\dots,Z_l)\in\mathfrak{H}_n^l$, and the volume of $\Gamma_1^{(n)}(N)\backslash\mathfrak{H}_n^l$ is finite. We call a holomorphic function F on $\mathfrak{H}_n^l$ a *Hilbert–Siegel modular form* of weight $k={}^t(k_1,\dots,k_l)$ $(\in\mathbb{Z}^l)$ with respect to $\Gamma_1^{(n)}(N)$ if the following conditions are satisfied:

$$F(\gamma(Z))=\prod_{i=1}^{l}\det(C^{(i)}Z_i+D^{(i)})^{k_i}F(Z)\tag{1.1}$$

for every $\gamma=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\Gamma_1^{(n)}(N)$ and $Z=(Z_1,\dots,Z_l)\in\mathfrak{H}_n^l$, and

$$F(Z)\text{ is finite at each cusp of }\Gamma_1^{(n)}(N).\tag{1.2}$$

We denote by $M_k(\Gamma_1^{(n)}(N))$ the space of all such modular forms.

2. Construction of Hilbert–Siegel modular forms of degree two. For $\alpha={}^t(\alpha_1,\dots,\alpha_q)\in F^q$, we set $\alpha^{(i)}={}^t(\alpha_1^{(i)},\dots,\alpha_q^{(i)})$ $(1\le i\le l)$. Then F^q can be embedded into $\mathbb{R}_l^q$ by $\alpha\to(\alpha^{(i)},\dots,\alpha^{(l)})$ for every $\alpha\in F^q$. Let S be a non-degenerate symmetric matrix over F. Let $S(\mathbb{R}^q)$ be the space of all rapidly decreasing functions on $\mathbb{R}^q$. For each $f=(f_1,\dots,f_l)\in\prod_{i=1}^l S(\mathbb{R}^q)$, we define an element $\gamma(\sigma,S)f$ in $\prod_{i=1}^l S(\mathbb{R}^q)$ by

$$\gamma(\sigma,S)f=\prod_{i=1}^{l}\gamma(\sigma_i,S^{(i)})f_i\qquad\text{for every }\sigma=(\sigma_1,\dots,\sigma_l)\in\mathrm{SL}_2(\mathbb{R})^l,$$

where $S^{(i)}=(S_{j,k}^{(i)})$ $(S=(S_{j,k}))$ and $\gamma(\sigma_i,S^{(i)})$ is the Weil representation given in [4]. We consider the following four matrices:

$$S=\begin{pmatrix}0&-1&0&0&0\\-1&0&0&0&0\\0&0&0&-1&0\\0&0&-1&0&0\\0&0&0&0&-2\end{pmatrix},\qquad S_1=\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&-2\end{pmatrix},$$

$$A=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&\sqrt2\end{pmatrix},\qquad A_1=\begin{pmatrix}1&0&0\\0&1&0\\0&0&\sqrt2\end{pmatrix}.$$

Let χ_0 be a character of $(\mathfrak{r}/(M))^\times$ with a positive integer M. For each $h = {}^t(\ldots, h_3, \ldots) \in \mathfrak{r}^5$, set

$$\widetilde{\chi}_0(h) = \begin{cases} \chi_0(h_3) & \text{if } (h_3, M) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $k = {}^t(k_1, \ldots, k_5) \in \mathbb{Z}^5$ $(k_i \geq 0)$ and $x = (x_1, \ldots, x_l) \in \mathbb{R}^5_l$, we define an element $g_k(x)$ of $\prod_{i=1}^l S(\mathbb{R}^5)$ by

$$g_k(x) = \prod_{i=1}^{l} ({}^t x_i S^{(i)} A^{-1}\,{}^t(-i, i, 1, -1, 0))^{k_i} \exp(-\pi\,{}^t x_i\,{}^t A A x_i).$$

We define a theta series

$$\Theta_k(z, g) = \prod_{i=1}^{l} (\Im(z_i))^{-(2k_i-1)/4} \sum_{h \in \mathfrak{r}^5} \widetilde{\chi}_0(h) \gamma(\sigma_z, S) g_k(\widetilde{\varrho}(g)^{-1} h) \tag{2.1}$$

for every $z = (z_1, \ldots, z_l) \in \mathfrak{H}^l$ and $g = (g_1, \ldots, g_l) \in \mathrm{Sp}(2, \mathbb{R})^l$, where $\widetilde{\varrho}(g) = (A^{-1}\varrho(g_1)A, \ldots, A^{-1}\varrho(g_l)A)$ and ϱ is an isomorphism of $\mathrm{Sp}(2,\mathbb{R})/\{\pm E_4\}$ onto $\mathrm{SO}({}^t A^{-1} S A^{-1})_0$ given in [3]. Before describing the transformation formula for $\Theta_k(z, g)$, we recall the definition of Hilbert modular forms of half integral weight.

We write $\Gamma_1(N)$ for $\Gamma_1^{(1)}(N)$ for simplicity. Throughout this paper, we treat only a congruence subgroup $\Gamma_1(N)$ such that every congruence subgroup of $\Gamma_1(N)$ is generated by its elements $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $d^{(i)} > 0$ $(1 \leq i \leq l)$ (see [9, Lemma 7.4] for the existence of such $\Gamma_1(N)$). Let $k = {}^t(k_1, \ldots, k_l)$ be an element of $\mathbb{Q}^l$ satisfying $2k \in \mathbb{Z}^l$. For a $\Gamma_1(N)$ as above, we denote by $S_{k/2}(\Gamma_1(N))$ the vector space of all holomorphic functions f on $\mathfrak{H}^l$ which satisfy

$$f(\gamma(z)) = \left(\frac{2c}{d\mathfrak{r}}\right)(cz + d)^{k/2} f(z) \quad \text{for all } \gamma \in \Gamma_1(N)$$

and which vanish at each cusp of $\Gamma_1(N)$. We refer to [9] for the definition of the symbols $\left(\frac{*}{*}\right)$ and $(*)^k$. By Shimura [9, Prop. 7.1], we can verify that $\Theta_k(z, g)$ admits the following transformation formula:

$$\begin{aligned} &\Theta_k(\gamma(z), g) \\ &\quad = \left(\frac{2c}{d\mathfrak{r}}\right)(cz + d)^{(2k-1)/2} \Theta_k(z, g) \qquad \left(\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_1(N)\right) \end{aligned} \tag{2.2}$$

for a suitable integer N. Furthermore, by [3, p. 67], we can prove the following transformation formula:

$$\Theta_k(z, g'g) = \Theta_k(z, g) \quad \text{for each } g' \in \Gamma_1^{(2)}(M). \tag{2.3}$$

Let N' be a positive integer satisfying $N \mid N'$. For $f \in S_{(2k-1)/2}(\Gamma_1(N))$, we define a function $\Psi_{N'}$ on $\mathfrak{H}_2^l$ by

(2.4) $$\Psi_{N'}(Z) = \prod_{i=1}^{l} \det(\sqrt{-1}\, C^{(i)} + D^{(i)})^{k_i} \int_{\Gamma_1(N')\backslash\mathfrak{H}^l} \prod_{i=1}^{l} v_i^{(2k_i-1)/2}\overline{\Theta_k(z,g)} f(z)\, dz$$

with $z = (z_1, \ldots, z_l) \in \mathfrak{H}^l, g(\sqrt{-1}\, E_2, \ldots, \sqrt{-1}\, E_2) = Z$ $\big(g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\big)$ and $dz = \prod_{i=1}^{l} v_i^{-2} du_i dv_i$ $(z_i = u_i + \sqrt{-1}\, v_i)$. By Shimura [10, Theorem 6.2], we see that $\Psi_{N'}$ is holomorphic on $\mathfrak{H}_2^l$. Therefore it follows from (2.3) that $\Psi_{N'}$ is a Hilbert–Siegel modular form of $M_k(\Gamma_1^{(2)}(M))$.

Next we shall determine explicitly the Fourier expansion of $\Psi_{N'}$. Put

$$H = \left\{ T = \begin{pmatrix} t_{11} & t_{12}/2 \\ t_{12}/2 & t_{22} \end{pmatrix} \,\middle|\, t_{ij} \in \mathfrak{r} \right\}.$$

We consider an equivalence relation on $H \times \mathfrak{r}$ defined by $(T,t) \sim_{N'} (T',t')$ if and only if $T' = \varepsilon T$ and $t' = \varepsilon^{-1} t$ for a suitable $\varepsilon \in U(N') = \{\varepsilon \mid \varepsilon$ is an unit element of $\mathfrak{r}$ and $\varepsilon \equiv 1 \pmod{N'}\}$. For a $(T,t) \in H \times \mathfrak{r}$, we define a coefficient $c_f(T,t)$ by

(2.5) $$c_f(T,t) = \sum_{\phi \in R(N')} \overline{\chi}_0(tc/\delta)\overline{W(\phi^{-1})} \sum_{h \in \mathfrak{r}^3/(c)^3} \overline{e[\mathrm{tr}_{F/\mathbb{Q}}(a^t h S_1 h/(2c))]} \times \prod_{i=1}^{l} ((t/\delta)^{(i)})^{k_i-1} \overline{e[\mathrm{tr}_{F/\mathbb{Q}}(\det(T)d/c)]} a(\det(T))$$

with $\phi^{-1} = \left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right)$ and

$$f|[\phi^{-1}]_{(2k-1)/2}(z) = (cz+d)^{-(2k-1)/2} f(\phi^{-1}(z)) = \sum_{\mu \in (1/N')} a(\mu) e\Big[\sum_{i=1}^{l} \mu^{(i)} z_i\Big]$$

with $\phi^{-1} = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$. The various symbols will be explained later.

Now the main theorem of this paper can be stated as

THEOREM. *Suppose that $k \in (2\mathbb{Z})^l$, $2 \mid N'$, $M\delta^2 \mid N'$ and N' satisfies the condition that f has an expression of the form in* (2.5) *for each $f \in S_{(2k-1)/2}(\Gamma_1(N))$ and for each $\phi \in R(N')$. Then $\Psi_{N'}$ has the Fourier expansion of the form*

$$\Psi_{N'}(Z) = c \sum c_f(T,t) e\Big[\sum_{i=1}^{l} \mathrm{tr}((t^{(i)}/(\delta^{(i)})^2) T^{(i)} Z_i)\Big],$$

where c $(\neq 0)$ is a constant not depending upon f, (δ) means the conjugate different from F, the sum $\sum$ is taken over $H \times \mathfrak{r}/\sim_{N'}$ and

$$T^{(i)} = \begin{pmatrix} (t_{11})^{(i)} & (t_{12})^{(i)}/2 \\ (t_{12})^{(i)}/2 & (t_{22})^{(i)} \end{pmatrix}.$$

3. Proof of Theorem. In this section, after preparing some theta series, we shall show that for an element

$$(3.1)\quad g = \left(\begin{pmatrix} \sqrt{Y^{(1)}} & 0 \\ 0 & \sqrt{Y^{(1)}}^{-1} \end{pmatrix}, \ldots, \begin{pmatrix} \sqrt{Y^{(l)}} & 0 \\ 0 & \sqrt{Y^{(l)}}^{-1} \end{pmatrix} \right) \in \mathrm{Sp}(2,\mathbb{R})^l,$$

$\Theta_k(z,g)$ can be split into the product of simpler theta functions. Set $Y^{(i)} = y^{(i)}Y_1^{(i)}$ with

$$Y_1^{(i)} = \begin{pmatrix} y_1^{(i)} & y_2^{(i)} \\ y_2^{(i)} & y_3^{(i)} \end{pmatrix}, \qquad \det(Y_1^{(i)}) = 1,$$

$y^{(i)} > 0$ and $y_1^{(i)} > 0$. Set also $Y = (Y^{(1)}, \ldots, Y^{(l)}), Y_1 = (Y_1^{(1)}, \ldots, Y_1^{(l)})$ and $y = (y^{(1)}, \ldots, y^{(l)})$. For $\varepsilon = {}^t(\varepsilon_1, \ldots, \varepsilon_l) \in \mathbb{Z}^l$ $(\varepsilon_i \geq 0)$ and $z = (z_1, \ldots, z_l) \in \mathfrak{H}^l$ $(z_i = u_i + \sqrt{-1}\, v_i)$, we define two theta series by

$$\begin{aligned}\Theta_{1,\varepsilon}(z, Y_1) = {} & \prod_{i=1}^{l} v_i^{(2-\varepsilon_i)/2} \sum_{h\in\mathfrak{r}^3} \prod_{i=1}^{l} H_{\varepsilon_i}([\sqrt{2\pi v}\,(y_1, -y_3, -2y_2)h]^{(i)}) \\ & \times e[([u^t h S_1 h + \sqrt{-1}\, v^t h R(Y_1) h]^{(i)})/2]\end{aligned}$$

and

$$\begin{aligned}\Theta_{2,\varepsilon}(z, y) = {} & \prod_{i=1}^{l} v_i^{(1-\varepsilon_i)/2} \sum_{m\in\mathfrak{r}} \sum_{n\in\mathfrak{r}} \chi_0(m) \prod_{i=1}^{l} \exp(-2\pi\sqrt{-1}\,[mnu]^{(i)} \\ & - \pi[v(y^2m^2 + y^{-2}n^2)]^{(i)}) H_{\varepsilon_i}([\sqrt{2\pi v}\,(my - ny^{-1})]^{(i)}),\end{aligned}$$

where

$[\sqrt{2\pi v}\,(y_1, -y_3, -2y_2)h]^{(i)}$ means $\sqrt{2\pi v_i}\,(y_1^{(i)}, -y_3^{(i)}, -2y_2^{(i)})h^{(i)}$,

$[u^t h S_1 h + \sqrt{-1}\, v^t h R(Y_1) h]^{(i)}$ means $u_i {}^t h^{(i)} S_1^{(i)} h^{(i)} + \sqrt{-1}\, v_i {}^t h^{(i)} R(Y_1^{(i)}) h^{(i)}$,

other symbols $[*]_{(i)}$ are similar symbols,

$$H_\varepsilon(x) = (-1)^\varepsilon \exp(x^2/2) \frac{d^\varepsilon}{dx^\varepsilon}(\exp(-x^2/2)), \qquad \widetilde{R}(Y_1^{(i)}) = A_1 R(Y_1^{(i)}) A_1$$

and $R(Y_1^{(i)})$ is the matrix in [3]. By the definition, we may derive

$$\begin{aligned}\Theta_k(z, g) = {} & \prod_{i=1}^{l} (\sqrt{2\pi})^{-\varepsilon_i} \sum_{\varepsilon_1=0}^{k_1} \cdots \sum_{\varepsilon_l=0}^{k_l} {}_{k_1}C_{\varepsilon_1} \cdots {}_{k_l}C_{\varepsilon_l} \\ & \times (-\sqrt{-1})^{\varepsilon_1+\ldots+\varepsilon_l} \Theta_{1,\varepsilon}(z, Y_1) \Theta_{2,k-\varepsilon}(z, y)\end{aligned}$$

for g in (3.1).

Now the Poisson summation formula gives an important expression of $\Theta_{2,\varepsilon}(z, y)$:

$$\Theta_{2,\varepsilon}(z,y)=(1/\sqrt{d})\prod_{i=1}^{l}(\sqrt{-2\pi})^{\varepsilon_i}\sum_{m\in\mathfrak{r}}\sum_{n\in(1/\delta)}\chi_0(m)\prod_{i=1}^{l}(m^{(i)}z_i+n^{(i)})^{\varepsilon_i}$$
$$\times\, y_i^{\varepsilon_i+1}v_i^{-\varepsilon_i}\exp(-\pi y_i^2|m^{(i)}z_i+n^{(i)}|^2/v_i),$$

where d is the discriminant of F. Pulling out greatest common divisors in the pair (m,n), we have

$$\{(m,n)\,|\,m\in\mathfrak{r},n\in(1/\delta)\}$$
$$=\{t(m'/\delta,n'/\delta)\,|\,(m',n')=1,\ m',n',t\in\mathfrak{r}\text{ and }\delta\,|\,tm'\}.$$

Take a pair $(c,d)\in\mathfrak{r}\times\mathfrak{r}$ such that $(c,d)=1$ and $(c,d)\in\mathfrak{r}/(N')\times\mathfrak{r}/(N')$. Then, for every pair $(m',n')\in\mathfrak{r}\times\mathfrak{r}$ ($(m',n')=1$, $m'\equiv c(N')$, $n'\equiv d(N')$), there exists a unique $\sigma\in\Gamma_{-d/c}\backslash\Gamma_1(N')$ satisfying $(m',n')=(c,d)\sigma$. Note that $\Gamma_{-d/c}$ is the stabilizer of the cusp $-d/c$ of $\Gamma_1(N)$. Throughout this paper, for the pair (c,d) as described above, we fix a matrix $\phi\in\mathrm{SL}_2(\mathfrak{r})$ with $(c,d)=(0,1)\phi$ and we denote by $R(N')$ the set of all such ϕ. Therefore, by the above arguments, we obtain the following lemma which plays an essential role in our later discussion.

Lemma 3.1. *Notations being as above, the theta series $\Theta_{2,\varepsilon}(z,y)$ coincides with*

$$(1/\sqrt{d})\prod_{i=1}^{l}(\sqrt{-2\pi})^{\varepsilon_i}\prod_{i=1}^{l}y_i^{\varepsilon_i+1}v_i^{-\varepsilon_i}$$
$$\times\sum\chi_0(tc/\delta)k(\phi\sigma(z);t/\delta,y)\prod_{i=1}^{l}\overline{(t/\delta)^{(i)}}^{\varepsilon_i}\,\overline{J(\phi\sigma,z_i)}^{\varepsilon_i},$$

where the sum $\sum$ is taken over all $(t,\phi,\sigma)\in\mathfrak{r}\times R(N')\times\Gamma_{-d/c}\backslash\Gamma_1(N')$ under the condition that $tc/\delta\in\mathfrak{r}$ with

$$\phi=\begin{pmatrix}* & *\\ c & d\end{pmatrix},\qquad k(z,n,y)=\exp\Big(-\pi\sum_{i=1}^{l}y_i^2|n^{(i)}|^2v_i^{-1}\Big),$$
$$J(g,z_i)=(g_3^{(i)}z_i+g_4^{(i)})\qquad\Big(g=\begin{pmatrix}* & *\\ g_3 & g_4\end{pmatrix}\in\mathrm{SL}_2(\mathfrak{r})\Big).$$

Put

$$W(\sigma)=(d^{3/2}N_{F/\mathbb{Q}}(c)^3 2^{1/2})^{-1}(c/i)^{1/2}ic\alpha(\sigma),$$
$$\alpha(\sigma)=\{(i/(c^2z+cd))^{1/2}\}^{-1}\{(c/i)^{1/2}\}^{-1}\{(cz+d)^{1/2}\}^{-1}$$

$\big(\sigma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathfrak{r})\ (c\neq0)\big)$, where the symbol $(*)^{1/2}$ is the same as in [9]. By a method similar to that in Shimura [9] and Friedberg [2], we can derive the following lemma.

Lemma 3.2. *Suppose that N is the same integer as in* (2.2). *Then $\Theta_{1,\varepsilon}(z,Y_1)$ admits the following transformation formula*:

$$\Theta_{1,\varepsilon}(\gamma(z),Y_1)=\left(\frac{2c}{d\mathfrak{r}}\right)(cz+d)^{(2\varepsilon-1)/2}\Theta_{1,\varepsilon}(z,Y_1) \tag{3.2}$$

$$\textit{for every } \gamma=\begin{pmatrix} * & * \\ c & d\end{pmatrix}\in\Gamma_1(N),$$

$$\begin{aligned}\Theta_{1,\varepsilon}(\sigma(z),Y_1)&=W(\sigma)(cz+d)^{(2\varepsilon-1)/2}\\ &\quad\times\sum_{h'\in\mathfrak{r}^3/(c)^3} e[\mathrm{tr}_{F/\mathbb{Q}}(a^t h' S_1 h'/(2c))]\widetilde{\Theta}_{1,\varepsilon}(z+d/c,Y_1)\end{aligned} \tag{3.3}$$

$$\textit{for every } \sigma=\begin{pmatrix} a & * \\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathfrak{r})\ (c\neq 0)$$

and

$$\begin{aligned}\widetilde{\Theta}_{1,\varepsilon}(z,Y_1)=\prod_{i=1}^{l} v_i^{(2-\varepsilon_i)/2}\sum_{h\in S_1^{-1}(1/\delta)^3}\prod_{i=1}^{l} H_{\varepsilon_i}([\sqrt{2\pi v}\,(y_1,-y_3,-2y_2)h]^{(i)})\\ \times e[[u^t h S_1 h+\sqrt{-1}\,v^t h\widetilde{R}(Y_1)h]^{(i)}/2].\end{aligned}$$

Proof. First we consider the case $\varepsilon_0=((\varepsilon_0)_1,\ldots,(\varepsilon_0)_l)$ $((\varepsilon_0)_i=0,1)$. By Shimura [9, (7.19) and (7.20)], we obtain the above statements. For an arbitrary ε, applying continuously several types of Maass differential operators $\lambda/(2\sqrt{-1}\,v_i)+\partial/\partial z_i$ ($\lambda\in\mathbb{Z}$, $1\le i\le l$) to $\Theta_{1,\varepsilon_0}(z,Y_1)$, we get the required assertion.

A direct calculation shows that

$$\begin{aligned}\Psi_{N'}(iY)&=\prod_{i=1}^{l} y_i^{-k_i}\sqrt{2\pi}^{-\varepsilon_i}\sum_{\varepsilon_1,\ldots,\varepsilon_l} {}_{k_1}C_{\varepsilon_1}\ldots{}_{k_l}C_{\varepsilon_l}(\sqrt{-1})^{\varepsilon_1+\ldots+\varepsilon_l}\\ &\quad\times\int_{\Gamma_1(N')\backslash\mathfrak{H}^l}\prod_{i=1}^{l} v_i^{(2k_i-1)/2} f(z)\overline{\Theta_{1,\varepsilon}(z,Y_1)\Theta_{2,k-\varepsilon}(z,y)}\,dz.\end{aligned} \tag{3.4}$$

Hence, by Lemma 3.1, we derive

$$\begin{aligned}&\int_{\Gamma_1(N')\backslash\mathfrak{H}^l}\prod_{i=1}^{l} v_i^{(2k_i-1)/2} f(z)\overline{\Theta_{1,\varepsilon}(z,Y_1)\Theta_{2,k-\varepsilon}(z,y)}\,dz\\ &=(1/\sqrt{d})\prod_{i=1}^{l}(-\sqrt{-2\pi})^{k_i-\varepsilon_i}\prod_{i=1}^{l} y_i^{k_i-\varepsilon_i+1}\\ &\quad\times\int_{\Gamma_1(N')\backslash\mathfrak{H}^l}\prod_{i=1}^{l} v_i^{(2\varepsilon_i-1)/2} f(z)\overline{\Theta_{1,\varepsilon}(z,Y_1)}\\ &\quad\times\sum\overline{\chi}_0(tc/\delta)k(\phi\sigma(z);t/\delta,y)\prod_{i=1}^{l}((t/\delta)^{(i)})^{k_i-\varepsilon_i}J(\phi\sigma,z_i)^{k_i-\varepsilon_i}\,dz.\end{aligned} \tag{3.5}$$

Therefore, the left hand side of (3.5) is equal to

$$(1/\sqrt{d})\prod_{i=1}^{l}(-\sqrt{-2\pi})^{k_i-\varepsilon_i}\prod_{i=1}^{l}y_i^{k_i-\varepsilon_i+1}\sum_{t\in\mathfrak{r}}\sum_{\phi\in R(N')}\overline{\chi}_0(tc/\delta)$$
$$\times\prod_{i=1}^{l}((t/\delta)^{(i)})^{k_i-\varepsilon_i}\int\limits_{\phi^{-1}\Gamma_\infty(N')\phi\backslash\mathfrak{H}^l}f(z)\overline{\Theta_{1,\varepsilon}(z,Y_1)}$$
$$\times\prod_{i=1}^{l}v_i^{(2\varepsilon_i-1)/2}J(\phi,z_i)^{k_i-\varepsilon_i}k(\phi(z);t/\delta,y)\,dz.$$

It is a consequence of Lemma 3.2 that the left hand side of (3.5) can be transformed into the following integral:

$$(1/\sqrt{d})\prod_{i=1}^{l}(-\sqrt{-2\pi})^{k_i-\varepsilon_i}\prod_{i=1}^{l}y_i^{k_i-\varepsilon_i+1}\int\limits_{\Gamma_\infty\backslash\mathfrak{H}^l}\prod_{i=1}^{l}v_i^{(2\varepsilon_i-1)/2}\overline{W(\phi^{-1})}$$
$$\times\sum_{h'\in\mathfrak{r}^3/(c)^3}\overline{e[\mathrm{tr}_{F/\mathbb{Q}}(a{}^th'S_1h'/(2c))]}k(z;t/\delta,y)\overline{\widetilde{\Theta}_{1,\varepsilon}(z+d/c,Y_1)}$$
$$\times\sum_{\mu\in(1/N')}a(\mu)e\Big[\sum_{i=1}^{l}\mu^{(i)}z_i\Big]\,dz.$$

In order to perform further evaluation, we need to consider an equivalence relation on $S_1^{-1}(1/\delta)^3\times\mathfrak{r}$:

$(h,t)\sim_{N'}(h',t')$ if and only if $h'=\varepsilon h$ and $t'=\varepsilon^{-1}t$ for some $\varepsilon\in U(N')$.

Hence, by easy evaluation, we can verify that

$$\sum\overline{\chi}_0(tc/\delta)\prod_{i=1}^{l}((t/\delta)^{(i)})^{k_i-\varepsilon_i}\overline{W(\phi^{-1})}\int\limits_{\Gamma_\infty\backslash\mathfrak{H}^l}\prod_{i=1}^{l}v_i^{(2\varepsilon_i-1)/2}k(z;t/\delta,y)$$
$$\times\overline{\widetilde{\Theta}_{1,\varepsilon}(z+d/c,Y_1)}\sum_{\mu\in(1/N')}a(\mu)e\Big[\sum_{i=1}^{l}\mu^{(i)}z_i\Big]\,dz$$
$$=\sum{}'\sum_{\phi\in R(N')}\overline{W(\phi^{-1})}\overline{\chi}_0(tc/\delta)\overline{e[\mathrm{tr}_{F/\mathbb{Q}}(d{}^thS_1h/(2c))]}a({}^thS_1h/2)$$
$$\times\int\limits_{(R^+)^l}\prod_{i=1}^{l}v_i^{(\varepsilon_i-3)/2}H_{\varepsilon_i}([\sqrt{2\pi v}\,(y_1,-y_3,-2y_2)h]^{(i)})$$
$$\times\exp\Big\{-\pi\sum_{i=1}^{l}[v({}^thS_1h+{}^th\widetilde{R}(Y_1)h)+y^2|t/\delta|^2(1/v)]^{(i)}\Big\}\prod_{i=1}^{l}dv_i,$$

where the sum $\sum$ (resp. $\sum'$) is taken over all $(t,\phi)\in\mathfrak{r}\times R(N')$ (resp. $(h,t)\in S_1^{-1}(1/\delta)^3\times\mathfrak{r}/\sim_{N'}$) under the condition that $tc/\delta\in\mathfrak{r}$ with $\phi^{-1}=\left(\begin{smallmatrix}a&*\\c&*\end{smallmatrix}\right)$. Observe that

$$\int_0^\infty v^{(\varepsilon-3)/2}\exp(-\alpha v-\beta v^{-1})H_\varepsilon(\sqrt{2\alpha v})\,dv=\beta^{(\varepsilon-1)/2}\sqrt{2}^{\varepsilon}\pi\exp(-2\sqrt{\alpha\beta})$$

for any $\alpha,\beta\in R^\times$. Consequently, we conclude that

$$\Psi_{N'}(iY)=c\sum c_f(T,t)\exp\Big(-2\pi\sum_{i=1}^{l}\operatorname{tr}((t^{(i)}/(\delta^{(i)})^2)T^{(i)}Y^{(i)})\Big),$$

where the sum $\sum$ is the same as in the Theorem and c is a constant. Therefore, by the same method as in [3, p. 72], we conclude the proof of the Theorem.

References

[1] T. Asai, *On the Doi–Naganuma lifting associated with imaginary quadratic fields*, Nagoya Math. J. 71 (1978), 149–167.

[2] S. Friedberg, *On the imaginary quadratic Doi–Naganuma lifting of modular forms of arbitrary level*, ibid. 92 (1983), 1–20.

[3] H. Kojima, *Siegel modular cusp forms of degree two*, Tôhoku Math. J. 33 (1981), 65–75.

[4] —, *On construction of Siegel modular forms of degree two*, J. Math. Soc. Japan 34 (1982), 393–411.

[5] S. Niwa, *Modular forms of half integral weight and the integral of certain theta-functions*, Nagoya Math. J. 56 (1974), 147–161.

[6] T. Oda, *On modular forms associated with indefinite quadratic forms of signature* $(2,n-2)$, Math. Ann. 231 (1977), 97–144.

[7] I. I. Piatetski-Shapiro, *On the Saito–Kurokawa lifting*, Invent. Math. 71 (1983), 309–338.

[8] S. Rallis and G. Schiffmann, *On a relation between* SL_2 *cusp forms and cusp forms on the tube domains associated to orthogonal groups*, Trans. Amer. Math. Soc. 263 (1981), 1–58.

[9] G. Shimura, *The arithmetic of certain zeta functions and automorphic forms on orthogonal groups*, Ann. of Math. 111 (1980), 313–375.

[10] —, *On certain zeta functions attached to Hilbert modular forms: 1. The case of Hecke characters*, ibid. 114 (1981), 127–164.

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