

Root systems and the Erdős–Szekeres Problem

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1. Introduction. An n -factor pure product is a polynomial which can be expressed in the form $\prod_{i=1}^n (1 - x^{\alpha_i})$ for some natural numbers $\alpha_1, \dots, \alpha_n$. We will use two different norms, both defined in terms of the expansion of a pure product. Given a pure product $\prod_{i=1}^n (1 - x^{\alpha_i})$ whose expansion as a polynomial is $\sum_{r=0}^{\infty} a_r x^r$, the 1-norm of $\prod_{i=1}^n (1 - x^{\alpha_i})$ is denoted by $\|\prod_{i=1}^n (1 - x^{\alpha_i})\|_1$ and is defined to be the sum of the absolute values of the coefficients in the expansion, i.e.

$$\left\| \sum_{r=0}^{\infty} a_r x^r \right\|_1 := \sum_{r=0}^{\infty} |a_r|.$$

The 2-norm is defined by

$$\left\| \sum_{r=0}^{\infty} a_r x^r \right\|_2 := \left(\sum_{r=0}^{\infty} a_r^2 \right)^{1/2}.$$

Two old problems to which norms of pure products are relevant are the Prouhet–Tarry–Escott Problem and the Erdős–Szekeres Problem. We use square brackets to delimit a *list*, and we call two lists (or, more precisely, k -lists) $[a_1, \dots, a_k]$ and $[b_1, \dots, b_k]$ *equal* if (a_1, \dots, a_k) is a permutation of (b_1, \dots, b_k) . That is, a list is like a set except that repeated elements are allowed, and a list is like a tuple except that the order of entries does not matter. (Some authors call a list a *multiset*.) Suppose we have two unequal lists of integers $[a_1, \dots, a_k]$ and $[b_1, \dots, b_k]$ such that

$$\sum_{i=1}^k a_i^r = \sum_{i=1}^k b_i^r$$

for $r = 1, \dots, d$. Then we say that $[a_1, \dots, a_k]$ and $[b_1, \dots, b_k]$ form a *multi-grade* of size k and degree d . The Prouhet–Tarry–Escott Problem is to find

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multigrades of the smallest possible size for each degree. It is known that for any degree up to $d = 9$, there is a multigrade of size $d + 1$, and it is conjectured (cf. [BI94, p. 10]) that this holds for all degrees. The connection to pure products is that any n -factor pure product of 1-norm $2k$ can be used to construct a multigrade of degree $n - 1$ and size k .

Erdős and Szekeres [ES58] asked how small the ∞ -norm of an n -factor pure product can be, where the ∞ -norm of a polynomial is defined by

$$\left\| \sum_{i=0}^d a_i x^i \right\|_{\infty} := \sup_{\{z \in \mathbb{C}: |z|=1\}} \left| \sum_{i=0}^d a_i z^i \right|,$$

where \mathbb{C} is the set of complex numbers and $|z|$ denotes the modulus (i.e. absolute value) of the complex number z . At the start of Section 3, we give some old inequalities relating the three different norms we have mentioned.

Throughout this paper, \mathbb{N} denotes the set of all natural numbers, and \mathbf{n} denotes the set $\{1, \dots, n\}$. If $\sum_{r=0}^{\infty} a_r x^r = \prod_{i=1}^n (1 - x^{\alpha_i})$, then for each r ,

$$a_r = \left| \left\{ I \subseteq \mathbf{n} : \sum_{i \in I} \alpha_i = r, |I| \text{ even} \right\} \right| - \left| \left\{ I \subseteq \mathbf{n} : \sum_{i \in I} \alpha_i = r, |I| \text{ odd} \right\} \right|.$$

So to make the 1-norm of $\prod_{i=1}^n (1 - x^{\alpha_i})$ as small as possible, we need to pick the exponents $\alpha_1, \dots, \alpha_n$ so as to maximise the number of disjoint pairs $\{I, J\}$ (that is, no two different pairs have the same I or the same J) such that $\sum_{i \in I} \alpha_i = \sum_{j \in J} \alpha_j$ where $I, J \subseteq \mathbf{n}$, $|I|$ even, and $|J|$ odd. The same criterion helps to make the 2-norm as small as possible, but one also has to consider the magnitude of individual coefficients. Among pure products having a given 1-norm, the ones with the smallest 2-norm will be those (if there are any) whose expansions have no coefficients other than 0, 1, and -1 . That is, we want to choose $\alpha_1, \dots, \alpha_n$ so that for every $r \geq 0$, the number of ways to express r as a sum of an even number of $\alpha_1, \dots, \alpha_n$ and the number of ways to express r as a sum of an odd number of $\alpha_1, \dots, \alpha_n$ differ by at most one.

For a definition of root systems, we refer the reader to Carter [C72]. For the present, we will just describe the properties of root systems which are useful in the context of constructing pure products of small norm. Positive root systems are sets of vectors which have the property that a small number of vectors can be represented as a sum of positive roots in exactly one way, and all other vectors can be represented as a sum of an even number of positive roots in exactly as many ways as they can be represented as a sum of an odd number of positive roots. We construct pure products of small norm essentially by projecting root systems down to one dimension and using the projections of the positive roots as exponents in pure products.

2. A construction from root systems. Let Π be a set of fundamental roots of a root system Φ . Let Φ^+ be the corresponding system of positive roots. For $r \in \Phi$, let H_r denote the hyperplane through the origin perpendicular to r , and let w_r denote the self-inverse bijection on points of the space consisting of reflection in the hyperplane H_r . The *Weyl group* W associated with Φ is the group of bijections generated by $\{w_\pi : \pi \in \Pi\}$ under composition of functions. A *chamber* is a connected subset of $\mathcal{V} \setminus (\bigcup_{r \in \Phi} H_r)$ where \mathcal{V} is the whole space. The intersection of the boundary of a chamber with one of the reflecting hyperplanes is called a *wall* of the chamber. Define

$$S := \frac{1}{2} \sum_{r \in \Phi^+} r.$$

The result that makes our construction of pure products useful is Theorem 1. Theorem 1 is essentially a well-known theorem of Weyl [C72, Theorem 10.1.8; M72, equation 0.1], which is usually expressed in the form of the equation

$$e^{-S} \prod_{r \in \Phi^+} (1 - e^r) = \sum_{w \in W} (-1)^{l(w)} e^{w(S)}.$$

In [M96], we state and prove the interesting part of this theorem in (we believe) a clearer presentation than those in the literature. In particular, we do not mention any ring, module, Lie algebra, or [C72] “alternating element of a rational group algebra”. We need one more definition before repeating our statement from [M96].

Call a chamber C *odd* if it satisfies the following two conditions: there is exactly one $\Omega \subseteq \Phi^+$ such that $S - \sum \Omega \in C$, and for every point p in a wall of C , exactly half the subsets $\Omega \subseteq \Phi^+$ such that $S - \sum \Omega = p$ have even cardinality. Theorem 1 is the interesting part of Weyl’s theorem.

THEOREM 1. *Every chamber is odd.* ■

For every $v \in \mathcal{V}$, define

$$a(v) := \left| \left\{ \Omega \subseteq \Phi^+ : \sum \Omega = v, |\Omega| \text{ even} \right\} \right| - \left| \left\{ \Omega \subseteq \Phi^+ : \sum \Omega = v, |\Omega| \text{ odd} \right\} \right|.$$

COROLLARY 2. *There are exactly $|W|$ points $v \in \mathcal{V}$ for which $|a(v)| = 1$, and $a(v) = 0$ for all other $v \in \mathcal{V}$.*

Proof. This is clear from Theorem 1 since there are exactly $|W|$ chambers [C72, Corollary 2.3.3]. ■

It is easy to see that for any natural numbers $\alpha_1, \dots, \alpha_n$ and a , $\| \prod_{i=1}^n (1 - x^{\alpha_i}) \| = \| \prod_{i=1}^n (1 - x^{a\alpha_i}) \|$ for the 1-norm, 2-norm, and ∞ -norm.

Every n -factor pure product has 1-norm at least $2n$. In [M94], S. Maltby answered the question: *For which natural numbers $\beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n$ does $\|\prod_{i=1}^n (1 - x^{b\beta_i + c\gamma_i})\|_1 = 2n$ for all $b, c \in \mathbb{N}$?* This question is equivalent to asking: *In 2-dimensional space \mathcal{V} with basis vectors π_1, π_2 , for what sets of n vectors $\{p_{1,1}\pi_1 + p_{1,2}\pi_2, \dots, p_{n,1}\pi_1 + p_{n,2}\pi_2\}$ does the construction in Lemma 3 produce a pure product of 1-norm $2n$ for all $d_1, d_2 \in \mathbb{N}$?* S. Maltby found that, in fact, the construction produces such pure products if and only if the $p_{i,j}$'s come from a 2-dimensional root system.

Let l be the rank of Φ and label the fundamental roots π_1, \dots, π_l . Let $n = |\Phi^+|$ and label the positive roots r_1, \dots, r_n . For $i = 1, \dots, n$ and $j = 1, \dots, l$, let $p_{i,j}$ be such that $\sum_{j=1}^l p_{i,j}\pi_j = r_i$. Since Π is a basis of \mathcal{V} , the $p_{i,j}$'s are uniquely determined. And since Φ^+ is a positive root system, the $p_{i,j}$'s are all non-negative integers [C72, Proposition 2.1.6]. For any given $\alpha_1, \dots, \alpha_n \in \mathbb{N}$, define for every $k \in \mathbb{N} \cup \{0\}$,

$$b(k) := \left| \left\{ I \subseteq \mathbf{n} : \sum_{i \in I} \alpha_i = k, |I| \text{ even} \right\} \right| - \left| \left\{ I \subseteq \mathbf{n} : \sum_{i \in I} \alpha_i = k, |I| \text{ odd} \right\} \right|.$$

It is easy to see that $\prod_{i=1}^n (1 - x^{\alpha_i}) = \sum_{k=0}^{\infty} b(k)x^k$.

Lemma 3 is intuitively clear if one thinks of the function p as a projection from the space \mathcal{V} to 1-dimensional space. Remember that Π is a basis of \mathcal{V} , so p in the lemma is well-defined.

LEMMA 3. *Let $d_1, \dots, d_l \in \mathbb{N}$. Define $p : \mathcal{V} \rightarrow \mathbb{R}$ by*

$$p\left(\sum_{i=1}^l c_i \pi_i\right) := \sum_{i=1}^l c_i d_i.$$

For $i = 1, \dots, n$, let $\alpha_i = p(r_i) = \sum_{j=1}^l p_{i,j}d_j$. Then for every $k \in \mathbb{N} \cup \{0\}$,

$$b(k) = \sum_{p(v)=k} a(v). \quad \blacksquare$$

We are now able to prove our desired result.

THEOREM 4. *Let $d_1, \dots, d_l \in \mathbb{N}$. For $i = 1, \dots, n$, let $\alpha_i = \sum_{j=1}^l p_{i,j}d_j$. Then*

$$\left\| \prod_{i=1}^n (1 - x^{\alpha_i}) \right\|_1 \leq |W|$$

where W is the Weyl group of Φ .

PROOF. We have

$$\left\| \prod_{i=1}^n (1 - x^{\alpha_i}) \right\|_1 = \left\| \sum_{k=0}^{\infty} b(k)x^k \right\|_1 = \sum_{k=0}^{\infty} |b(k)|.$$

Since $a(v) = 0$ whenever $p(v) \notin \mathbb{N} \cup \{0\}$, Lemma 3 tells us that

$$\sum_{k=0}^{\infty} |b(k)| = \sum_{k=0}^{\infty} \left| \sum_{p(v)=k} a(v) \right| \leq \sum_{k=0}^{\infty} \sum_{p(v)=k} |a(v)| = \sum_{v \in \mathcal{V}} |a(v)|.$$

And Corollary 2 tells us that

$$\sum_{v \in \mathcal{V}} |a(v)| = |W|. \blacksquare$$

There is a reason why one could allow the d_j 's in Theorem 4 to be any integers (not necessarily positive) which do not yield any $\alpha_i = 0$, but there is another reason why the resulting new permitted values of d_j 's would be redundant. Suppose one allows polynomials to include terms with negative exponents, and suppose also that one adjusts the definition of the 1-norm and 2-norm to include coefficients of negative exponents the same as coefficients of positive ones. Then, for any natural numbers $\alpha_1, \dots, \alpha_n$, it is easy to see that the following equalities hold for the 1-norm, the 2-norm, and the ∞ -norm:

$$\begin{aligned} \left\| (1 - x^{-\alpha_1}) \prod_{i=2}^n (1 - x^{\alpha_i}) \right\| &= \left\| x^{-\alpha_1} (x^{\alpha_1} - 1) \prod_{i=2}^n (1 - x^{\alpha_i}) \right\| \\ &= \left\| -x^{-\alpha_1} \prod_{i=1}^n (1 - x^{\alpha_i}) \right\| = \left\| \prod_{i=1}^n (1 - x^{\alpha_i}) \right\|. \end{aligned}$$

From this we see that if we allow negative exponents in the factors of a pure product, we can replace them with their absolute values and the expansion will have the same coefficients, but associated with different exponents, and perhaps with reversed signs, so that the norms are not affected. Because of this, it would make sense to allow the d_j 's to be any integers not yielding any $\alpha_i = 0$. The resulting pure product $\prod_{i=1}^n (1 - x^{\alpha_i})$ would be equivalent, for our purposes, to the pure product $\prod_{i=1}^n (1 - x^{|\alpha_i|})$. Notice that while replacing some α_i 's with the corresponding $-\alpha_i$'s does not change the expansion in any important way, replacing some d_j 's with the corresponding $-d_j$'s is a more complicated matter and it may appear that allowing negative d_j 's would result in pure products not attainable with only positive d_j 's. We now explain why the new pure products that this would allow are not different in any important way from the ones attainable using only positive d_j 's.

THEOREM 5. *Suppose $d_1, \dots, d_l \in \mathbb{Z}$ are such that for $i = 1, \dots, n$, $\alpha_i \neq 0$ where $\alpha_i = |\sum_{j=1}^l p_{i,j} d_j|$. Then there exist $d'_1, \dots, d'_l \in \mathbb{N}$ such that if $\alpha'_i = \sum_{j=1}^l p_{i,j} d'_j$ for $i = 1, \dots, n$, then $(\alpha'_1, \dots, \alpha'_n)$ is a permutation of $(\alpha_1, \dots, \alpha_n)$.*

PROOF. We define dot products by

$$(x_1, \dots, x_k) \cdot (y_1, \dots, y_k) := \sum_{i=1}^k x_i y_i.$$

Since some of the tuples in this proof represent vectors relative to the non-orthonormal basis Π , dot products involving these tuples do not satisfy the usual lengths-cosine relationship. So, for instance, saying that $(x_1, \dots, x_k) \cdot (y_1, \dots, y_k) = 0$ is not equivalent to saying that $x_1\pi_1 + \dots + x_k\pi_k$ is perpendicular to $y_1\pi_1 + \dots + y_k\pi_k$.

For $i = 1, \dots, n$, if $(p_{i,1}, \dots, p_{i,l}) \cdot (d_1, \dots, d_l) > 0$ then let $\varepsilon_i = 1$; and if $(p_{i,1}, \dots, p_{i,l}) \cdot (d_1, \dots, d_l) < 0$ then let $\varepsilon_i = -1$. So we have the equation

$$\begin{bmatrix} \varepsilon_1 p_{1,1} & \cdots & \varepsilon_1 p_{1,l} \\ \varepsilon_2 p_{2,1} & \cdots & \varepsilon_2 p_{2,l} \\ \vdots & & \vdots \\ \varepsilon_n p_{n,1} & \cdots & \varepsilon_n p_{n,l} \end{bmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_l \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

For $i = 1, \dots, n$, let $r'_i = \varepsilon_i r_i$. So for $i = 1, \dots, n$, $r'_i = \sum_{j=1}^l \varepsilon_i p_{i,j} \pi_j$ and r'_i is a root in the root system Φ ; r'_i is either r_i or $-r_i$ —whichever one has positive dot product with (d_1, \dots, d_l) . That is, all the r'_i 's lie on the same side of the hyperplane of vectors having zero dot product with (d_1, \dots, d_l) (which are not necessarily perpendicular to (d_1, \dots, d_l) since the coordinates $(p_{i,1}, \dots, p_{i,l})$ of each r_i are with respect to the basis Π which is not orthonormal). This means that $\{r'_i : i \in I\}$ is a positive root system of the same type as Φ^+ —call this positive root system Φ'^+ . Denote the fundamental roots of Φ'^+ by π'_1, \dots, π'_l and put $\Pi' = \{\pi'_1, \dots, \pi'_l\}$. For $i = 1, \dots, n$, let $(p'_{i,1}, \dots, p'_{i,l})$ be the coordinates of r'_i relative to Π' . That is, each $r'_i = \sum_{j=1}^l p'_{i,j} \pi'_j$. Since Φ^+ and Φ'^+ are positive root systems of the same type, we know that the matrices

$$\begin{bmatrix} p_{1,1} & \cdots & p_{1,l} \\ p_{2,1} & \cdots & p_{2,l} \\ \vdots & & \vdots \\ p_{n,1} & \cdots & p_{n,l} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} p'_{1,1} & \cdots & p'_{1,l} \\ p'_{2,1} & \cdots & p'_{2,l} \\ \vdots & & \vdots \\ p'_{n,1} & \cdots & p'_{n,l} \end{bmatrix}$$

have the same rows, but not necessarily in the same order. So we prove the theorem if we show that there exist $d'_1, \dots, d'_l \in \mathbb{N}$ satisfying

$$\begin{bmatrix} p'_{1,1} & \cdots & p'_{1,l} \\ p'_{2,1} & \cdots & p'_{2,l} \\ \vdots & & \vdots \\ p'_{n,1} & \cdots & p'_{n,l} \end{bmatrix} \begin{bmatrix} d'_1 \\ \vdots \\ d'_l \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

For $j = 1, \dots, l$, let i'_j be so that $r'_{i'_j} = \pi'_j$. So for $j = 1, \dots, l$, $p'_{i'_j,1}, \dots, p'_{i'_j,l}$ are all 0 except for $p'_{i'_j,j} = 1$. Hence, the only assignment of d'_1, \dots, d'_l that can satisfy the matrix multiplication above is to let each $d'_j = \alpha_{i'_j}$

since we require

$$\begin{aligned} \alpha_{i'} &= (p'_{i',1}, \dots, p'_{i',l}) \cdot (d'_1, \dots, d'_l) \\ &= (0, \dots, 0, 1, 0, \dots, 0) \cdot (d'_1, \dots, d'_{j-1}, d'_j, d'_{j+1}, \dots, d'_l) = d'_j. \end{aligned}$$

Notice that this means we are choosing d'_1, \dots, d'_l to be natural numbers.

Now we just need to verify that for $i = 1, \dots, n$, $\alpha_i = \sum_{j=1}^l p'_{i,j} d'_j$. This is actually fairly easy to see from equalities we have already established. The details are as follows. For $i = 1, \dots, n$, $r'_i = \sum_{j=1}^l p'_{i,j} \pi'_j$, so $(p'_{i,1}, \dots, p'_{i,l}) = \sum_{j=1}^l p'_{i,j} (p'_{i',1}, \dots, p'_{i',l})$ and $(\varepsilon_i p_{i,1}, \dots, \varepsilon_i p_{i,l}) = \sum_{j=1}^l p'_{i,j} (\varepsilon_{i'} p_{i',1}, \dots, \varepsilon_{i'} p_{i',l})$. Now

$$\begin{aligned} \sum_{j=1}^l p'_{i,j} d'_j &= (p'_{i,1}, \dots, p'_{i,l}) \cdot (d'_1, \dots, d'_l) \\ &= \left(\sum_{j=1}^l p'_{i,j} (p'_{i',1}, \dots, p'_{i',l}) \right) \cdot (d'_1, \dots, d'_l) \\ &= \sum_{j=1}^l p'_{i,j} ((p'_{i',1}, \dots, p'_{i',l}) \cdot (d'_1, \dots, d'_l)) \\ &= \sum_{j=1}^l p'_{i,j} \alpha_{i'} = \sum_{j=1}^l p'_{i,j} ((\varepsilon_{i'} p_{i',1}, \dots, \varepsilon_{i'} p_{i',l}) \cdot (d_1, \dots, d_l)) \\ &= \left(\sum_{j=1}^l p'_{i,j} (\varepsilon_{i'} p_{i',1}, \dots, \varepsilon_{i'} p_{i',l}) \right) \cdot (d_1, \dots, d_l) \\ &= (\varepsilon_i p_{i,1}, \dots, \varepsilon_i p_{i,l}) \cdot (d_1, \dots, d_l) = \alpha_i. \quad \blacksquare \end{aligned}$$

In the next theorem, we choose the d_j 's so that the restriction of p to $\{\sum \Omega : \Omega \subseteq \Phi^+\}$ is one-to-one. One can do this by choosing $d_j = (C + 1)^j$ where $C = \max\{c_1, \dots, c_l\}$ and $\sum_{j=1}^l c_j \pi_j = \sum \Phi^+$, but we will use slightly smaller d_j 's. Notice that when we make the restriction of p to $\{\sum \Omega : \Omega \subseteq \Phi^+\}$ one-to-one, the reason this makes the 2-norm small is that the coefficients in the expansion of the pure product we construct are each 0, 1, or -1 . Few methods are known for constructing such products.

THEOREM 6. *Let c_1, \dots, c_l be such that $\sum_{j=1}^l c_j \pi_j = \sum \Phi^+$. Let $d_1 = 1$ and assign d_2, \dots, d_l inductively by $d_j = d_{j-1}(c_{j-1} + 1)$. (That is, $d_j = \prod_{k=1}^{j-1} (c_k + 1)$.) For $i = 1, \dots, n$, let $\alpha_i = \sum_{j=1}^l p_{i,j} d_j$. Then*

$$\left\| \prod_{i=1}^n (1 - x^{\alpha_i}) \right\|_2 = \sqrt{|W|}.$$

Proof. Similarly to the proof of Theorem 4, we can say that

$$\begin{aligned} \left\| \prod_{i=1}^n (1 - x^{\alpha_i}) \right\|_2 &= \left\| \sum_{k=0}^{\infty} b(k)x^k \right\|_2 \\ &= \left(\sum_{k=0}^{\infty} |b(k)|^2 \right)^{1/2} = \left(\sum_{k=0}^{\infty} \left| \sum_{p(v)=k} a(v) \right|^2 \right)^{1/2}. \end{aligned}$$

Observe by the following that if $k \in \mathbb{N} \cup \{0\}$ and $v \in \{\sum \Omega : \Omega \subseteq \Phi^+\}$ are such that $p(v) = k$, then v is uniquely determined. We know $v = \sum_{j=1}^l v_j \pi_j$ for some $v_1, \dots, v_l \in \mathbb{N} \cup \{0\}$ such that each $v_j \leq c_j$. Since $d_l > \sum_{j=1}^{l-1} c_j d_j$, we know that $v_l = \lfloor k/d_l \rfloor$, and we find v_{l-1} similarly after reducing k modulo d_l , and so on. So for every $k \in \mathbb{N} \cup \{0\}$ we have $|\sum_{p(v)=k} a(v)| \in \{0, 1\}$ and, hence, $|\sum_{p(v)=k} a(v)|^2 = |\sum_{p(v)=k} a(v)|$. Together with Corollary 2, this tells us that

$$\left(\sum_{k=0}^{\infty} \left| \sum_{p(v)=k} a(v) \right|^2 \right)^{1/2} = \left(\sum_{k=0}^{\infty} \left| \sum_{p(v)=k} a(v) \right| \right)^{1/2} = |W|^{1/2}. \blacksquare$$

Every indecomposable root system is of one of the following types. This table (except for the last column) is copied from Carter’s book [C72, p. 43]. (A decomposable root system is one of the form $\bigcup_{i \in I} \Phi_i$ where $2 \leq |I|$ finite, each Φ_i is an indecomposable root system of rank l_i , and the Φ_i ’s are embedded in $(\sum_{i \in I} l_i)$ -space so that $r \perp s$ for all distinct $i, j \in I$ and all $r \in \Phi_i$ and $s \in \Phi_j$.)

Table 1. All indecomposable root systems

Root system Φ	$n = \Phi^+ $	$ W $	$ W $
A_l ($l \geq 1$)	$\frac{1}{2}l(l+1)$	$(l+1)!$	$(\sqrt{2n+1/4}+1/2)!$
B_l ($l \geq 2$)	l^2	$2^l l!$	$2^{\sqrt{n}} \sqrt{n}!$
C_l ($l \geq 3$)	l^2	$2^l l!$	ditto
D_l ($l \geq 4$)	$l(l-1)$	$2^{l-1} l!$	$2^{\sqrt{n+1/4}-1/2} (\sqrt{n+1/4}+1/2)!$
G_2	6	12	
F_4	24	$2^7 \cdot 3^2$	
E_6	36	$2^7 \cdot 3^4 \cdot 5$	
E_7	63	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	
E_8	120	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	

Some readers may wonder about the less restrictive definition of root systems which allows a type known as BC . The root system BC_l has $l(l+1)$ positive roots and the number corresponding to $|W|$ for our purposes is

$(l + 1)!2^l$. This makes BC -type root systems less useful to us than B - and C -type root systems.

Carter [C72, Section 3.6] explicitly lists the roots of each root system in terms of an orthonormal basis. From these descriptions, we get the following α_i 's using the construction in Theorem 4. For A_l ,

$$\{\alpha_i : 1 \leq i \leq n\} = \{d_i + \dots + d_j : 1 \leq i \leq j \leq l\}.$$

For B_l ,

$$\begin{aligned} \{\alpha_i : 1 \leq i \leq n\} &= \{d_i + \dots + d_j : 1 \leq i \leq j \leq l\} \\ &\cup \{d_i + \dots + d_{j-1} + 2(d_j + \dots + d_l) : 1 \leq i < j \leq l\}. \end{aligned}$$

For C_l ,

$$\begin{aligned} \{\alpha_i : 1 \leq i \leq n\} &= \{d_i + \dots + d_{j-1} : 1 \leq i < j \leq l\} \\ &\cup \{d_i + \dots + d_{j-1} + 2(d_j + \dots + d_{l-1}) + d_l : 1 \leq i < j \leq l\} \\ &\cup \{2(d_i + \dots + d_{l-1}) + d_l : 1 \leq i \leq l\}. \end{aligned}$$

For D_l ,

$$\begin{aligned} \{\alpha_i : 1 \leq i \leq n\} &= \{d_i + \dots + d_{j-1} : 1 \leq i < j \leq l\} \\ &\cup \{d_i + \dots + d_{j-1} + 2(d_j + \dots + d_{l-2}) + d_{l-1} + d_l : 1 \leq i < j \leq l\}. \end{aligned}$$

From Table 1, we see that using B -type root systems in Theorems 7 and 9, we can construct n -factor pure products $\prod_{i=1}^n (1 - x^{\alpha_i})$ so that

$$\left\| \prod_{i=1}^n (1 - x^{\alpha_i}) \right\|_1 \leq 2^{\sqrt{n}} \sqrt{n!} \leq (2\sqrt{n})^{\sqrt{n}}$$

and

$$\left\| \prod_{i=1}^n (1 - x^{\alpha_i}) \right\|_2 = \sqrt{2^{\sqrt{n}} \sqrt{n!}} \leq (2\sqrt{n})^{\sqrt{n/4}}.$$

We now compare these bounds to those obtainable by methods in the literature.

3. Comparisons with old results. The relevant results in the literature are concerned mainly with the ∞ -norms of pure products. The 1-norm, 2-norm, and ∞ -norm are related by the following easily verified inequalities (cf. [BI94]). For any polynomial $p(x)$,

$$\frac{\|p(x)\|_1}{\sqrt{\deg p(x) + 1}} \leq \|p(x)\|_2 \leq \|p(x)\|_\infty \leq \|p(x)\|_1 \leq \|p(x)\|_2^2.$$

If one starts searching for pure products of small norm, one is likely to get the notion that for general $n \in \mathbb{N}$, $\prod_{i=1}^n (1 - x^i)$ will have small norm, but this example turns out to be disappointing. According to Erdős & Szekeres [ES58],

$$\lim_{n \rightarrow \infty} \left(\left\| \prod_{i=1}^n (1 - x^i) \right\|_{\infty} \right)^{1/n} > 1.$$

In other words, there is a constant $c > 1$ so that for large n , the ∞ -norm of the n -factor pure product $\prod_{i=1}^n (1 - x^i)$ is roughly c^n . Atkinson [A61] and Dobrowolski [D79] showed by different methods that for any natural number k ,

$$\left\| \prod_{i=1}^{k-1} (1 - x^i)^{k-i} \right\|_{\infty} \leq k^{k/2}.$$

(Borwein & Ingalls [BI94] incorrectly gave the outer exponent in this pure product as $k - i - 1$ instead of $k - i$.) Letting $n = k(k - 1)/2$, so that $k = 1/2 + \sqrt{2n + 1/4}$, we can say that this construction yields an n -factor pure product having ∞ -norm at most $(1/2 + \sqrt{2n + 1/4})^{1/4 + \sqrt{n/2 + 1/16}}$. If we allow the approximation $k \approx \sqrt{2n}$, this bound simplifies to $\sqrt{2n} \sqrt{n/2}$. (In [BI94], Borwein & Ingalls say incorrectly that $k \leq \sqrt{2n}$.) This pure product can be constructed by using A_{k-1} as the root system in Theorem 4 and letting $d_1 = \dots = d_{k-1} = 1$. Borwein & Ingalls [BI94] observed, essentially, that Dobrowolski's method achieves the same result for any values of the d_i 's. The method relies on expressing the pure product in terms of a Vandermonde determinant and is not useful for root systems other than A -type.

The best bounds in the literature are only existence results—the authors do not give a construction of pure products witnessing their bounds, they only show that such pure products exist. Also, they do not express these bounds precisely, they only give the order. The exponent in Odlyzko's bound [O82] is the cube root $\sqrt[3]{n(\ln n)^4}$, improving on the square root in the exponent of $\sqrt{2n} \sqrt{n/2}$. Kolountzakis [K94] improved on Odlyzko's bound, getting an exponent of $\sqrt[3]{n}$. According to Odlyzko [O95], recent unpublished work of S. V. Konyagin and A. S. Belov makes a major improvement on these results, showing that there exists a constant c so that $A_{\infty}(n) \leq e^{c(\ln n)^4}$ for all $n \in \mathbb{N}$. Our results will not approach these bounds, but our examples have the advantage of being explicit constructions.

If we construct a pure product $\prod_{i=1}^n (1 - x^{\alpha_i})$ from A_{k-1} as in Theorem 4, we know that $\left\| \prod_{i=1}^n (1 - x^{\alpha_i}) \right\|_1 \leq (\sqrt{2n + 1/4} + 1/2)!$. Furthermore, by choosing the d_i 's as in Theorem 6, we see that this bound is tight. Clearly, this bound is an improvement on the bound of $\sqrt{2n} \sqrt{2n}$ resulting from Borwein & Ingalls's inequalities $\|p\|_1 \leq \|p\|_{\infty}^2$ and $\left\| \prod_{i=1}^n (1 - x^{\alpha_i}) \right\|_{\infty} \leq$

$\sqrt{2n}\sqrt{n/2}$. However, one is really only interested in how small the *smallest* 1-norm of an n -factor pure product can be. In this sense, we cannot improve on Dobrowolski’s example in which $d_1 = \dots = d_{k-1} = 1$ so that $\prod_{i=1}^n (1 - x^{\alpha_i}) = \prod_{i=1}^k (1 - x^i)^{k-i}$. Then, at least for $k \geq 4$,

$$\begin{aligned} \left\| \prod_{i=1}^n (1 - x^{\alpha_i}) \right\|_1 &\leq \left\| \prod_{i=1}^k (1 - x^i)^{k-i} \right\|_\infty \sqrt{1 + \deg \prod_{i=1}^k (1 - x^i)^{k-i}} \\ &\leq k^{k/2} \sqrt{1 + \sum_{i=1}^k i(k-i)} < k^{k/2} \sqrt{\sum_{i=1}^k (k/2)^2} \\ &= k^{k/2} \sqrt{k(k/2)^2} = k^{k/2} \frac{k^{3/2}}{2} = \frac{1}{2} k^{(k+3)/2} \\ &\approx \sqrt{2n}^{(\sqrt{2n}+3)/2} \approx \sqrt{2n} \sqrt{n/2}. \end{aligned}$$

For the 2-norm, Theorem 6 improves on the literature. Using Borwein & Ingalls’s inequalities $\|p\|_2 \leq \|p\|_\infty$ and $\|\prod_{i=1}^n (1 - x^{\alpha_i})\|_\infty \leq \sqrt{2n}\sqrt{n/2}$, we get the bound $\|\prod_{i=1}^n (1 - x^{\alpha_i})\|_2 \leq \sqrt{2n}\sqrt{n/2}$. Using A -type root systems in Theorem 6 to construct the same pure product, we get the small improvement

$$\left\| \prod_{i=1}^n (1 - x^{\alpha_i}) \right\|_2 = \sqrt{(\sqrt{2n} + 1/4 + 1/2)!}.$$

Better than this, using B - or C -type root systems in Theorem 6, we get (for $n \geq 36$)

$$\left\| \prod_{i=1}^n (1 - x^{\alpha_i}) \right\|_2 = \sqrt{2\sqrt{n}\sqrt{n}!} < \sqrt{\sqrt{n}\sqrt{n}} = \sqrt{n}\sqrt{n/4}.$$

Using Theorem 6 with B - or C -type root systems is better than any other method known for constructing pure products of small 2-norm.

We now consider pure products having particular numbers of factors. In Table 2, each p_n is an n -factor pure product of minimal known 1-norm. A table of pure products of minimal known 1-norm appeared in Borwein & Ingalls’s paper [BI94]. We found a few errors and corrections for their table, but it seems pointless to report these now since Borwein has since improved on these results. The results in Table 2 are Borwein’s [B96] for $n = 15, 24, 28, \text{ and } 30$ through 70. Some of the other pure products in this table are also Borwein’s [B96], but they are not actually improvements on other results in terms of the size of the 1-norm. Other results are in [BI94], [M97], and [M96]. We include 2-norms in the table since, as far as we know, no search

has been made specifically for pure products of small 2-norm. We use two kinds of ellipsis in the table. The single ellipsis “...” means that every number between the bounds appears exactly once, and the double ellipsis “:::” means that every *odd* number between the bounds appears exactly once.

Table 2. Exponents of pure products of least known 1-norm

n	$\ p_n\ _1$	$\ p_n\ _2^2$	Exponents of p_n
3	6	6	[1, 2, 3]
4	8	10	[1, 2, 3, 4]
5	10	10	[1, 3, 4, 5, 7]
6	12	12	[1, 2, 3, 4, 5, 7]
7	16	16	[1, 2, 3, 4, 5, 7, 11]
8	16	16	[2, 3, 5, 7, 8, 11, 13, 19]
9	20	20	[2, 3, 5, 7, 8, 11, 13, 17, 19]
10	24	24	[1, ..., 5, 7, 9, 11, 13, 17]
11	28	28	[1, 2, 3, 5, 7, 8, 9, 11, 13, 17, 19]
12	36	40	[1, ..., 9, 11, 13, 17]
13	48	52	[1, ..., 9, 11, 13, 17, 19]
14	56	56	[1, 3, ..., 7, 10, 11, 13, 16, 17, 19, 23, 29]
15	52	52	[1, 3, ..., 7, 9, 10, 11, 13, 16, 17, 19, 23, 29]
16	60	68	[1, ..., 11, :::, 19, 23]
17	68	68	[1, ..., 7, 9, 10, 11, 13, 14, 16, 17, 19, 23, 29]
18	84	94	[1, ..., 11, 13, 14, 16, 17, 19, 22, 23]
19	100	108	[1, ..., 11, :::, 25, 29]
20	116	132	[1, ..., 11, :::, 27, 31]
21	130	142	[1, ..., 11, :::, 31]
22	140	152	[1, ..., 9, :::, 33, 37]
23	156	168	[1, ..., 11, :::, 33, 37]
24	192	208	[1, ..., 11, 13, 14, 15, :::, 31, 35, 37]
25	188	204	[1, ..., 11, :::, 37, 41]
26	228	256	[1, ..., 11, :::, 41]
27	276	324	[1, ..., 13, :::, 41]
28	292	350	[1, ..., 13, :::, 41, 47]
29	392	696	[1, 1, 2, 2, 3, ..., 27]
30	396	760	[1, 1, 2, 2, 3, 3, 4, ..., 27]
31	414	714	[1, 1, 2, 2, 3, ..., 29]
32	456	852	[1, 1, 2, 2, 3, 3, 4, ..., 29]
33	482	930	[1, 1, 2, 2, 3, 3, 4, ..., 29, 31]
34	516	974	[1, 1, 2, 2, 3, 3, 4, ..., 31]
35	604	1248	[1, 1, 2, 2, 3, 3, 4, ..., 31, 33]
36	616	1200	[1, 1, 2, 2, 3, 3, 4, ..., 33]

Table 2 (cont.)

n	$\ p_n\ _1$	$\ p_n\ _2^2$	Exponents of p_n
37	688	1440	[1, 1, 2, 2, 3, 3, 4, ..., 33, 35]
38	664	1338	[1, 1, 2, 2, 3, 3, 4, ..., 35]
39	740	1460	[1, 1, 2, 2, 3, 3, 4, ..., 35, 37]
40	804	1564	[1, 1, 2, 2, 3, 3, 4, ..., 37]
41	908	1976	[1, 1, 2, 2, 3, 3, 4, ..., 37, 39]
42	924	1996	[1, 1, 2, 2, 3, 3, 4, ..., 39]
43	952	2032	[1, 1, 2, 2, 3, 3, 4, ..., 39, 41]
44	1028	2144	[1, 1, 2, 2, 3, 3, 4, ..., 41]
45	1132	2516	[1, 1, 2, 2, 3, 3, 4, ..., 41, 43]
46	1176	2866	[1, 1, 2, 2, 3, 3, 4, ..., 43]
47	1254	2910	[1, 1, 2, 2, 3, 3, 4, 4, 5, ..., 43]
48	1288	3180	[1, 1, 2, 2, 3, 3, 4, 4, 5, ..., 44]
49	1484	3692	[1, 1, 2, 2, 3, 3, 4, 4, 5, ..., 45]
50	1604	4400	[1, 1, 2, 2, 3, 3, 4, 4, 5, ..., 46]
51	1632	4416	[1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, ..., 45, 47]
52	1740	4586	[1, 1, 2, 2, 3, 3, 4, 5, 5, 6, ..., 47, 49]
53	1602	3686	[1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, ..., 47, 49]
54	1784	4616	[1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 7, 7, 8, ..., 47, 49]
55	1916	4924	[1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, ..., 50]
56	2144	5928	[1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, ..., 51]
57	2188	3988	[1, 2, 2, 3, ..., 25, :::, 33, 34, 35, :::, 85]
58	2352	4828	[1, 2, 2, 3, ..., 27, :::, 33, 34, 35, :::, 85]
59	2256	4068	[1, 2, 2, 3, ..., 25, :::, 33, 34, 35, :::, 89]
60	2456	4780	[1, 2, 2, 3, ..., 27, :::, 33, 34, 35, :::, 89]
61	2742	5486	[1, 2, 2, 3, ..., 27, :::, 33, 34, 35, :::, 89, 101]
62	2760	5164	[1, 2, 2, 3, ..., 27, :::, 93, 101]
63	2992	5940	[1, 2, 2, 3, 4, 4, 5, ..., 27, :::, 93, 101]
64	3232	6884	[1, 2, 2, 3, ..., 29, :::, 93, 101, 107]
65	3130	6190	[1, 2, 2, 3, ..., 29, :::, 95, 101, 107]
66	3108	5904	[1, 2, 2, 3, ..., 29, :::, 97, 101, 107]
67	3508	7272	[1, 2, 2, 3, ..., 29, :::, 37, 38, 39, :::, 97, 101, 107]
68	3864	8224	[1, 2, 2, 3, ..., 29, :::, 97, 101, 107, 109, 111]
69	4464	10968	[1, 2, 2, 3, ..., 29, :::, 63, 64, 65, :::, 97, 101, 107, 109, 111]
70	4592	11444	[1, 2, 2, 3, ..., 29, :::, 45, 46, 47, :::, 63, 64, 65, :::, 97, 101, 107, 109, 111]

Theorem 4 gives an upper bound on the 1-norm of any pure product constructed from root systems as described. It seems likely that good choices of the d_i 's in this construction will actually provide 1-norms much smaller than this upper bound. The most ambitious goal is to see if we can use the

root system constructions to match the smallest known 1-norms, and for this purpose it seems best to begin by comparing the bounds from Theorem 4 with the best known 1-norms to see if any are already close. Since Borwein (and Ingalls's) results do not go up to 120-factor pure products, we take p_{120} to be $(p_{60})^2$.

Table 3. Comparing best known norms to norms from root systems

n	System	$ W $	$\ p_n\ _1$	$\ p_n\ _2^2$
1	A_1	2	2	2
3	A_2	6	6	6
6	A_3	24	12	12
10	A_4	120	24	24
15	A_5	720	52	52
4	B_2	8	8	10
9	B_3, C_3	48	20	20
16	B_4, C_4	384	60	68
25	B_5, C_5	3840	188	204
12	D_4	192	36	40
20	D_5	1920	116	132
30	D_6	23,040	396	760
6	G_2	12	12	12
24	F_4	1152	192	208
36	E_6	51,840	616	1200
63	E_7	2,903,040	2992	5940
120	E_8	696,729,600	1,440,480	1,025,261,796

We see from Table 3 that constructions according to Theorem 4 always match the best known 1-norms when the root system used is A_1 , A_2 , B_2 , or G_2 . This does not hold in other cases but it would be interesting to know if we can always match the best known (or even best possible) results using the construction in Theorem 4 if we choose suitable values for the d_i 's.

From [M97], we know that the least possible 1-norm of a 9-factor pure product is 20 and that there are only five ways (up to multiplication of all the exponents by a constant) to achieve this—the list of exponents $[\alpha_1, \dots, \alpha_9]$ must be one of the following:

$$\begin{aligned}
 & [1, 2, 3, 4, 5, 7, 9, 11, 13], \\
 & [1, 2, 3, 5, 7, 8, 9, 11, 13], \\
 & [1, 2, 3, 5, 7, 8, 11, 13, 19], \\
 & [1, 4, 5, 6, 7, 9, 11, 13, 17], \\
 & [2, 3, 5, 7, 8, 11, 13, 17, 19].
 \end{aligned}$$

The only root systems that produce 9 factors in the construction in Theorem 4 are B_3 and C_3 . Can B_3 or C_3 produce any of the 9-exponent lists above using this construction? No. The details of the tedious case examinations appear in [M96].

For a given number of roots, decomposable root systems tend to have larger Weyl groups than indecomposable root systems do, so one might think it hopeless to ask if any of these 9-factor pure products of minimal 1-norm can be constructed from a decomposable root system. In fact, it is trivial to construct any pure product $\prod_{i=1}^n (1 - x^{\alpha_i})$ from a decomposable root system—just use the root system constructed from n copies of A_1 and let $d_i = \alpha_i$ for $i = 1, \dots, n$. However, any resulting smallness of the norm will not be due to properties of root systems so this is not an enlightening answer.

The list of exponents of p_{10} is $[1, 2, 3, 4, 5, 7, 9, 11, 13, 17]$ and $\|p_{10}\|_1 = 24$. The only root system that produces 10-factor pure products is A_4 , and A_4 does not produce p_{10} , but it does come tantalisingly close. Using A_4 in the construction in Theorem 4 and letting $(d_1, d_2, d_3, d_4) = (4, 1, 2, 10)$, we get $[\alpha_1, \dots, \alpha_{10}] = [1, 2, 3, 4, 5, 7, 10, 12, 13, 17]$, which differs from the list in question only in that 9 and 11 are replaced by 10 and 12. Unfortunately, this tiny change has a substantial effect on the 1-norm, upping it from 24 to 60.

The preceding examples discourage hope that root systems are always the best way to construct pure products of small norm, but perhaps if we look at Table 3 comparing the best known results with the worst possible by root systems, we can find a largish example where these quantities are not far apart. We seem to have such an example in F_4 which produces 24-factor pure products of 1-norm at most $2^7 \cdot 3^2 = 1152$, while the least known 1-norm of a 24-factor pure product is 192. Because of the freedom we have in choosing d_1, d_2, d_3, d_4 , we might expect to be able to produce a pure product using F_4 in Theorem 4 which has 1-norm much less than the worst case of 1152. In fact, a substantial improvement is possible, but not to the point of breaking any records as far as we can see. The least 1-norm we have found for a 24-factor pure product constructed from A_4 by the method in Theorem 4 is 328 and the exponents $[\alpha_i]$ in the pure product are

$$[1, 2, 3, 4, 6, 7, 8, 9, 10, 16, 20, 22, 23, 24, 25, 26, 28, 29, 30, 31, 32, 34, 38, 54].$$

Insofar as 120-factor pure products have been investigated at all in the literature, the construction in Theorem 4 using E_8 is an improvement, at least in terms of producing a small 2-norm. The norms for p_{120} in Table 3 are based on taking $p_{120} = (p_{60})^2$, which is about the best we can extrapolate from the literature. (Borwein & Ingalls [BI94] only searched for pure products of small 1-norm up to 100 factors, and the largest good one

they found had only 60 factors. Borwein [B96] decided to only keep track of n -factor pure products having 1-norm less than n^2 , of which he found examples only up to $n = 70$.) Another 120-factor pure product from the literature is that of Atkinson [A61] and Dobrowolski [D79] which is the A_{15} construction using $d_1 = \dots = d_{15} = 1$. This yields 1-norm 10,642,137,208 and 2-norm square root of 672,106,633,395,797,608, both much worse than the norms from E_8 . Furthermore, remember that $|W|$ given in Table 3 for E_8 is an upper bound on the 1-norm and that with appropriate choices of d_1, \dots, d_8 , we expect that a much smaller 1-norm is possible.

In Theorem 6, we see that with an appropriate choice of the d_i 's in Theorem 4, we can use any root system with n positive roots and associated Weyl group W to construct an n -factor pure product having $|W|$ coefficients 1 or -1 and all other coefficients 0. One might wonder if there is always a choice of the d_i 's so that this construction produces a polynomial in which all coefficients are ± 1 (and none are zero). Considering A_2 , we can see quickly that the answer is no. The pure product we get from A_2 is

$$(1 - x^{d_1})(1 - x^{d_2})(1 - x^{d_1+d_2}) = 1 - x^{d_1} - x^{d_2} + x^{2d_1+d_2} + x^{d_1+2d_2} - x^{2d_1+2d_2}.$$

To get the coefficient of x to be ± 1 , we need $d_1 = 1$ and $d_2 \geq 2$ (assuming, without loss of generality, that $d_2 \geq d_1$). Then, to get the coefficient of x^2 to be ± 1 , we need $d_2 = 2$. The resulting pure product is $1 - x - x^2 + x^4 + x^5 - x^6$ in which x^3 has coefficient 0.

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