

Diagonal cubic equations

by

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1. Introduction. Let λ_j be positive integers. We shall be concerned with solutions of the diophantine equation

$$(1.1) \quad \lambda_1 x_1^3 + \lambda_2 x_2^3 + \dots + \lambda_7 x_7^3 = 0$$

which are small in the following sense. Let $A = \lambda_1 \lambda_2 \dots \lambda_7$. Then we seek for solutions to (1.1) which satisfy

$$(1.2) \quad 0 < \sum_{i=1}^7 \lambda_i |x_i|^3 \ll A^g$$

with g as small as possible.

A first positive answer to the problem was found by R. C. Baker [1]. He established the solubility of (1.1) and (1.2) whenever $g = 61$. In this paper we prove the following theorem.

THEOREM. *Let $g = 14$. Then there are solutions to (1.1), (1.2).*

The key for improvement is the mean value estimate for a class of exponential sums presented in [2]. We state the special case we require as a lemma. Let X be a large parameter, and write

$$(1.3) \quad \theta_1 = 15/113, \quad \theta_2 = 14/113, \quad \eta = 84/113.$$

We note that $\theta_1 + \theta_2 + \eta = 1$. Put

$$(1.4) \quad M_1 = X^{\theta_1}, \quad M_2 = X^{\theta_2}, \quad Y = X^\eta.$$

As usual we write $e(\alpha) = \exp(2\pi i\alpha)$. Now define the exponential sums

$$(1.5) \quad g(\alpha, P) = \sum_{P < X \leq 2P} e(\alpha x^3),$$

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$$(1.6) \quad G(\alpha, X) = \sum_{M_1 < p_1 \leq 2M_1} \sum_{M_2 < p_2 \leq 2M_2} g(\alpha(p_1 p_2)^3, Y),$$

where, here and throughout, p, p_i denote prime numbers.

LEMMA 1. *In the notation introduced above, for any $\varepsilon > 0$ we have*

$$\int_0^1 |G(\alpha, X)|^6 d\alpha \ll X^{3+2\theta_1+\varepsilon}.$$

This is the case $l = 3$ of the Theorem in Brüdern [2].

2. Preliminaries. Now define

$$(2.1) \quad \chi_3(X) = \{p_1 p_2 y : X^{\theta_1} < p_1 \leq 2X^{\theta_1}, X^{\theta_2} < p_2 \leq 2X^{\theta_2}, X^\eta < y \leq 2X^\eta\}$$

and

$$(2.2) \quad \chi_4(X) = \{py : X^\theta < p \leq 2X^\theta, y \in \chi_3(X^{1-\theta})\}$$

in which $\theta = 233/1815$.

Given positive integers μ_1, μ_2 , we write $S(\mu_1, \mu_2, B, E)$ for the number of solutions of the equation

$$(2.3) \quad \mu_1 x_1^3 + \mu_2 p^3 (y_1^3 + y_2^3) = \mu_1 x_2^3 + \mu_2 p^3 (y_3^3 + y_4^3)$$

in integers $x_1, x_2, y_1, y_2, y_3, y_4$ and primes p satisfying

$$(2.4) \quad B < x_1, x_2 \leq 2B, \quad p \nmid \mu_1 x_1 x_2, \quad y_i \in \chi_3(E^{1-\theta}) \quad (1 \leq i \leq 4),$$

$$(2.5) \quad E^\theta < p \leq 2E^\theta, \quad p \equiv 2 \pmod{3}.$$

LEMMA 2. *Let $B \geq 1, E \geq 1$. Let μ_1, μ_2 be positive integers, and*

$$(2.6) \quad \mu_2 E^3 \ll \mu_1 B^3 \ll \mu_2 E^3, \quad \mu_1 \gg \mu_2, \quad E^\theta \leq B^{1/7}.$$

Then

$$(2.7) \quad S(\mu_1, \mu_2, B, E) \ll B^{1+\varepsilon} E^{2-\theta}.$$

Proof. By the argument of Vaughan [7], p. 125, we have $S(\mu_1, \mu_2, B, E) \ll E^\varepsilon S_0$, where S_0 is the number of solutions to (2.3) subject to (2.4), (2.5) and $x_1 \equiv x_2 \pmod{p^3}$. The solutions with $x_1 = x_2$ contribute to S_0 at most $BE^\theta E^{2(1-\theta)+\varepsilon} = BE^{2-\theta+\varepsilon}$. For the solutions with $x_1 \neq x_2$ we again follow through the argument in Vaughan [7] and find that there are at most $2S_1$ remaining solutions where S_1 equals the number of solutions to

$$\mu_1 h(3x^2 + h^2 p^6) = 4\mu_2 (y_1^3 + y_2^3 - y_3^3 - y_4^3),$$

where $x \leq 4B$, $E^\theta < p \leq 2E^\theta$, $0 < h \leq H$, $y_i \in \chi_3(E^{1-\theta})$, and $H = CBE^{-3\theta}$ for some constant C . On writing

$$\Omega(\alpha) = \sum_{E^\theta < p \leq 2E^\theta} \sum_{h \leq H} \sum_{x \leq 4B} e(\alpha h(3x^2 + h^2 p^6))$$

we have, by Hölder's inequality,

$$\begin{aligned} S_1 &= \int_0^1 \Omega(\mu_1 \alpha) |G(4\mu_2 \alpha, E^{1-\theta})|^4 d\alpha \\ &\leq \left(\int_0^1 |\Omega(\mu_1 \alpha)|^3 d\alpha \right)^{1/3} \left(\int_0^1 |G(\alpha, E^{1-\theta})|^6 d\alpha \right)^{2/3}. \end{aligned}$$

The argument of Vaughan [8], p. 39 shows that for $B^{1/8} < E^\theta \leq B^{1/7}$ we have

$$\int_0^1 |\Omega(\alpha)|^3 d\alpha \ll H^2 (E^\theta B)^{3/2+\varepsilon}.$$

Since $\Omega(\alpha)$ has period 1, we have

$$\int_0^1 |\Omega(\mu_1 \alpha)|^3 d\alpha = \frac{1}{\mu_1} \int_0^{\mu_1} |\Omega(\alpha)|^3 d\alpha = \int_0^1 |\Omega(\alpha)|^3 d\alpha$$

and by Lemma 1 we have

$$\int_0^1 |G(\alpha, E^{1-\theta})|^6 d\alpha \ll (E^{1-\theta})^{3+2\theta_1+\varepsilon}.$$

The lemma follows easily.

3. Outline of the method. For the proof of the Theorem, as in Baker [1] it suffices to prove the Theorem in the special case when $\lambda_1, \dots, \lambda_7$ are cube-free, and no prime divides more than four of the λ_i . We may suppose that

$$(3.1) \quad \lambda_5 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_6 \geq \lambda_7 \geq \lambda_1.$$

Let

$$(3.2) \quad N = C_0 A^g, \quad P = N^{1/3}, \quad P_j = P \lambda_j^{-1/3},$$

where C_0 is sufficiently large. Let δ denote a sufficiently small positive absolute constant, and let $\varepsilon = \delta^2$.

Let $R(\lambda_1, \dots, \lambda_7)$ denote the number of solutions of the equation

$$(3.3) \quad \lambda_1 x^3 - \lambda_2 p^3 y^3 + \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 z_3^3 + \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 z_4^3 + \lambda_5 u^3 + \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 v_6^3 + \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 v_7^3 = 0$$

in integers $x, y, z_3, z_4, u, v_6, v_7$ and primes p, p_{ij} satisfying

$$(3.4) \quad P_1 < x \leq 2P_1, \quad W < y \leq 2W, \quad R_i < z_i \leq 2R_i,$$

$$(3.5) \quad U < u \leq 2U, \quad V_i < v_i \leq 2V_i,$$

$$(3.6) \quad p \nmid x, \quad p_{ij} \nmid y \quad (3 \leq i \leq 4, 1 \leq j \leq 3), \quad p_{ij} \nmid u \quad (6 \leq i \leq 7, 1 \leq j \leq 3),$$

$$(3.7) \quad Y < p \leq 2Y, \quad p \equiv 2 \pmod{3}, \quad p \nmid \Lambda,$$

$$Z_{ij} < p_{ij} \leq 2Z_{ij}, \quad p_{ij} \equiv 2 \pmod{3},$$

$$(i = 3, 4, 6, 7, 1 \leq j \leq 3), \quad p_{ij} \nmid \Lambda.$$

Here

$$(3.8) \quad Y = C_1 P_1^{1/7}, \quad W = P_2 Y^{-1}, \quad R_i = \frac{1}{200} (P_i Y^{-1})^{1176/1815} \quad (i = 3, 4),$$

$$(3.9) \quad U = \frac{1}{200} P_5, \quad V_i = \frac{1}{200} P_i^{1176/1815} \quad (i = 6, 7),$$

$$(3.10) \quad Z_{i1} = 2^{-i} (P_i Y^{-1})^{233/1815}, \quad Z_{i2} = 2^{-i} (P_i Y^{-1})^{210/1815},$$

$$Z_{i3} = 2^{-i} (P_i Y^{-1})^{196/1815} \quad (i = 3, 4),$$

$$(3.11) \quad Z_{i1} = 2^{-i} P_i^{233/1815}, \quad Z_{i2} = 2^{-i} P_i^{210/1815},$$

$$Z_{i3} = 2^{-i} P_i^{196/1815} \quad (i = 6, 7),$$

where C_1 is sufficiently large.

The inequalities

$$(3.12) \quad N \ll \lambda_1 P_1^3, \lambda_2 Y^3 W^3, \lambda_3 Y^3 Z_{31}^3 Z_{32}^3 Z_{33}^3 R_3^3, \lambda_4 Y^3 Z_{41}^3 Z_{42}^3 Z_{43}^3 R_4^3,$$

$$\lambda_5 U^3, \lambda_6 Z_{61}^3 Z_{62}^3 Z_{63}^3 V_6^3, \lambda_7 Z_{71}^3 Z_{72}^3 Z_{73}^3 V_7^3 \ll N$$

are easily verified.

Let (a_1, a_2) or more generally (a_1, \dots, a_n) denote greatest common divisor, while $[a_1, \dots, a_n]$ denotes least common multiple. We write

$$(3.13) \quad f_d(X, \alpha) = \sum_{\substack{X < x \leq 2X \\ (x, d) = 1}} e(\alpha x^3)$$

and use the notation f in place of f_1 .

Let \mathbf{p} denote the ordered set $p, p_{31}, p_{32}, p_{33}, p_{41}, p_{42}, p_{43}, p_{61}, p_{62}, p_{63}, p_{71}, p_{72}, p_{73}$ and let

$$(3.14) \quad A = \prod_{i=3}^4 \prod_{j=1}^3 p_{ij}, \quad B = \prod_{i=6}^7 \prod_{j=1}^3 p_{ij},$$

$$(3.15) \quad F(\mathbf{p}, \alpha) = f_p(P_1, \lambda_1 \alpha) \overline{f_A(W, \lambda_2 p^3 \alpha)} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha)$$

$$\times f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha)$$

$$\times f_B(U, \lambda_5 \alpha) f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) f(V_7, \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 \alpha),$$

$$(3.16) \quad F(\alpha) = \sum_{\substack{\mathbf{p} \\ (3.6),(3.7)}} F(\mathbf{p}; \alpha).$$

The summation here is over ordered sets \mathbf{p} satisfying (3.6), (3.7); we often use this type of notation below.

Let \mathfrak{R} be the unit interval $[LN^{-1}, 1 + LN^{-1}]$. Then clearly

$$(3.17) \quad R(\lambda_1, \dots, \lambda_7) = \int_{\mathfrak{R}} F(\alpha) d\alpha.$$

Here

$$(3.18) \quad L = P_1/10.$$

When $1 \leq a \leq q \leq L$ and $(a, q) = 1$, we take $M(q, a)$ to be the interval $\{\alpha : |\alpha - a/q| \leq q^{-1}LN^{-1}\}$, and let M denote the union of all such $M(q, a)$. Then $M(q, a)$ are disjoint subsets of M , as we easily verify using (3.2) and (3.18). Let

$$(3.19) \quad m = \mathfrak{R} \setminus M.$$

We shall prove that

$$(3.20) \quad \int_M F(\alpha) d\alpha \gg \lambda_1^{2/21} A^{-1/3-\varepsilon} N^{26/21} (\log N)^{-13},$$

$$(3.21) \quad \int_m F(\alpha) d\alpha \ll \lambda_1^{2/21} A^{-1/3} N^{26/21-\varepsilon}.$$

It follows from (3.19)–(3.21) and (3.17) that

$$(3.22) \quad R(\lambda_1, \dots, \lambda_7) \gg \lambda_1^{2/21} A^{-1/3-\varepsilon} N^{26/21} (\log N)^{-13}.$$

Thus the Theorem will follow once (3.20)–(3.21) have been proved.

We recall the standard notations

$$(3.23) \quad S(q, b) = \sum_{r=1}^q e(br^3/q), \quad J(\beta, X) = \int_X^{2X} e(\beta x^3) dx.$$

The estimate

$$(3.24) \quad S(q, b) \ll q^{1+\varepsilon} \psi(q), \quad (q, b) = 1,$$

where $\psi(q)$ is the multiplicative function with

$$(3.25) \quad \psi(p^{3h+r}) = p^{-h-r/2} \quad (h = 0, 1, \dots; 0 \leq r \leq 2),$$

follows from Lemmas 4.3 and 4.4 of [5]. By partial integration,

$$(3.26) \quad J(\beta, X) \ll \frac{X}{1 + X^3|\beta|}$$

for positive X and real β .

LEMMA 3. Let d be an integer with $\ll 1$ divisors. Let

$$(3.27) \quad s_d(q, c) = q^{-1} \sum_{b|d} \mu(d) \frac{S(q, b^3 c)}{b}$$

and let λ be a positive integer. Then

$$(3.28) \quad f_d(X, \lambda\alpha) = s_d(q, \lambda\alpha) J(\lambda(\alpha - a/q), X) \\ + O(q^{1/2+\varepsilon} (1 + \lambda X^3 |\alpha - a/q|)^{1/2})$$

for any $X > 0$, real α and rational number a/q .

This is Lemma 2 of [1].

For $\alpha \in M(q, a)$, we introduce the notations

$$(3.29) \quad s(p, \alpha) = s_p(q, \lambda_1 a) J(\lambda_1(\alpha - a/q), P_1),$$

$$(3.30) \quad g(p, \alpha) = s_A(q, \lambda_2 p^3 a) J(\lambda_2(\alpha - a/q) p^3, W).$$

It is convenient to write

$$(3.31) \quad \Delta = \Delta(\alpha, a, q) = 1 + N|\alpha - a/q|.$$

LEMMA 4. Let \mathbf{p} satisfy (3.6) and (3.7). Let $\alpha \in M(q, a)$ where $1 \leq a \leq q \leq L$, $(a, q) = 1$. Then

$$(3.32) \quad f_p(P_1, \lambda_1 \alpha) - s(p, \alpha) \ll q^{1/2+\varepsilon} \Delta^{1/2},$$

$$(3.33) \quad f_A(W, \lambda_2 p^3 \alpha) - g(p, \alpha) \ll q^{1/2+\varepsilon} \Delta^{1/2}.$$

Proof. The proof is similar to that of Lemma 3 of [1]; see [1] and [6].

4. The minor arcs. In this section we use the notations

$$(4.1) \quad H = C_2 P_1 Y^{-3}, \quad Q = P_1 Y^{-1},$$

where C_2 is a sufficiently large absolute constant; also

$$(4.2) \quad M = L(2Y)^{-3}.$$

Further, let

$$(4.3) \quad S(\alpha) = \left| \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} \sum_{\substack{p_{41}, p_{42}, p_{43} \\ (3.7)}} f_A(W, \lambda_2 \alpha) \right. \\ \left. \times f(R_3, \lambda_3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^2$$

and

$$(4.4) \quad \Phi_p(\alpha) = \sum_{\substack{P_1 < y \leq 2P_1 \\ p|y}} 1 + 2\operatorname{Re} \sum_{h \leq H} \sum_{\substack{2P_1 + hp^3 < y \leq 4P_1 - hp^3 \\ p \nmid y, y \equiv h \pmod{2}}} e\left(\frac{\lambda_1 \alpha}{4}(3hy^2 + h^3 p^6)\right).$$

Let n denote the set of real numbers in $(0, 1]$ with the property that whenever $|\alpha - a/q| \leq q^{-1}LN^{-1}$ and $(a, q) = 1$, we have $q > M$. Let

$$(4.5) \quad T = \int_n \sum_p \Phi_p(\alpha) S(\alpha) d\alpha. \tag{3.6}$$

LEMMA 5. *Let*

$$J_1 = \int_0^1 \left| \sum_{\substack{p_{61}, p_{62}, p_{63} \\ (3.7)}} \sum_{\substack{p_{71}, p_{72}, p_{73} \\ (3.7)}} f_B(U, \lambda_5 \alpha) \times f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) f(V_7, \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 \alpha) \right|^2 d\alpha.$$

Then $J_1 \ll P_5 P_6^{2\theta} P_7^{2+\epsilon}$.

Proof. By considering the underlying diophantine equation we have

$$J_1 \ll \int_0^1 \left| \sum_{\substack{p_{61}, p_{62}, p_{63} \\ (3.7)}} \sum_{\substack{p_{71}, p_{72}, p_{73} \\ (3.7)}} f_{p_{61} p_{71}}(U, \lambda_5 \alpha) f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) \times f(V_7, \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 \alpha) \right|^2 d\alpha.$$

From the inequality $|zw| \leq \frac{1}{2}|z|^2 + \frac{1}{2}|w|^2$, we have

$$J_1 \ll \sum_{k=6}^7 \int_0^1 \left(\sum_{\substack{p_{61} \\ (3.7)}} \sum_{\substack{p_{71} \\ (3.7)}} |f_{p_{61} p_{71}}(U, \lambda_5 \alpha)| \sum_{\substack{p_{k2}, p_{k3} \\ (3.7)}} f(V_k, \lambda_k p_{k1}^3 p_{k2}^3 p_{k3}^3 \alpha) \right)^2 d\alpha.$$

Hence, by Cauchy's inequality,

$$J_1 \ll Z_{61} Z_{71} \sum_{k=6}^7 \int_0^1 \sum_{\substack{p_{61} \\ (3.7)}} \sum_{\substack{p_{71} \\ (3.7)}} |f_{p_{61} p_{71}}(U, \lambda_5 \alpha)|^2 \times \left| \sum_{\substack{p_{k2}, p_{k3} \\ (3.7)}} f(V_k, \lambda_k p_{k1}^3 p_{k2}^3 p_{k3}^3 \alpha) \right|^4 d\alpha \ll Z_{71}^2 Z_{61} S(\lambda_5, \lambda_6, P_5, P_6) + Z_{61}^2 Z_{71} S(\lambda_5, \lambda_7, P_5, P_7).$$

By (3.1) and (3.2) we have

$$P_6 \leq P_7, \quad P_6^\theta \leq P_7^\theta \leq P_5^{1/7}.$$

By (3.11), (3.12) and Lemma 2,

$$J_1 \ll P_5 P_6^{2\theta} P_7^{2+\varepsilon}.$$

This completes the proof of Lemma 5.

LEMMA 6. *Let*

$$J_2 = \int_m \left| \sum_{(3.6)}^P \sum_{(3.7)}^{P_{31}, P_{32}, P_{33}} \sum_{(3.7)}^{P_{41}, P_{42}, P_{43}} f_p(P_1, \lambda_1 \alpha) f_A(W, \lambda_2 p^3 \alpha) \right. \\ \left. \times f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^2 d\alpha.$$

Then $J_2 \ll YT$.

Proof. The proof is similar to that of Lemma 15 of [1].

Let

$$(4.6) \quad F(\beta, \gamma; h) = \sum_{\substack{2P_1 < y \leq 4P_1 \\ y \equiv h \pmod{2}}} e\left(\frac{3}{4}\beta y^2 - \gamma y\right),$$

$$(4.7) \quad G_h(\varrho, \sigma) = \sum_{\substack{Y < p \leq 2Y, p \equiv 2 \pmod{3} \\ p \leq (2P_1/h)^{1/3}}} e\left(\frac{1}{4}\varrho p^6 + \sigma p^3\right),$$

$$(4.8) \quad F_p(\alpha; h) = \sum_{\substack{2P_1 + hp^3 < y \leq 4P_1 - hp^3 \\ y \equiv h \pmod{2}}} e\left(\frac{3}{4}\lambda_1 \alpha h y^2\right),$$

$$(4.9) \quad \Psi_p(\alpha) = 2\text{Re} \sum_{h \leq H} F_p(\alpha; h) e\left(\frac{1}{4}\lambda_1 \alpha h^3 p^6\right),$$

$$(4.10) \quad D_p(\alpha; h) = \sum_{\substack{2P_1/p + hp^2 < y \leq 4P_1/p - hp^2 \\ y \equiv h \pmod{2}}} e\left(\frac{3}{4}\lambda_1 \alpha h p^2 y^2\right),$$

$$(4.11) \quad \Xi_p(\alpha) = 2\text{Re} \sum_{h \leq H} D_p(\alpha; h) e\left(\frac{1}{4}\lambda_1 \alpha h^3 p^6\right),$$

$$(4.12) \quad T_1(p) = \int_n \Psi_p(\alpha) S(\alpha) d\alpha,$$

$$(4.13) \quad T_2(p) = \int_n \Xi_p(\alpha) S(\alpha) d\alpha,$$

$$(4.14) \quad T_3 = \int_0^1 S(\alpha) d\alpha.$$

By (4.4),

$$(4.15) \quad \Phi_p(\alpha) = \Psi_p(\alpha) - \Xi_p(\alpha) + O(P_1).$$

Hence

$$(4.16) \quad T = \sum_{\substack{p \\ (3.6)}} (T_1(p) - T_2(p)) + O(P_1 Y T_3)$$

from (4.5).

We estimate T_3 via Lemma 2. By (3.1), (3.2), (3.8) and (3.10) we have

$$(P_3 Y^{-1})^\theta \leq (P_4 Y^{-1})^\theta \leq (P_2 Y^{-1})^{1/7}.$$

Just as in Lemma 5, we have

$$(4.17) \quad T_3 \ll (P_2 Y^{-1})(P_3 Y^{-1})^{2\theta} (P_4 Y^{-1})^{2+\varepsilon} \ll P_2 P_3^{2\theta} P_4^{2+\varepsilon} Y^{-3-2\theta}.$$

Let

$$(4.18) \quad T_5(\gamma, \theta) = \int \sum_{n, h \leq H} |F(\alpha \lambda_1 h, \gamma; h) G_h(\alpha \lambda_1 h^3, \theta \gamma h)| S(\alpha) d\alpha.$$

By a very minor adaptation of the proof of (5.26) of [7], one finds that

$$(4.19) \quad \sum_{\substack{p \\ (3.6)}} T_1(p) \ll (\log P_1) \sup_{\substack{0 \leq \gamma \leq 1 \\ \theta = \pm 1}} T_5(\gamma, \theta) + N^{(26+12\theta)/21} \Lambda^{-1}.$$

We omit the details.

LEMMA 7. *Suppose that $\alpha \in \mathbb{R}$ and*

$$\left| \alpha - \frac{a}{q} \right| \leq \frac{1}{24qHP_1}, \quad (a, q) = 1.$$

Then

$$\sum_{h \leq H} |F(\alpha h, \gamma; h)|^2 \ll P_1^\delta \left[\frac{q^{-1} H P_1^2}{1 + Q^3 |\alpha - a/q|} + H P_1 + q \right].$$

This is Lemma 16 of [1].

LEMMA 8. *Let $\alpha, \gamma \in \mathbb{R}$. Suppose that*

$$\left| \alpha - \frac{a}{q} \right| \leq q^{-1} Q^{-3} H^{3/4}, \quad (a, q) = 1, \quad q \leq Q^3 H^{-3/4}.$$

Then

$$\sum_{h \leq H} |G_h(\alpha h^3, \gamma h)|^2 \ll P_1^\varepsilon \left[\frac{H Y^2 q^{-1/3}}{(1 + Q^3 |\alpha - a/q|)^{1/3}} + H^{3/4} Y^2 \right].$$

This is essentially Lemma 8 of [7].

LEMMA 9. Let $\gamma \in \mathbb{R}$ and $\theta \in \{-1, 1\}$. Then

$$T_5(\gamma, \theta) \ll P_1^{1+\delta} P_2 P_3^{2\theta} P_4^{2+\varepsilon} Y^{-2-2\theta} + \lambda_1^{1+\varepsilon} P_1^{2+\varepsilon} P_3^{1+\varepsilon} P_4^{1+\varepsilon} Y^{-1} \\ + \lambda_1^{1/2} P_1^{1+\varepsilon} P_2^{1/2+\varepsilon} P_3^{5/4+\varepsilon} P_4^{5/4+\varepsilon} Y^{-1/2}.$$

PROOF. Let n_1 denote the set of α in n with the property that whenever

$$(4.20) \quad |\lambda_1 \alpha - r/b| \leq r^{-1} H^{7/4} Q^{-3}, \quad (b, r) = 1,$$

one has $r > H^{7/4}$. Let $\alpha \in n_1$. We apply Lemma 7 with $\lambda_1 \alpha$ in place of α . By Dirichlet's theorem there exist integers b, r satisfying (4.20) with $r \leq Q^3 H^{-7/4}$; here we must have $r > H^{7/4}$. Now the condition of Lemma 7 is easily verified, thus

$$(4.21) \quad \sum_{h \leq H} |F(\alpha h \lambda_1, \gamma; h)|^2 \ll P_1^\delta (P_1^2 H^{-3/4} + H P_1 + Q^3 H^{-7/4}) \\ \ll P_1^{2+\delta} H^{-3/4}.$$

We may choose c, s so that $(c, s) = 1, s \leq Q^3 H^{-3/4}$, and

$$|\lambda_1 \alpha - c/s| \leq s^{-1} H^{3/4} Q^{-3}.$$

Then $|\lambda_1 \alpha - c/s| \leq s^{-1} H^{7/4} Q^{-3}$, so that $s > H^{7/4} > H^{3/4}$. By Lemma 8, applied to $\lambda_1 \alpha$ in place of α , we have

$$\sum_{h \leq H} |G_h(\lambda_1 \alpha h^3, \theta \gamma h)|^2 \ll P_1^\varepsilon H^{3/4} Y^2.$$

Hence by Cauchy's inequality, (4.14), (4.17) and (4.21),

$$(4.22) \quad \int \sum_{n_1} \sum_{h \leq H} |F(\alpha h \lambda_1, \gamma; h) G_h(\lambda_1 \alpha h^3, \theta \gamma h)| S(\alpha) d\alpha \\ \ll P_1^{1+\delta} Y T_3 \ll P_1^{1+\delta} P_2 P_3^{2\theta} P_4^{2+\varepsilon} Y^{-2-2\theta}.$$

It remains to consider $n \setminus n_1$. Let $\alpha \in n \setminus n_1$. There are integers b, r satisfying

$$(4.23) \quad |\lambda_1 \alpha - b/r| \leq r^{-1} H^{7/4} Q^{-3}, \quad (b, r) = 1, \quad r \leq H^{7/4}.$$

We write $b/(\lambda_1 r) = a/q$ in lowest terms. Clearly $q = rd$ where $d = \lambda_1/(\lambda_1, b)$. Also, $(r, \lambda_1/d) = (r, (\lambda_1, b)) = 1$. Because $\alpha \in n$, we know that either $|\alpha - a/(rd)| > r^{-1} d^{-1} L N^{-1}$, or $rd > M$, or both.

Let d be a given divisor of λ_1 . Let

$$N_1(d, r, a) = \left\{ \alpha : \left| \alpha - \frac{a}{rd} \right| \leq r^{-1} d^{-1} L N^{-1} \right\}, \\ N_2(d, r, a) = \left\{ \alpha : \left| \alpha - \frac{a}{rd} \right| \leq r^{-1} H^{7/4} Q^{-3} \lambda_1^{-1} \right\}.$$

It is clear from (3.2), (3.8), (3.18), (4.1) that $N_2(d, r, a)$ contains $N_1(d, r, a)$.

Let $N_1(d)$ denote the union of $N_1(d, r, a)$ with

$$(4.24) \quad 1 \leq a \leq rd, \quad (a, rd) = (r, \lambda_1/d) = 1,$$

$$(4.25) \quad r \leq Md^{-1}.$$

Let $N_2(d)$ denote the union of $N_2(d, r, a)$ with (4.24) and $r \leq H^{7/4}$. By the discussion following (4.22) we have, modulo one,

$$n \setminus n_1 \subset \bigcup_{d|\lambda_1} \{N_2(d) \setminus N_1(d)\}.$$

Let $L(d, r, a)$ denote $N_2(d, r, a)$ when $Md^{-1} < r \leq H^{7/4}$, and (4.24) holds; and denote $N_2(d, r, a) \setminus N_1(d, r, a)$ when $r \leq Md^{-1}$ and (4.24) holds. For fixed d , we have

$$N_2(d) \setminus N_1(d) = \bigcup_{r \leq H^{7/4}; a} L(d, r, a).$$

Let $\alpha \in L(d, r, a)$. The fraction $\lambda_1 a / (rd)$ may be written in the form b/r with $(b, r) = 1$. Since

$$(\lambda_1 a, rd) = d(\lambda_1 a/d, r) = d,$$

by Lemma 7, with $\lambda_1 \alpha$ in place of α , and

$$\left| \lambda_1 \alpha - \frac{b}{r} \right| = \lambda_1 \left| \alpha - \frac{a}{rd} \right| \leq r^{-1} H^{7/4} Q^{-3} \leq \frac{1}{24rHP_1},$$

we have

$$(4.26) \quad \sum_{h \leq H} |F(\alpha h \lambda_1, \gamma; h)|^2 \ll P_1^\delta \left[\frac{r^{-1} HP_1^2}{1 + Q^3 |\lambda_1 \alpha - b/r|} + HP_1 + r \right] \\ \ll \frac{r^{-1} HP_1^{2+\delta}}{1 + NY^{-3} |\alpha - a/(rd)|}.$$

Here we require the observations that

$$Q^3 \lambda_1 = NY^{-3}, \quad HP_1 + r \ll HP_1 \ll r^{-1} HP_1^2, \\ (HP_1 + r)rQ^3 |\lambda_1 \alpha - b/r| \ll HP_1 H^{7/4} \ll HP_1^2.$$

Choose c, s so that $|\lambda_1 \alpha - c/s| \leq s^{-1} H^{3/4} Q^{-3}$, $(c, s) = 1$ and $s \leq Q^3 H^{-3/4}$. If $b/r = c/s$, then by the definition of $L(d, r, a)$, we know that either

$$|\lambda_1 \alpha - c/s| > s^{-1} d^{-1} LN^{-1} \lambda_1 \geq s^{-1} LN^{-1},$$

or $s > Md^{-1}$, or both. By (3.1), (3.2) we have either $s > H^{3/4}$ or $Q^3 s |\lambda_1 \alpha - c/s| \geq H^{3/4}$, or both. If $b/r \neq c/s$, then

$$\frac{1}{rs} \leq \left| \frac{b}{r} - \frac{c}{s} \right| \leq (H^{3/4} s^{-1} + H^{7/4} r^{-1}) Q^{-3} \leq \frac{1}{2sr} + H^{7/4} r^{-1} Q^{-3},$$

whence $s > \frac{1}{2}Q^3H^{-7/4} > H^{3/4}$ once more. Now we may apply Lemma 8 with $\lambda_1\alpha$ in place of α , obtaining

$$(4.27) \quad \sum_{h \leq H} |G_h(\lambda_1\alpha h^3, \theta\gamma h)|^2 \ll P_1^\varepsilon H^{3/4} Y^2.$$

Therefore, by (4.26) and Cauchy's inequality

$$(4.28) \quad \sum_{h \leq H} |F(\alpha h \lambda_1, \gamma; h) G_h(\lambda_1\alpha h^3, \theta\gamma h)| \ll \frac{Y H^{7/8} P_1^{1+\delta} r^{-1/2}}{(1 + NY^{-3} |\alpha - a/(rd)|)^{1/2}}.$$

It follows that

$$(4.29) \quad \int_{N_2(d) \setminus N_1(d)} \sum_{h \leq H} |F(\alpha h \lambda_1, \gamma; h) G_h(\lambda_1\alpha h^3, \theta\gamma h)| S(\alpha) d\alpha \\ \ll P_1^{1+\delta} H^{7/8} Y \int_{N_2(d) \setminus N_1(d)} r^{-1/2} \left(1 + NY^{-3} \left| \alpha - \frac{a}{rd} \right| \right)^{-1/2} S(\alpha) d\alpha.$$

We write

$$\Gamma = \Gamma(\alpha, a, q) = q^{1/2+\varepsilon} (1 + NY^{-3} |\alpha - a/q|)^{1/2}.$$

By Lemma 3, we have

$$f_A(W, \lambda_2\alpha) = s_A(q, \lambda_2a) J(\lambda_2(\alpha - a/q), W) + O(\Gamma) = \Gamma_1 + O(\Gamma).$$

Hence by (3.26) we have

$$|f_A(W, \lambda_2\alpha)|^2 \ll \Gamma_1^2 + \Gamma^2 \ll |s_A(q, \lambda_2a) J(\lambda_2(\alpha - a/q), W)|^2 + \Gamma^2 \\ \ll \frac{s_A^2(q, \lambda_2a) W^2}{(1 + NY^{-3} |\alpha - a/q|)^2} + \Gamma^2.$$

Hence

$$(4.30) \quad \int_{N_2(d) \setminus N_1(d)} r^{-1/2} \left(1 + NY^{-3} \left| \alpha - \frac{a}{rd} \right| \right)^{-1/2} S(\alpha) d\alpha \\ \ll \int_{N_2(d) \setminus N_1(d)} r^{-1/2} \left(1 + NY^{-3} \left| \alpha - \frac{a}{rd} \right| \right)^{-1/2} (\Gamma_1^2 + \Gamma^2) \\ \times \left| \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} \sum_{\substack{p_{41}, p_{42}, p_{43} \\ (3.7)}} f(R_3, \lambda_3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^2 d\alpha.$$

By Hua's inequality we have

$$\int_0^1 \left| \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} f(R_3, \lambda_3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) \right|^8 d\alpha \ll (P_3 Y^{-1})^{5+\varepsilon},$$

$$\int_0^1 \left| \sum_{\substack{p_{41}, p_{42}, p_{43} \\ (3.7)}} f(R_4, \lambda_4 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^8 d\alpha \ll (P_4 Y^{-1})^{5+\varepsilon}$$

and

$$\begin{aligned} & \int_0^1 \left| \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} \sum_{\substack{p_{41}, p_{42}, p_{43} \\ (3.7)}} f(R_3, \lambda_3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^2 d\alpha \\ & \ll \left(\int_0^1 \left| \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} f(R_3, \lambda_3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) \right|^4 d\alpha \right)^{1/2} \\ & \quad \times \left(\int_0^1 \left| \sum_{\substack{p_{41}, p_{42}, p_{43} \\ (3.7)}} f(R_4, \lambda_4 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^4 d\alpha \right)^{1/2} \\ & \ll ((P_3 Y^{-1})^{2+\varepsilon} (P_4 Y^{-1})^{2+\varepsilon})^{1/2}. \end{aligned}$$

By the definition of Γ ,

$$\Gamma^2 = q^{1+\varepsilon} \left(1 + NY^{-3} \left| \alpha - \frac{a}{q} \right| \right) = (rd)^{1+\varepsilon} \left(1 + NY^{-3} \left| \alpha - \frac{a}{rd} \right| \right).$$

For $\alpha \in N_2(d) \setminus N_1(d)$,

$$r^{1/2+\varepsilon} d^{1+\varepsilon} \left(1 + NY^{-3} \left| \alpha - \frac{a}{rd} \right| \right)^{1/2} \ll d^{1+\varepsilon} H^{7/8+\varepsilon} \ll \lambda_1^{1+\varepsilon} H^{7/8+\varepsilon}.$$

Hence

$$\begin{aligned} (4.31) \quad & \int_{N_2(d) \setminus N_1(d)} r^{-1/2} \left(1 + NY^{-3} \left| \alpha - \frac{a}{rd} \right| \right)^{-1/2} \Gamma^2 \\ & \times \left| \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} \sum_{\substack{p_{41}, p_{42}, p_{43} \\ (3.7)}} f(R_3, \lambda_3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^2 d\alpha \\ & \ll \int_{N_2(d) \setminus N_1(d)} r^{1/2+\varepsilon} d^{1+\varepsilon} \left(1 + NY^{-3} \left| \alpha - \frac{a}{rd} \right| \right)^{1/2} \\ & \quad \times \left| \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} \sum_{\substack{p_{41}, p_{42}, p_{43} \\ (3.7)}} f(R_3, \lambda_3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^2 d\alpha \\ & \ll \lambda_1^{1+\varepsilon} H^{7/8+\varepsilon} (P_3 P_4 Y^{-2})^{1+\varepsilon}. \end{aligned}$$

Now we consider

$$(4.32) \quad \int_{N_2(d) \setminus N_1(d)} \Gamma_1^4 d\alpha = \int_{N_2(d) \setminus N_1(d)} \frac{s_A^4(q, \lambda_2 a) W^4}{(1 + NY^{-3} |\alpha - a/(rd)|)^4} d\alpha \\ \ll W^4 \sum_{r \leq H^{7/4}} \sum_{a \leq rd} \int_{L(d, r, a)} \frac{s_A^4(q, \lambda_2 a)}{(1 + NY^{-3} |\alpha - a/(rd)|)^4} d\alpha.$$

Let $\kappa(q)$ and $\kappa^*(q)$ be the multiplicative functions defined by

$$\kappa(p^{3k}) = p^{-k}, \quad \kappa(p^{3k+1}) = 2p^{-k-1/2}, \quad \kappa(p^{3k+2}) = p^{-k-1}, \\ \kappa^*(p) = 2p^{-1/2}, \quad \kappa^*(p^2) = p^{-3/4}, \quad \kappa^*(p^l) = p^{-l/3} \quad (l \geq 3).$$

Then $\kappa(q) \leq \kappa^*(q)$, and by Lemmas 4.3, 4.4 and Theorem 4.2 of Vaughan [5],

$$q^{-1} S(q, a) \ll \kappa(q) \quad \text{when } (a, q) = 1.$$

Now we consider

$$\sum_{q \leq \Xi} q \kappa^* \left(\frac{q}{(q, \lambda_2)} \right)^4 \\ \ll \prod_{p \leq \Xi} \left(1 + \sum_{l=1}^{\infty} p^l \kappa^* \left(\frac{p^l}{(p^l, \lambda_2)} \right)^4 \right) \\ \ll \prod_{\substack{p \nmid \lambda_2 \\ p \leq \Xi}} \left(1 + 16p^{-1} + p^{-1} + \sum_{l=3}^{\infty} p^{-l/3} \right) \\ \times \prod_{\substack{p \parallel \lambda_2 \\ p \leq \Xi}} \left(1 + p + 16 + 1 + 1 + \sum_{l=1}^{\infty} p^{-l/3} \right) \\ \times \prod_{\substack{p^2 \parallel \lambda_2 \\ p \leq \Xi}} \left(1 + p + p^2 + 16p + p + p + p^{2/3} + p^{1/3} + \sum_{l=0}^{\infty} p^{-l/3} \right).$$

We have

$$\prod_{\substack{p \parallel \lambda_2 \\ p \leq \Xi}} (20 + p) \ll \prod_{\substack{p \parallel \lambda_2 \\ p \leq C(\varepsilon)}} (20 + p) \prod_{\substack{p \parallel \lambda_2 \\ p \geq C(\varepsilon)}} (20 + p) \ll \prod_{p \parallel \lambda_2} p^{1+\varepsilon}.$$

Here we use the fact that for $p \geq C(\varepsilon)$ we have $20 + p \leq p^{1+\varepsilon}$. Similarly

$$\prod_{\substack{p^2 \parallel \lambda_2 \\ p \leq \Xi}} (p^2 + 22p + 1) \ll \prod_{p^2 \parallel \lambda_2} p^{2+\varepsilon}.$$

The constant implied by \ll depends only on ε . So we have

$$(4.33) \quad \sum_{q \leq \Xi} q \kappa^* \left(\frac{q}{(q, \lambda_2)} \right)^4 \ll \lambda_2^{1+\varepsilon} (\log \Xi)^{20}.$$

By (3.9) of [1],

$$s_A(q, \lambda_2 a) \ll q^\varepsilon \kappa \left(\frac{q}{(q, \lambda_2 a)} \right) \ll q^\varepsilon \kappa^* \left(\frac{q}{(q, \lambda_2)} \right),$$

so we have

$$(4.34) \quad \int_{N_2(d) \setminus N_1(d)} \Gamma_1^4 d\alpha \ll W^4 Y^3 N^{-1+\varepsilon} \lambda_2.$$

For $\alpha \in N_2(d) \setminus N_1(d)$, we have

$$r^{-1/2} \left(1 + NY^{-3} \left| \alpha - \frac{a}{rd} \right| \right)^{-1/2} \ll M^{-1/2} d^{1/2}.$$

Hence by Schwarz's inequality,

$$(4.35) \quad \int_{N_2(d) \setminus N_1(d)} r^{-1/2} \left(1 + NY^{-3} \left| \alpha - \frac{a}{rd} \right| \right)^{-1/2} \Gamma_1^2 \\ \times \left| \sum_{\substack{P_{31}, P_{32}, P_{33} \\ (3.7)}} \sum_{\substack{P_{41}, P_{42}, P_{43} \\ (3.7)}} f(R_3, \lambda_3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^2 d\alpha \\ \ll (W^4 Y^3 N^{-1+\varepsilon} \lambda_2)^{1/2} M^{-1/2} d^{1/2} (P_3 Y^{-1})^{(5+\varepsilon)/4} (P_4 Y^{-1})^{(5+\varepsilon)/4}.$$

By (4.29)–(4.35) we have

$$(4.36) \quad \int_{n \setminus n_1} \sum_{h \leq H} |F(\alpha h \lambda_1, \gamma; h) G_h(\lambda_1 \alpha h^3, \theta \gamma h)| S(\alpha) d\alpha \\ \ll P_1^{1+\delta} H^{7/8} Y [\lambda_1^{1+\varepsilon} H^{7/8+\varepsilon} (P_3 P_4 Y^{-2})^{1+\varepsilon} \\ + (W^4 Y^3 N^{-1+\varepsilon} \lambda_2)^{1/2} M^{-1/2} d^{1/2} (P_3 P_4 Y^{-2})^{(5+\varepsilon)/4}] \\ \ll \lambda_1^{1+\varepsilon} P_1^{2+\varepsilon} P_3^{1+\varepsilon} P_4^{1+\varepsilon} Y^{-1} + \lambda_1^{1/2} P_1^{1+\varepsilon} P_2^{1/2+\varepsilon} P_3^{5/4+\varepsilon} P_4^{5/4+\varepsilon} Y^{-1/2}.$$

By (4.22), (4.36) the lemma follows.

As in Section 8 of [1], for the estimation of $\sum_{p, (3.6)} T_2(p)$ we have the same upper bound for $T_5(\gamma, \theta)$.

5. First steps on the major arcs. In this section we show that $R(\lambda_1, \dots, \lambda_7)$ is well approximated by integrals similar to (3.17), but with $f_p(P_1, \lambda_1 \alpha)$ and $f_A(W, \lambda_2 p^3 \alpha)$ replaced by suitable approximations. Let

$$D(\alpha) = q^{1/2} (1 + N |\alpha - a/q|)^{1/2}.$$

By Lemma 4,

$$\begin{aligned} f_p(P_1, \lambda_1 \alpha) - s(p, \alpha) &\ll q^{1/2+\varepsilon} \Delta^{1/2} \ll D(\alpha) q^\varepsilon, \\ f_A(W, \lambda_2 \alpha) - g(p, \alpha) &\ll q^{1/2+\varepsilon} \Delta^{1/2} \ll D(\alpha) q^\varepsilon. \end{aligned}$$

Here

$$\Delta = 1 + N|\alpha - a/q|.$$

Now we introduce the numbers

$$\begin{aligned} v_1 &= \int_M \left| \sum_{\substack{p \\ (3.6)}} \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} \sum_{\substack{p_{41}, p_{42}, p_{43} \\ (3.7)}} (f_p(P_1, \lambda_1 \alpha) - s(p, \alpha)) f_A(W, \lambda_2 p^3 \alpha) \right. \\ &\quad \left. \times f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^2 d\alpha, \\ v_2 &= \int_M \left| \sum_{\substack{p \\ (3.6)}} \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} \sum_{\substack{p_{41}, p_{42}, p_{43} \\ (3.7)}} s(p, \alpha) (f_A(W, \lambda_2 \alpha) - g(p, \alpha)) \right. \\ &\quad \left. \times f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^2 d\alpha. \end{aligned}$$

We use Cauchy's inequality and note that $D(\alpha) \ll L^{1/2}$ for $\alpha \in M$. Hence by (4.17),

$$\begin{aligned} (5.1) \quad v_1 &\ll Y L^{1+\varepsilon} \sum_{Y < p \leq 2Y} \int_0^1 \left| \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} \sum_{\substack{p_{41}, p_{42}, p_{43} \\ (3.7)}} f_A(W, \lambda_2 p^3 \alpha) \right. \\ &\quad \left. \times f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^2 d\alpha \\ &\ll Y^2 L^{1+\varepsilon} \int_0^1 \left| \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} \sum_{\substack{p_{41}, p_{42}, p_{43} \\ (3.7)}} f_A(W, \lambda_2 \alpha) \right. \\ &\quad \left. \times f(R_3, \lambda_3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^2 d\alpha \\ &\ll Y^2 L^{1+\varepsilon} T_3 \ll L^{1+\varepsilon} P_2 P_3^{2\theta} P_4^{2+\varepsilon} Y^{-1-2\theta}. \end{aligned}$$

Now we consider v_2 . By Cauchy's inequality and Schwarz's inequality we have

$$\begin{aligned} v_2 &\ll Y P^\varepsilon \sum_{Y < p \leq 2Y} \int_M \left| s(p, \alpha) D(\alpha) \right. \\ &\quad \left. \times \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} \sum_{\substack{p_{41}, p_{42}, p_{43} \\ (3.7)}} f(R_3, \lambda_3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^2 d\alpha \\ &\ll Y P^\varepsilon (I_1 I_2)^{1/2}, \end{aligned}$$

where

$$I_1 = \sum_{Y < p \leq 2Y} \int_M |s(p, \alpha)D(\alpha)|^4 d\alpha,$$

$$I_2 = \sum_{Y < p \leq 2Y} \int_0^1 \left| \sum_{\substack{P_{31}, P_{32}, P_{33} \\ (3.7)}} \sum_{\substack{P_{41}, P_{42}, P_{43} \\ (3.7)}} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) \right. \\ \left. \times f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^4 d\alpha.$$

By Hua's inequality and Cauchy's inequality,

$$I_2 \ll Y(P_3 Y^{-1})^{(5+\varepsilon)/2} (P_4 Y^{-1})^{(5+\varepsilon)/2} = (P_3 P_4)^{(5+\varepsilon)/2} Y^{-4}.$$

As in (4.34), by (3.29), (3.26) and the definition of $D(\alpha)$ we have

$$I_1 \ll Y \lambda_1 N^{-1} P_1^{4+\varepsilon} L^2.$$

Hence

$$(5.2) \quad v_2 \ll Y P_1^\varepsilon (P_3^{(5+\varepsilon)/2} P_4^{(5+\varepsilon)/2} Y^{-4})^{1/2} (Y \lambda_1 N^{-1} P_1^{4+\varepsilon} L^2)^{1/2} \\ \ll Y^{-1/2} P_1^{2+\varepsilon} P_3^{5/4+\varepsilon} P_4^{5/4+\varepsilon} \lambda_1^{1/2} L N^{-1/2}.$$

6. The second step on the major arcs. Let M_1 denote the set of all major arcs $M(q, a)$ subject to $1 \leq a \leq q \leq X, (a, q) = 1$, where

$$(6.1) \quad X = P^{4/7}.$$

Let

$$(6.2) \quad v^* = \int_{M \setminus M_1} \left| \sum_{\substack{P \\ (3.6)}} \sum_{\substack{P_{31}, P_{32}, P_{33} \\ (3.7)}} \sum_{\substack{P_{41}, P_{42}, P_{43} \\ (3.7)}} s(p, \alpha) g(p, \alpha) \right. \\ \left. \times f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^2 d\alpha,$$

where $s(p, \alpha)$ and $g(p, \alpha)$ are given by (3.29) and (3.30).

As in [3], by Lemmas 4.3 and 4.4 of Vaughan [5] when $p \nmid a$ one has

$$(6.3) \quad S(p^l, a) = \begin{cases} p^{[2l/3]}, & l \not\equiv 1 \pmod{3}, \\ 0, & l \equiv 1 \pmod{3}, p \equiv 2 \pmod{3}, \\ p^{2(l-1)/3} S(p, a), & l \equiv 1 \pmod{3}, p \equiv 1 \pmod{3}. \end{cases}$$

Also, standard methods show that

$$(6.4) \quad S(q, ab^3) = S(q, a) \quad \text{when } (b, q) = 1;$$

$$(6.5) \quad S(qr, a) = S(q, ar^2) S(r, aq^2) \quad \text{when } (r, q) = (qr, a) = 1.$$

When $p \nmid q$, by (3.9) of [1],

$$s_A(q, \lambda_2 p^3 a) \ll q^\varepsilon \kappa\left(\frac{q}{(\lambda_2 a p^3, q)}\right) \ll q^\varepsilon \kappa\left(\frac{q}{(\lambda_2, q)}\right),$$

$$s_p(q, \lambda_1 a) = q^{-1} \{S(q, \lambda_1 a) - p^{-1} S(q, \lambda_1 p^3 a)\} \ll \kappa\left(\frac{q}{(\lambda_1, q)}\right).$$

When $p \parallel q$, write $q = pr$ with $p \nmid r$. By (3.9) of [1],

$$s_A(q, \lambda_2 p^3 a) \ll q^\varepsilon \kappa\left(\frac{q}{(\lambda_2 a p^3, q)}\right) \ll q^\varepsilon \kappa\left(\frac{r}{(\lambda_2, r)}\right),$$

$$S(q, \lambda_1 a) = (q, \lambda_1) S\left(\frac{q}{(q, \lambda_1)}, \frac{\lambda_1}{(q, \lambda_1)} a\right) = (q, \lambda_1) S\left(\frac{pr}{(r, \lambda_1)}, \frac{\lambda_1}{(r, \lambda_1)} a\right).$$

By (6.3) and (6.5) this vanishes. Similarly,

$$S(q, \lambda_1 p^3 a) \ll pr \kappa\left(\frac{r}{(r, \lambda_1)}\right),$$

so

$$s_p(q, \lambda_1 a) \ll p^{-1} \kappa\left(\frac{r}{(r, \lambda_1)}\right).$$

When $p^2 \mid q$, let $q = p^l r$, $p \nmid r$ and $l \geq 2$. Then

$$S(q, \lambda_1 a) = (r, \lambda_1) S\left(p^l \frac{r}{(r, \lambda_1)}, \frac{\lambda_1}{(r, \lambda_1)} a\right)$$

$$= (r, \lambda_1) S\left(p^l, \frac{\lambda_1}{(r, \lambda_1)} a \frac{r^2}{(r, \lambda_1)^2}\right) S\left(\frac{r}{(r, \lambda_1)}, \frac{\lambda_1 a p^{2l}}{(r, \lambda_1)}\right).$$

When $l = 2$,

$$p^{-1} S(q, \lambda_1 a p^3) = p(\lambda_1, r) S\left(\frac{r}{(r, \lambda_1)}, \frac{\lambda_1 a p}{(r, \lambda_1)}\right).$$

By (6.3),

$$S\left(p^2, \frac{\lambda_1}{(r, \lambda_1)} a \frac{r^2}{(r, \lambda_1)^2}\right) = p,$$

$$S\left(\frac{r}{(r, \lambda_1)}, \frac{\lambda_1 a p^{2l}}{(r, \lambda_1)}\right) = S\left(\frac{r}{(r, \lambda_1)}, \frac{\lambda_1 a p}{(r, \lambda_1)}\right),$$

so $s_p(q, \lambda_1 a) = 0$.

When $l \geq 3$, by (6.3)–(6.5) we have

$$p^{-1} S(p^l r, \lambda_1 a p^3)$$

$$= p^2(\lambda_1, r) S\left(p^{l-3} \frac{r}{(r, \lambda_1)}, \frac{\lambda_1 a}{(r, \lambda_1)}\right)$$

$$= p^2(\lambda_1, r) S\left(\frac{r}{(r, \lambda_1)}, \frac{\lambda_1 a p^{2l-6}}{(r, \lambda_1)}\right) S\left(p^{l-3}, \frac{\lambda_1 a}{(r, \lambda_1)} \cdot \frac{r^2}{(r, \lambda_1)^2}\right),$$

hence $s_p(q, \lambda_1 a) = 0$. Combining this result with (3.29) and (3.30) we have

$$s(p, \alpha)g(p, \alpha) \ll \begin{cases} 0 & \text{if } p^2 \mid q, \\ q^\varepsilon p^{-1} \kappa\left(\frac{r}{(r, \lambda_1)}\right) \kappa\left(\frac{r}{(r, \lambda_2)}\right) P_1 W \left(1 + N \left|\alpha - \frac{a}{q}\right|\right)^{-2} & \text{if } q = pr, p \nmid r, \\ q^\varepsilon \kappa\left(\frac{q}{(q, \lambda_1)}\right) \kappa\left(\frac{q}{(q, \lambda_2)}\right) P_1 W \left(1 + N \left|\alpha - \frac{a}{q}\right|\right)^{-2} & \text{if } p \nmid q. \end{cases}$$

Note that

$$4p^{-1} \kappa(r)^2 = \kappa(pr)^2 \quad \text{for } p \nmid r.$$

For $p^2 \nmid q$ we have

$$(6.6) \quad s(p, \alpha)g(p, \alpha) \ll q^\varepsilon \kappa\left(\frac{q}{(q, \lambda_1)}\right) \kappa\left(\frac{q}{(q, \lambda_2)}\right) P_1 W \left(1 + N \left|\alpha - \frac{a}{q}\right|\right)^{-2}.$$

Hence

$$(6.7) \quad \sum_{\substack{Y < p \leq 2Y \\ (3.6)}} \int_{M \setminus M_1} \left| \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} \sum_{\substack{p_{41}, p_{42}, p_{43} \\ (3.7)}} s(p, \alpha)g(p, \alpha) \right. \\ \left. \times f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^2 d\alpha \\ \ll P_1^2 W^2 L^\varepsilon \sum_{\substack{Y < p \leq 2Y \\ (3.6)}} \sum_{\substack{X < q \leq L \\ p^2 \nmid q}} \kappa^2\left(\frac{q}{(q, \lambda_1)}\right) \kappa^2\left(\frac{q}{(q, \lambda_2)}\right) \\ \times \sum_{\substack{a=1 \\ (a, q)=1}}^q \int_{M(q, a)} \left(1 + N \left|\alpha - \frac{a}{q}\right|\right)^{-4} \\ \times \left| \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} \sum_{\substack{p_{41}, p_{42}, p_{43} \\ (3.7)}} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^2 d\alpha \\ \ll P_1^2 W^2 L^\varepsilon \sum_{\substack{Y < p \leq 2Y \\ (3.6)}} \left(\sum_{\substack{X < q \leq L \\ p^2 \nmid q}} \kappa^4\left(\frac{q}{(q, \lambda_1)}\right) \sum_{\substack{a=1 \\ (a, q)=1}}^q \int_{M(q, a)} \left(1 + N \left|\alpha - \frac{a}{q}\right|\right)^{-4} \right. \\ \left. \times \left| \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) \right|^4 d\alpha \right)^{1/2}$$

$$\begin{aligned} & \times \left(\sum_{\substack{X < q \leq L \\ p^2 \nmid q}} \kappa^4 \left(\frac{q}{(q, \lambda_2)} \right) \sum_{\substack{a=1 \\ (a,q)=1}}^q \int_{M(q,a)} \left(1 + N \left| \alpha - \frac{a}{q} \right| \right)^{-4} \right. \\ & \left. \times \left| \sum_{\substack{p_{41}, p_{42}, p_{43} \\ (3.7)}} f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^4 d\alpha \right)^{1/2} \end{aligned}$$

Now we consider

$$\begin{aligned} & \sum_{\substack{X < q \leq L \\ p^2 \nmid q}} \kappa^4 \left(\frac{q}{(q, \lambda_1)} \right) \sum_{\substack{a=1 \\ (a,q)=1}}^q \int_{M(q,a)} \left(1 + N \left| \alpha - \frac{a}{q} \right| \right)^{-4} \\ & \times \left| \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) \right|^4 d\alpha \\ & \ll \sum_{\substack{X < q \leq L \\ p^2 \nmid q}} \kappa^4 \left(\frac{q}{(q, \lambda_1)} \right) \\ & \times \int_{-1/2}^{1/2} (1 + N|\beta|)^{-4} \sum_{a=1}^q \left| \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} f \left(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \left(\frac{a}{q} + \beta \right) \right) \right|^4 d\beta. \end{aligned}$$

Now,

$$\begin{aligned} & \sum_{a=1}^q \left| \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} f \left(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \left(\frac{a}{q} + \beta \right) \right) \right|^4 \\ & = \sum_{x_1, \dots, x_4 \in \chi_4(P_3 Y^{-1})} \sum_{a=1}^q e \left(\lambda_3 p^3 \left(\frac{a}{q} + \beta \right) (x_1^3 + x_2^3 - x_3^3 - x_4^3) \right) \\ & = q \sum_{\substack{x_i \in \chi_4(P_3 Y^{-1}) \\ \lambda_3 p^3 (x_1^3 + x_2^3 - x_3^3 - x_4^3) \equiv 0 \pmod{q}}} e(\lambda_3 p^3 \beta (x_1^3 + x_2^3 - x_3^3 - x_4^3)) \ll q \Psi(q, p), \end{aligned}$$

where $\Psi(q, p)$ denotes the number of solutions of the congruence

$$\lambda_3 p^3 (x_1^3 + x_2^3 - x_3^3 - x_4^3) \equiv 0 \pmod{q}$$

subject to $x_i \in \chi_4(P_3 Y^{-1})$. Now we have

$$\sum_{\substack{X < q \leq L \\ p^2 \nmid q}} \kappa^4 \left(\frac{q}{(q, \lambda_1)} \right) \sum_{\substack{a=1 \\ (a,q)=1}}^q \int_{M(q,a)} \left(1 + N \left| \alpha - \frac{a}{q} \right| \right)^{-4}$$

$$\begin{aligned} & \times \left| \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) \right|^4 d\alpha \\ & \ll N^{-1} \sum_{\substack{X < q \leq L \\ p^2 \nmid q}} q \kappa^4 \left(\frac{q}{(q, \lambda_1)} \right) \Psi(q, p) = N^{-1} \left(\sum_0 + \sum_1 \right), \end{aligned}$$

where \sum_j is the part of the remaining sum with $p^j \parallel q$.

When $p \nmid q$ we have $\Psi(q, p) = \Psi(q, 1) = \Psi(q)$, say. Thus by a simple splitting up argument and Cauchy's inequality,

$$\begin{aligned} (6.8) \quad \sum_0 & \ll (\log L) \sum_{\Xi < q \leq 2\Xi} q \kappa^4 \left(\frac{q}{(q, \lambda_1)} \right) \Psi(q) \\ & \ll (\log L) \left(\sum_{\Xi < q \leq 2\Xi} q \kappa^8 \left(\frac{q}{(q, \lambda_1)} \right) \right)^{1/2} \left(\sum_{q \leq 2\Xi} q \Psi^2(q) \right)^{1/2} \end{aligned}$$

for some Ξ in the range $X < \Xi \leq L$.

Let $\varrho(h)$ denote the number of solutions to

$$x_1^3 + x_2^3 - x_3^3 - x_4^3 = h$$

subject to $x_i \in \chi_4(P_3 Y^{-1})$. Then

$$\Psi(q) = \sum_{|l| \leq (P_3 Y^{-1})^3 (q/(q, \lambda_3))^{-1}} \varrho \left(\frac{lq}{(q, \lambda_3)} \right),$$

and by Cauchy's inequality,

$$\Psi(q)^2 \ll \left(2(P_3 Y^{-1})^3 \left(\frac{q}{(q, \lambda_3)} \right)^{-1} + 1 \right) \sum_{|l| \leq (P_3 Y^{-1})^3 (q/(q, \lambda_3))^{-1}} \varrho^2 \left(\frac{lq}{(q, \lambda_3)} \right).$$

Of course, $\varrho(0) \ll (P_3 Y^{-1})^{2+\varepsilon}$ so that

$$\begin{aligned} (6.9) \quad & \sum_{q \leq 2\Xi} q \Psi(q)^2 \\ & \ll \Xi (P_3 Y^{-1})^{7+\varepsilon} + (P_3 Y^{-1})^3 \sum_{0 < lq/(q, \lambda_3) \leq (P_3 Y^{-1})^3} (q, \lambda_3) \varrho^2 \left(\frac{lq}{(q, \lambda_3)} \right) \\ & \ll \Xi (P_3 Y^{-1})^{7+\varepsilon} + (P_3 Y^{-1})^{3+\varepsilon} \lambda_3 \sum_{h \leq (P_3 Y^{-1})^3} \varrho^2(h) \end{aligned}$$

and the final sum is bounded by the number of solutions of

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = x_5^3 + x_6^3 + x_7^3 + x_8^3$$

subject to $x_i \in \chi_4(P_3 Y^{-1})$. By Hua's inequality (Lemma 2.5 of [5]) this

number is bounded by $O((P_3 Y^{-1})^{5+\varepsilon})$. We now have

$$\sum_{q \leq 2\varepsilon} q \Psi(q)^2 \ll \varepsilon (P_3 Y^{-1})^{7+\varepsilon} + (P_3 Y^{-1})^{8+\varepsilon} \lambda_3.$$

Note that $\kappa(q) \ll q^{-1/3+\varepsilon}$. Hence from (4.33),

$$(6.10) \quad \sum_{\varepsilon < q \leq 2\varepsilon} q \kappa^8 \left(\frac{q}{(q, \lambda_1)} \right) \ll \lambda_1^{7/3+\varepsilon} \varepsilon^{-4/3+\varepsilon}.$$

From (6.8) we have

$$\begin{aligned} \sum_0 &\ll (\log L) (\lambda_1^{7/3+\varepsilon} \varepsilon^{-4/3+\varepsilon})^{1/2} (\varepsilon (P_3 Y^{-1})^{7+\varepsilon} + (P_3 Y^{-1})^{8+\varepsilon} \lambda_3)^{1/2} \\ &\ll (\log L) \lambda_1^{7/6+\varepsilon} X^{-2/3+\varepsilon} (P_3 Y^{-1})^{4+\varepsilon} \lambda_3^{1/2}. \end{aligned}$$

An estimate for \sum_1 is obtained along similar lines. We write $q = pr$ with $p \nmid r$. Then $\Psi(pr, p) = \Psi(r, 1) = \Psi(r)$. Therefore since κ is multiplicative,

$$\sum_1 \ll p^{-1} \sum_{Xq^{-1} < r \leq Lq^{-1}} r \kappa^4 \left(\frac{r}{(r, \lambda_1)} \right) \Psi(r).$$

By the same argument as before, the sum is

$$\ll (\log L) \lambda_1^{7/6+\varepsilon} (P_3 Y^{-1})^{4+\varepsilon} \lambda_3^{1/2} X^{-2/3+\varepsilon} p^{-1/3}.$$

So the bound for \sum_1 is even stronger than the bound for \sum_0 . Hence we have

$$\begin{aligned} \sum_{\substack{X < q \leq L \\ p^2 \nmid q}} \kappa^4 \left(\frac{q}{(q, \lambda_1)} \right) \sum_{\substack{a=1 \\ (a,q)=1}}^q \int_{M(q,a)} \left(1 + N \left| \alpha - \frac{a}{q} \right| \right)^{-4} \\ \times \left| \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) \right|^4 d\alpha \\ \ll N^{-1+\varepsilon} \lambda_1^{7/6+\varepsilon} X^{-2/3} (P_3 Y^{-1})^{4+\varepsilon} \lambda_3^{1/2}. \end{aligned}$$

Thus

$$\begin{aligned} (6.11) \quad \sum_{Y < p \leq 2Y} \int_{M \setminus M_1} \left| \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} \sum_{\substack{p_{41}, p_{42}, p_{43} \\ (3.7)}} s(p, \alpha) g(p, \alpha) \right. \\ \left. \times f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \right|^2 d\alpha \\ \ll P_1^2 W^2 Y L^\varepsilon (N^{-1+\varepsilon} \lambda_1^{7/6+\varepsilon} X^{-2/3} (P_3 Y^{-1})^{4+\varepsilon} \lambda_3^{1/2})^{1/2} \\ \times (N^{-1+\varepsilon} \lambda_2^{7/6+\varepsilon} X^{-2/3} (P_4 Y^{-1})^{4+\varepsilon} \lambda_4^{1/2})^{1/2} \\ \ll P_1^2 W^2 N^{-1+\varepsilon} \lambda_1^{7/12+\varepsilon} \lambda_2^{7/12+\varepsilon} Y X^{-2/3} (P_3 P_4 Y^{-2})^{2+\varepsilon} \lambda_3^{1/4} \lambda_4^{1/4}. \end{aligned}$$

By (6.2) and Cauchy's inequality we have

$$(6.12) \quad v^* \ll P_1^2 W^2 N^{-1+\varepsilon} \lambda_1^{7/12+\varepsilon} \lambda_2^{7/12+\varepsilon} Y^{-2} X^{-2/3} (P_3 P_4)^{2+\varepsilon} \lambda_3^{1/4} \lambda_4^{1/4}.$$

7. The major arcs. For $\alpha \in M(q, a)$, let

$$h_i(\alpha) = s(q, \lambda_i p^3 p_{i1}^3 p_{i2}^3 p_{i3}^3 a) J\left(\lambda_i \left(\alpha - \frac{a}{q}\right) p^3 p_{i1}^3 p_{i2}^3 p_{i3}^3, R_i\right) \quad (i = 3, 4),$$

$$k(\alpha) = s_B(q, \lambda_5 a) J\left(\lambda_5 \left(\alpha - \frac{a}{q}\right), U\right),$$

$$l_i(\alpha) = s(q, \lambda_i p_{i1}^3 p_{i2}^3 p_{i3}^3 a) J\left(\lambda_i \left(\alpha - \frac{a}{q}\right) p_{i1}^3 p_{i2}^3 p_{i3}^3, V_i\right) \quad (i = 6, 7).$$

Then as in Lemma 3 of [1], for $\alpha \in M(q, a)$, where $1 \leq a \leq q \leq X$, $(a, q) = 1$, we have

$$\begin{aligned} f(R_i, \lambda_i p^3 p_{i1}^3 p_{i2}^3 p_{i3}^3 \alpha) - h_i(\alpha) &\ll q^{1/2+\varepsilon} \Delta^{1/2} \quad (i = 3, 4), \\ f_B(U, \lambda_5 \alpha) - k(\alpha) &\ll q^{1/2+\varepsilon} \Delta^{1/2}, \\ f(V_i, \lambda_i p_{i1}^3 p_{i2}^3 p_{i3}^3 \alpha) - l_i(\alpha) &\ll q^{1/2+\varepsilon} \Delta^{1/2} \quad (i = 6, 7). \end{aligned}$$

Let

$$\begin{aligned} F^*(\mathbf{p}, \alpha) &= s(p, \alpha) \overline{g(p, \alpha)} h_3(\alpha) h_4(\alpha) k(\alpha) l_6(\alpha) l_7(\alpha), \\ F^*(\alpha) &= \sum_{\substack{\mathbf{p} \\ (3.6), (3.7)}} F^*(\mathbf{p}, \alpha), \end{aligned}$$

$$\begin{aligned} A_1 &= q^{-1/3} P_1 \Delta^{-1}, \quad A_2 = (q, p^3)^{1/3} q^{-1/3} W \Delta^{-1}, \quad A_5 = q^{-1/3} U \Delta^{-1}, \\ A_i &= q^{-1/3} (q, p^3)^{1/3} (q, p_{i1}^3)^{1/3} (q, p_{i2}^3)^{1/3} (q, p_{i3}^3)^{1/3} R_i \Delta^{-1} \quad (i = 3, 4), \\ A_i &= q^{-1/3} (q, p_{i1}^3)^{1/3} (q, p_{i2}^3)^{1/3} (q, p_{i3}^3)^{1/3} V_i \Delta^{-1} \quad (i = 6, 7). \end{aligned}$$

Then as in Lemma 5 of [1] we have

$$h_i(\alpha) \ll \lambda_i^{1/3} q^\varepsilon A_i \quad (i = 3, 4), \quad k(\alpha) \ll \lambda_5^{1/3} q^\varepsilon A_5, \quad l_i(\alpha) \ll \lambda_i^{1/3} q^\varepsilon A_i \quad (i = 6, 7)$$

and

$$\begin{aligned} &s(p, \alpha) \overline{g(p, \alpha)} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \\ &\quad \times f_B(U, \lambda_5 \alpha) f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) f(V_7, \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 \alpha) - F^*(\mathbf{p}, \alpha) \\ &\ll \lambda_1^{1/3} \lambda_2^{1/3} q^{2\varepsilon} A_1 A_2 \sum_{i=3}^7 q^{1/2+\varepsilon} \Delta^{1/2} \prod_{\substack{j=3 \\ j \neq i}}^7 (\lambda_j^{1/3} q^\varepsilon A_j + q^{1/2+\varepsilon} \Delta^{1/2}). \end{aligned}$$

By Lemma 4 of [1] we see that

$$\begin{aligned}
 & \sum_{\substack{p_{ij} \\ (3.7)}} |s(p, \alpha) \overline{g(p, \alpha)}| f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \\
 & \quad \times f_B(U, \lambda_5 \alpha) f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) f(V_7, \lambda_3 p_{71}^3 p_{72}^3 p_{73}^3 \alpha) - F^*(\mathbf{p}, \alpha) | \\
 & \ll \left\{ \prod_{(3.7)} Z_{ij} \right\} \lambda_1^{1/3} \lambda_2^{1/3} q^{-2/3+2\varepsilon} P_1 W \Delta^{-2}(q, p^3) \\
 & \quad \times [(q^{1/2+\varepsilon} \Delta^{1/2})^5 + (q^{1/2+\varepsilon} \Delta^{1/2})^4 q^{-1/3} U \Delta^{-1} \\
 & \quad + (q^{1/2+\varepsilon} \Delta^{1/2})^3 q^{-2/3} UV_7 \Delta^{-2} + (q^{1/2+\varepsilon} \Delta^{1/2})^2 q^{-1} UV_6 V_7 \Delta^{-3} \\
 & \quad + (q^{1/2+\varepsilon} \Delta^{1/2}) q^{-4/3} UV_6 V_7 R_4 \Delta^{-4}]
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_{M_1} \sum_{\substack{\mathbf{p} \\ (3.6), (3.7)}} |s(p, \alpha) \overline{g(p, \alpha)}| f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \\
 & \quad \times f_B(U, \lambda_5 \alpha) f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) f(V_7, \lambda_3 p_{71}^3 p_{72}^3 p_{73}^3 \alpha) - F^*(\mathbf{p}, \alpha) | d\alpha \\
 & = \sum_{\substack{\mathbf{p} \\ (3.6), (3.7)}} \sum_{q \leq X} \sum_{\substack{a \leq q \\ (a, q) = 1}} \int_{M(q, a)} |s(p, \alpha) \overline{g(p, \alpha)}| f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) \\
 & \quad \times f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) f_B(U, \lambda_5 \alpha) \\
 & \quad \times f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) f(V_7, \lambda_3 p_{71}^3 p_{72}^3 p_{73}^3 \alpha) - F^*(\mathbf{p}, \alpha) | d\alpha \\
 & \ll \left\{ \prod_{(3.7)} Z_{ij} \right\} \lambda_1^{1/3} \lambda_2^{1/3} P_1 W N^{-1} \sum_{\substack{p \\ (3.6)}} \sum_{q \leq X} (q, p^3) [q^{4/3+\varepsilon} L^{3/2} + U \log N q^{2+\varepsilon} \\
 & \quad + q^{7/6+\varepsilon} UV_7 + q^{1/3+\varepsilon} UV_6 V_7 + q^{-1/2+\varepsilon} UV_6 V_7 R_4] \\
 & \ll \left\{ \prod_{(3.7)} Z_{ij} \right\} \lambda_1^{1/3} \lambda_2^{1/3} P_1 W N^{-1} [X^{7/3+\varepsilon} L^{3/2} + U \log N X^{3+\varepsilon} + UV_7 X^{13/6+\varepsilon} \\
 & \quad + UV_6 V_7 X^{4/3+\varepsilon} + UV_6 V_7 R_4 X^{1/2+\varepsilon}].
 \end{aligned}$$

By (3.1), (3.2), (3.8)–(3.11) and (6.1), the above is

$$\ll Y \left\{ \prod_{(3.7)} Z_{ij} \right\} P_1 W R_3 R_4 UV_6 V_7 N^{-1-\varepsilon} \ll \lambda_1^{2/21} \Lambda^{-1/3} N^{26/21-\varepsilon}.$$

Now, we write

$$S(q, a, h) = \sum_{\substack{x=1 \\ (x,q,h)=1}}^q e(ax^3/q),$$

where h, q are natural numbers and a is an integer.

Let

$$\chi(q, p) = \begin{cases} 0 & \text{if } p \mid q, \\ 1 & \text{otherwise.} \end{cases}$$

Then as in Lemma 6 of [1], by induction we have

$$\begin{aligned} s_p(q, c) &= \left(1 - \frac{1}{p}\right)^{\chi(q,p)} q^{-1} S(q, c, p), \\ s_A(q, c) &= \prod_{i=3}^4 \prod_{j=1}^3 \left(1 - \frac{1}{p_{ij}}\right)^{\chi(q,p_{ij})} q^{-1} S(q, c, A), \\ s_B(q, c) &= \prod_{i=6}^7 \prod_{j=1}^3 \left(1 - \frac{1}{p_{ij}}\right)^{\chi(q,p_{ij})} q^{-1} S(q, c, B). \end{aligned}$$

Now we consider $\int_{M_1} F^*(\alpha) d\alpha$. We write

$$\begin{aligned} S(\mathbf{p}, q) &= \frac{1}{q^7} \sum_{\substack{a=1 \\ (a,q)=1}}^q S(q, \lambda_1 a, p) S(q, \lambda_2 a p^3, A) S(q, \lambda_3 a p^3 p_{31}^3 p_{32}^3 p_{33}^3) \\ &\quad \times S(q, \lambda_4 a p^3 p_{41}^3 p_{42}^3 p_{43}^3) S(q, \lambda_5 a, B) \\ &\quad \times S(q, \lambda_6 a p_{61}^3 p_{62}^3 p_{63}^3) S(q, \lambda_7 a p_{71}^3 p_{72}^3 p_{73}^3) \end{aligned}$$

and for $q \leq X, |\beta| \leq q^{-1} L N^{-1}$,

$$\sum_{\substack{a=1 \\ (a,q)=1}}^q F^*(\mathbf{p}, \beta + a/q) = \theta(q, \mathbf{p}) S(\mathbf{p}, q) \varphi(\mathbf{p}, \beta).$$

Here

$$\theta(q, \mathbf{p}) = \prod_{\substack{\omega \in \{p, p_{ij}\} \\ \omega \nmid q}} \left(1 - \frac{1}{\omega}\right)$$

and

$$\begin{aligned} \varphi(\mathbf{p}, \beta) &= J(\lambda_1 \beta, P_1) J(-\lambda_2 p^3 \beta, W) J(\lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \beta, R_3) \\ &\quad \times J(\lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \beta, R_4) J(\lambda_5 \beta, U) \\ &\quad \times J(\lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \beta, V_6) J(\lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 \beta, V_7). \end{aligned}$$

As in the proof of Lemma 10 of [1],

$$\int_{M_1} F^*(\alpha) d\alpha = \sum_{\substack{\mathbf{p} \\ (3.6),(3.7)}} \sum_{q \leq X} \theta(q, \mathbf{p}) S(\mathbf{p}, q) \int_{-LN^{-1}}^{LN^{-1}} \varphi(\mathbf{p}, \beta) d\beta.$$

In view of Lemmas 8 and 9 of [1], we may insert the summation condition

$$(q, pAB) = 1$$

in the last summation over q . For these values of q we have

$$\theta(q, \mathbf{p}) = \prod_{\omega \in \{p, p_{ij}\}} \left(1 - \frac{1}{\omega}\right).$$

As in Lemma 11 of [1], by (3.2) we have

$$\sum_{\substack{q > X \\ (q, pAB)=1}} |S(\mathbf{p}, q)| \ll \Lambda^{1/3} X^{\varepsilon-1/3} \ll \Lambda^{-2\varepsilon}$$

and

$$S_X(\mathbf{p}) = \sum_{\substack{q \leq X \\ (q, pAB)=1}} S(\mathbf{p}, q) \gg \Lambda^{-\varepsilon}.$$

As in the proof of (2.7) of [1] we have

$$\int_{M_1} F^*(p, \alpha) d\alpha \gg \lambda_1^{2/21} \Lambda^{-1/3-\varepsilon} N^{26/21} (\log N)^{-13}.$$

8. The proof of Theorem

LEMMA 10. *Let $\lambda_1 \geq \dots \geq \lambda_s \geq 1$ be real numbers, let $\kappa_1, \dots, \kappa_s$ be non-negative real numbers, and suppose that the real number c satisfies the inequalities*

$$\kappa_1 + \dots + \kappa_t \leq tc \quad \text{for } 1 \leq t \leq s.$$

Then

$$\lambda_1^{\kappa_1} \dots \lambda_s^{\kappa_s} \leq (\lambda_1 \dots \lambda_s)^c.$$

This is Lemma 5 of [4].

Now, by Lemmas 6 and 9, (4.16) and (4.19), $J_2 \ll YT$. By (3.1), (3.2) and Lemma 10, it is easy to check that

$$J_2 \ll P_1^{1+\delta} P_2 P_3^{2\theta} P_4^{2+\varepsilon} Y^{-1-2\theta} + \lambda_1^{1/2} P_1^{1+\varepsilon} P_2^{1/2+\varepsilon} P_3^{5/4+\varepsilon} P_4^{5/4+\varepsilon} Y^{1/2}.$$

By Cauchy's inequality and Lemma 5,

$$\begin{aligned} \int_M F(\alpha) d\alpha &\ll J_2^{1/2} J_1^{1/2} \\ &\ll (P_1^{1+\delta} P_2 P_3^{2\theta} P_4^{2+\varepsilon} Y^{-1-2\theta} \\ &\quad + \lambda_1^{1/2} P_1^{1+\varepsilon} P_2^{1/2+\varepsilon} P_3^{5/4+\varepsilon} P_4^{5/4+\varepsilon} Y^{1/2})^{1/2} (P_5 P_6^{2\theta} P_7^{2+\varepsilon})^{1/2} \\ &\ll \lambda_1^{2/21} \Lambda^{-1/3} N^{26/21+\varepsilon} \\ &\quad \times (\lambda_1^{(2+\theta)/21} \lambda_2^{1/6} \lambda_3^{(1-\theta)/3} \lambda_5^{1/6} \lambda_6^{(1-\theta)/3} N^{-(2-13\theta)/21} \\ &\quad + \lambda_1^{13/42} \lambda_2^{1/4} \lambda_3^{1/8} \lambda_4^{1/8} \lambda_5^{1/6} \lambda_6^{(1-\theta)/3} N^{-(5-28\theta)/84}). \end{aligned}$$

By Lemma 10,

$$\begin{aligned} \lambda_1^{(2+\theta)/21} \lambda_2^{1/6} \lambda_3^{(1-\theta)/3} \lambda_5^{1/6} \lambda_6^{(1-\theta)/3} &\ll \Lambda^{(2-\theta)/9}, \\ \lambda_1^{13/42} \lambda_2^{1/4} \lambda_3^{1/8} \lambda_4^{1/8} \lambda_5^{1/6} \lambda_6^{(1-\theta)/3} &\ll \Lambda^{5/24} \end{aligned}$$

and by (3.2) we have

$$\begin{aligned} \lambda_1^{(2+\theta)/21} \lambda_2^{1/6} \lambda_3^{(1-\theta)/3} \lambda_5^{1/6} \lambda_6^{(1-\theta)/3} N^{-(2-13\theta)/21} \\ + \lambda_1^{13/42} \lambda_2^{1/4} \lambda_3^{1/8} \lambda_4^{1/8} \lambda_5^{1/6} \lambda_6^{(1-\theta)/3} N^{-(5-28\theta)/84} \ll N^{-3\varepsilon} \end{aligned}$$

so (3.21) follows.

By Lemma 5 and Cauchy's inequality,

$$\begin{aligned} \int_M \sum_{\mathbf{p}}_{(3.6),(3.7)} (f_p(P_1, \lambda_1 \alpha) - s(p, \alpha)) \overline{f_A(W, \lambda_2 p^3 \alpha)} \\ \times f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \\ \times f_B(U, \lambda_5 \alpha) f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) f(V_7, \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 \alpha) d\alpha \ll v_1^{1/2} J_1^{1/2}. \end{aligned}$$

As above, by Lemma 10 and (3.2), (5.1) the above is

$$\ll (P_1^{1+\varepsilon} P_2 P_3^{2\theta} P_4^{2+\varepsilon} Y^{-1-2\theta})^{1/2} (P_5 P_6^{2\theta} P_7^{2+\varepsilon})^{1/2} \ll \lambda_1^{2/21} \Lambda^{-1/3} N^{26/21-\varepsilon}$$

hence

$$\begin{aligned} \int_M F(\alpha) d\alpha &= \int_M \sum_{\mathbf{p}}_{(3.6),(3.7)} s(p, \alpha) \overline{f_A(W, \lambda_2 p^3 \alpha)} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) \\ &\quad \times f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \\ &\quad \times f_B(U, \lambda_5 \alpha) f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) f(V_7, \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 \alpha) d\alpha \\ &\quad + O(\lambda_1^{2/21} \Lambda^{-1/3} N^{26/21-\varepsilon}). \end{aligned}$$

By Lemma 5 and Cauchy's inequality,

$$\begin{aligned} \int_M \sum_{\substack{\mathbf{P} \\ (3.6),(3.7)}} s(p, \alpha) (\overline{f_A(W, \lambda_2 p^3 \alpha)} - \overline{g(p, \alpha)}) \\ \times f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \\ \times f_B(U, \lambda_5 \alpha) f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) f(V_7, \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 \alpha) d\alpha \\ \ll v_2^{1/2} J_1^{1/2}. \end{aligned}$$

By Lemmas 5 and 10 and (5.2), the above is

$$\begin{aligned} &\ll (\lambda_1^{1/2} P_1^{2+\varepsilon} P_3^{5/4+\varepsilon} P_4^{5/4+\varepsilon} Y^{-1/2} L N^{-1/2})^{1/2} (P_5 P_6^{2\theta} P_7^{2+\varepsilon})^{1/2} \\ &\ll \lambda_1^{2/21} \Lambda^{-1/3} N^{26/21+\varepsilon} (\lambda_2^{1/3} \lambda_3^{1/8} \lambda_4^{1/8} \lambda_5^{1/6} \lambda_6^{(1-\theta)/3} N^{-(7-28\theta)/84}) \\ &\ll \lambda_1^{2/21} \Lambda^{-1/3} N^{26/21+\varepsilon} (\Lambda^{1/4} N^{-(7-28\theta)/84}) \\ &\ll \lambda_1^{2/21} \Lambda^{-1/3} N^{26/21-\varepsilon}. \end{aligned}$$

Hence

$$\begin{aligned} \int_M F(\alpha) d\alpha &= \int_M \sum_{\substack{\mathbf{P} \\ (3.6),(3.7)}} s(p, \alpha) \overline{g(p, \alpha)} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) \\ &\quad \times f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) f_B(U, \lambda_5 \alpha) f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) \\ &\quad \times f(V_7, \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 \alpha) d\alpha + O(\lambda_1^{2/21} \Lambda^{-1/3} N^{26/21-\varepsilon}). \end{aligned}$$

By Lemma 5 and Cauchy's inequality,

$$\begin{aligned} \int_{M \setminus M_1} \sum_{\substack{\mathbf{P} \\ (3.6),(3.7)}} s(p, \alpha) \overline{g(p, \alpha)} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \\ \times f_B(U, \lambda_5 \alpha) f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) f(V_7, \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 \alpha) d\alpha \\ \ll (v^*)^{1/2} J_1^{1/2}. \end{aligned}$$

By (6.12) and Lemma 10 the above is

$$\begin{aligned} &\ll (P_1^2 W^2 N^{-1+\varepsilon} \lambda_1^{7/12+\varepsilon} \lambda_2^{7/12+\varepsilon} Y^{-2} X^{-2/3} (P_3 P_4)^{2+\varepsilon} \lambda_3^{1/4} \lambda_4^{1/4})^{1/2} \\ &\quad \times (P_5 P_6^{2\theta} P_7^{2+\varepsilon})^{1/2} \\ &\ll \lambda_1^{2/21} \Lambda^{-1/3} N^{26/21+\varepsilon} (\lambda_1^{179/504} \lambda_2^{7/24} \lambda_3^{1/8} \lambda_4^{1/8} \lambda_5^{1/6} \lambda_6^{(1-\theta)/3} N^{-(4-21\theta)/63}) \\ &\ll \lambda_1^{2/21} \Lambda^{-1/3} N^{26/21+\varepsilon} (\Lambda^{11/48} N^{-(4-21\theta)/63}) \\ &\ll \lambda_1^{2/21} \Lambda^{-1/3} N^{26/21-\varepsilon}. \end{aligned}$$

So

$$\begin{aligned} \int_M F(\alpha) d\alpha &= \int_{M_1} \sum_{\substack{\mathbf{p} \\ (3.6), (3.7)}} s(p, \alpha) \overline{g(p, \alpha)} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) \\ &\quad \times f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) f_B(U, \lambda_5 \alpha) f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) \\ &\quad \times f(V_7, \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 \alpha) d\alpha + O(\lambda_1^{2/21} \Lambda^{-1/3} N^{26/21-\varepsilon}) \\ &= \int_{M_1} F^*(\alpha) d\alpha + O(\lambda_1^{2/21} \Lambda^{-1/3} N^{26/21-\varepsilon}). \end{aligned}$$

Hence the Theorem follows.

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