Exact $m$-covers and the linear form $\sum_{s=1}^{k} x_s/n_s$

by

ZHI-WEI SUN (Nanjing)

1. Introduction. For $a, n \in \mathbb{Z}$ with $n > 0$, we let

$$a + n\mathbb{Z} = \{\ldots, a - 2n, a - n, a, a + n, a + 2n, \ldots\}$$

and call it an arithmetic sequence. Given a finite system

$$(1) \quad A = \{a_s + n_s\mathbb{Z}\}_{s=1}^{k}$$

of arithmetic sequences, we assign to each $x \in \mathbb{Z}$ the corresponding covering multiplicity $\sigma(x) = |\{1 \leq s \leq k : x \in a_s + n_s\mathbb{Z}\}|$ ($|S|$ means the cardinality of a set $S$), and call $m(A) = \inf_{x \in \mathbb{Z}} \sigma(x)$ the covering multiplicity of $A$. Apparently

$$(2) \quad \sum_{s=1}^{k} \frac{1}{n_s} = \frac{1}{N} \sum_{x=0}^{N-1} \sigma(x) \geq m(A)$$

where $N$ is the least common multiple of those common differences (or moduli) $n_1, \ldots, n_k$. For a positive integer $m$, (1) is said to be an $m$-cover of $\mathbb{Z}$ if its covering multiplicity is not less than $m$, and an exact $m$-cover of $\mathbb{Z}$ if $\sigma(x) = m$ for all $x \in \mathbb{Z}$. Note that $k \geq m$ if (1) forms an $m$-cover of $\mathbb{Z}$. Clearly the covering function $\sigma : \mathbb{Z} \to \mathbb{Z}$ is constant if and only if (1) forms an exact $m$-cover of $\mathbb{Z}$ for some $m = 1, 2, \ldots$. An exact 1-cover of $\mathbb{Z}$ is a partition of $\mathbb{Z}$ into residue classes.

P. Erdős ([E]) proposed the concept of cover (i.e., 1-cover) of $\mathbb{Z}$ in the 1930’s, Š. Porubský ([P]) introduced the notion of exact $m$-cover of $\mathbb{Z}$ in the 1970’s, and the author ([Su3]) studied $m$-covers of $\mathbb{Z}$ for the first time. The most challenging problem in this field is to describe those $n_1, \ldots, n_k$ in an $m$-cover (or exact $m$-cover) (1) of $\mathbb{Z}$ (cf. [Gu]). In [Su2, Su3, Su4] the author revealed some connections between (exact) $m$-covers of $\mathbb{Z}$ and

---

1991 Mathematics Subject Classification: Primary 11B25; Secondary 11A07, 11B75, 11D68.

This research is supported by the National Natural Science Foundation of P.R. China.
Egyptian fractions. Here we concentrate on exact $m$-covers of $\mathbb{Z}$. In [Su3, Su4] results for exact $m$-covers of $\mathbb{Z}$ were obtained by studying general $m$-covers of $\mathbb{Z}$ and noting that an exact $m$-cover (1) of $\mathbb{Z}$ is an $m$-cover of $\mathbb{Z}$. In Section 4 of the present paper we shall directly characterize exact $m$-covers of $\mathbb{Z}$ in various ways. (Note that in the famous book [Gu] R. K. Guy wrote that characterizing exact 1-covers of $\mathbb{Z}$ is a main outstanding unsolved problem in the area.) This enables us to make further progress. With the help of the linear form $\sum_{s=1}^{k} x_s/n_s$ (studied in the next section), we will provide some new properties of exact $m$-covers of $\mathbb{Z}$ (see Section 3). The fifth section is devoted to proofs of the main theorems stated in Section 3.

For a complex number $x$ and nonnegative integer $n$, as usual,

$$\binom{x}{n} := \frac{1}{n!} \prod_{j=0}^{n-1} (x - j)$$

($\binom{x}{0}$ is 1). For real $x$ we use $[x]$ and $\{x\}$ to represent the integral part and the fractional part of $x$ respectively. For two integers $a, b$ not both zero, $(a, b)$ denotes the greatest common divisor of $a$ and $b$.

Now we state our central results for an exact $m$-cover (1) of $\mathbb{Z}$:

(I) For $a = 0, 1, 2, \ldots$ and $t = 1, \ldots, k$ there are at least $(m - 1)\lfloor a/n_t \rfloor$ subsets $I$ of $\{1, \ldots, k\}$ for which $t \notin I$ and $\sum_{s \in I} 1/n_s = a/n_t$, where the lower bounds are best possible.

(II) If $\emptyset \neq I \subseteq \{1, \ldots, k\}$ and $(n_s, n_t) \mid a_s - a_t$ for all $s, t \in I$, then

$$\left\{ \left\{ \frac{1}{n_s} \right\} : J \subseteq \{1, \ldots, k\} \setminus I \right\} \supseteq \left\{ \frac{r}{[n_s]_{s \in I}} : r = 0, 1, \ldots, [n_s]_{s \in I} - 1 \right\}$$

where $[n_s]_{s \in I}$ is the least common multiple of those $n_s$ with $s \in I$.

(III) For any rational $c$, the number of solutions of the equation $\sum_{s=1}^{k} x_s/n_s = c$ with $x_s \in \{0, 1, \ldots, n_s - 1\}$ for $s = 1, \ldots, k$, is the sum of finitely many (not necessarily distinct) prime factors of $n_1, \ldots, n_k$ if $c \neq 0, 1, 2, \ldots$, and at least $(k - m)/n$ if $c$ equals a nonnegative integer $n$.

2. On the linear form $\sum_{s=1}^{k} x_s/n_s$. In this section we shall say something general about the linear form $\sum_{s=1}^{k} x_s/n_s$ where $n_1, \ldots, n_k$ are positive integers.

Let us first introduce more notations. For $x, y$ in the rational field $\mathbb{Q}$, if $x - y \in \mathbb{Z}$ then we write $x \equiv y \pmod{1}$. For $n = 1, 2, \ldots$ we set $R(n) = \{0, \ldots, n - 1\}$. When we deal with a finite collection $\{n_s\}_{s \in I}$ of positive integers, the least common multiple $[n_s]_{s \in I}$ and the product $\prod_{s \in I} n_s$ will be regarded as 1 if $I$ is empty.
Definition. Two (finite) sequences \( \{n_s\}_{s=1}^k \) and \( \{m_t\}_{t=1}^l \) of positive integers are said to be equivalent if \( k = l \) and \( (n_s, n_t) = (m_s, m_t) \) for all \( s, t = 1, \ldots, k \) with \( s \neq t \). We call \( \{n_s\}_{s=1}^k \) a normal sequence if \( n_t \) divides \( n_s \) for every \( t = 1, \ldots, k \).

Proposition 2.1. Let \( n_1, \ldots, n_k \) be arbitrary positive integers. Then \( \{(n_t, [n_s]_{s=1}^k, s \neq t)\}_{t=1}^k \) is the only normal sequence equivalent to \( \{n_s\}_{s=1}^k \).

Proof. For each \( t = 1, \ldots, k \) we let
\[
n'_t = (n_t, [n_s]_{s=1}^k, s \neq t) = ([n_s, n_t])_{s=1}^k.
\]
Clearly \( n'_t \) divides \( [n_s]_{s=1}^k \) because \( (n_s, n_t) \) \( n'_s \) for all \( s = 1, \ldots, k \) with \( s \neq t \). For \( i, j = 1, \ldots, k \) with \( i \neq j \), \( (n'_i, n'_j) = (n_i, n_j) \) since \( n_i \) \( n'_s \) \( n_j \) \( n'^s \), which gives the normality. Hence \( \{n'_s\}_{s=1}^k \) is normal and equivalent to \( \{n_s\}_{s=1}^k \). If so is \( \{m_s\}_{s=1}^k \) where \( m_1, \ldots, m_k \) are positive integers, then
\[
m_t = (m_t, [m_s]_{s=1}^k, s \neq t) = ([m_s, m_t])_{s=1}^k = ([n_s, n_t])_{s=1}^k = n'_t
\]
for every \( t = 1, \ldots, k \). We are done.

Proposition 2.2. Let \( n_1, \ldots, n_k \) be positive integers. For \( \theta \in \mathbb{Q} \) the equation
\[
\sum_{s=1}^k \frac{x_s}{n_s} \equiv \theta \pmod{1} \quad \text{with } x_s \in R(n_s) \text{ for } s = 1, \ldots, k
\]
is solvable if and only if \( [n_1, \ldots, n_k] \theta \in \mathbb{Z} \), and in the solvable case the number of solutions is \( n_1 \ldots n_k / [n_1, \ldots, n_k] \), which does not change if we replace \( \{n_s\}_{s=1}^k \) by an equivalent sequence.

Proof. We argue by induction. The case \( k = 1 \) is trivial. Let \( k > 1 \) and assume Proposition 2.2 for smaller values of \( k \). Observe that
\[
\frac{1}{[n_1, \ldots, n_k]} \mathbb{Z} = \left(\frac{[n_1, \ldots, n_{k-1}]}{n_k} \mathbb{Z} + \frac{1}{[n_1, \ldots, n_{k-1}]} \mathbb{Z} \right).
\]
So \( [n_1, \ldots, n_k] \theta \in \mathbb{Z} \) if and only if \( [n_1, \ldots, n_{k-1}] \left( \theta - x/n_k \right) \in \mathbb{Z} \) for some \( x \in \mathbb{Z} \). For any \( a \in \mathbb{Z} \) with \( 0 \leq a < n_k \), the congruence
\[
\sum_{s=1}^{k-1} \frac{x_s}{n_s} \equiv \theta \frac{a}{n_k} \pmod{1}
\]
is solvable if and only if
\[
[n_1, \ldots, n_{k-1}] \left( \theta - \frac{a}{n_k} \right) \in \mathbb{Z},
\]
i.e.
\[
[n_1, \ldots, n_{k-1}] a \equiv [n_1, \ldots, n_{k-1}] n_k \theta \pmod{n_k}.
\]
Hence (3) is solvable if and only if $[n_1, \ldots, n_k] \theta \in \mathbb{Z}$. In the solvable case there are exactly

$$\binom{n_1 \ldots n_k - 1}{[n_1, \ldots, n_k]}$$

numbers $a \in R(n_k)$ satisfying the last congruence, thus by the induction hypothesis (3) has exactly

$$\frac{n_1 \ldots n_k}{[n_1, \ldots, n_k]} \binom{n_1 \ldots n_k}{[n_1, \ldots, n_k]}$$

solutions. As $n_1 \ldots n_k - 1/[n_1, \ldots, n_k]$ depends only on those $(n_i, n_j)$ with $1 \leq i < j < k$, the number $n_1 \ldots n_k/[n_1, \ldots, n_k]$ depends only on the $(n_s, n_t)$, $1 \leq s < t \leq k$. This ends the proof.

**Corollary 2.1.** Let $a$ be an integer and $n_1, \ldots, n_k$ positive integers. Then $a/[n_1, \ldots, n_k]$ can be written uniquely in the form $q + \sum_{s=1}^{k} x_s/n_s$ with $q \in \mathbb{Z}$ and $x_s \in R(n_s)$ for $s = 1, \ldots, k$ if and only if $(n_s, n_t) = 1$ for all $s, t = 1, \ldots, k$ with $s \neq t$.

**Proof.** By Proposition 2.2, equation (3) with $\theta = a/[n_1, \ldots, n_k]$ has a unique solution if and only if $n_1 \ldots n_k/[n_1, \ldots, n_k]$. So the desired result follows.

**Corollary 2.2.** Let $n_1, \ldots, n_k$ be positive integers. Then the number of solutions of the equation

$$\sum_{s=1}^{k} \frac{x_s}{n_s} \equiv 0 \pmod{1} \quad \text{with } x_s \in \mathbb{Z} \text{ and } 0 < x_s < n_s \text{ for } s = 1, \ldots, k$$

equals

$$(-1)^k + \sum_{t=1}^{k} (-1)^{k-t} \sum_{1 \leq i_1 < \ldots < i_t \leq k} \binom{n_{i_1} \ldots n_{i_t}}{[n_{i_1}, \ldots, n_{i_t}]}$$

which depends only on those $(n_s, n_t)$ with $1 \leq s < t \leq k$.

**Proof.** For $I \subseteq \{1, \ldots, k\}$ let $#I$ denote the number of solutions of the diophantine equation $\sum_{s \in I} x_s/n_s \equiv 0 \pmod{1}$ with $x_s \in \{1, \ldots, n_s - 1\}$ for $s \in I$, and consider $#\emptyset$ to be 1. By Proposition 2.2, $\sum_{J \subseteq I} #J = \prod_{s \in I} n_s/[n_s]_{s \in I}$ for all $I \subseteq \{1, \ldots, k\}$, therefore #$\{1, \ldots, k\}$ coincides with

$$\sum_{J \subseteq \{1, \ldots, k\}} (-1)^{k - |J|} \binom{k - |J|}{s} \#J = \sum_{J \subseteq \{1, \ldots, k\}} \sum_{J \subseteq I \subseteq \{1, \ldots, k\}} (-1)^{k - |J|} \#J$$
The fact that the number does not vary if we replace \(Q. \text{Sun}, D.-Q. \text{Wan} \text{ and} D.-G. \text{Ma} \text{[SWM]} \) with much more complicated
tained by R. Lidl and H. Niederreiter [LN], R. Stanly (cf. C. Small [Sm]),
over a finite field. The formula for the number of solutions of (4) was ob-
{\text{section}}

\[\text{Proposition 2.2,}\]

\[\begin{aligned}
\text{we have}
\left(5\right)
\end{aligned}\]

\[\begin{aligned}
\text{In view of Proposition 2.2, the number}\ # \\{1, \ldots, k\}\ \text{remains the same if an}
equivalent sequence is substituted for}\ \{n_s\}_{s=1}^k. \text{The proof is now complete.}
\end{aligned}\]

\text{Remark 1. Equation (4) is closely related to diagonal hypersurfaces}

\[\text{Corollary 2.3. Let (1) be a system of arithmetic sequences with}
\end{aligned}\]

\[\begin{aligned}
(\text{for any } \theta \in \mathbb{Q} \text{ with } 0 \leq \theta < 1 \text{ we have})
\end{aligned}\]

\[\begin{aligned}
\text{Proof. By the Chinese Remainder Theorem in general form, the inter-
section}\ \bigcap_{s=1}^k a_s + n_s \mathbb{Z}\ \text{is nonempty if and only if}\ a_s + n_s \mathbb{Z} \cap a_t + n_t \mathbb{Z} \neq \emptyset
\end{aligned}\]

\[\begin{aligned}
\text{vanishes if}\ [n_1, \ldots, n_k] \theta \notin \mathbb{Z}, \text{and otherwise equals}\ \frac{n_1 \ldots n_k}{[n_1, \ldots, n_k]} e^{2\pi i \theta}\text{. So (5) holds.}
\end{aligned}\]

\[\begin{aligned}
\text{To conclude this section we make a few comments. For system (1),}
M(A) = \sup\{\sigma(x) \mid x \in \mathbb{Z}\}
\end{aligned}\]

\[\begin{aligned}
\text{does not change if an equivalent sequence takes the}
\text{place of}\ \{n_s\}_{s=1}^k, \text{because for}\ \emptyset \neq I \subseteq \{1, \ldots, k\} \text{the set}\ \bigcap_{s \in I} a_s + n_s \mathbb{Z}
\text{is nonempty if and only if}\ (n_s, n_t) | a_s - a_t \text{for all } s, t \in I. \text{Observe that (1)}
\end{aligned}\]
forms an exact \( m \)-cover of \( \mathbb{Z} \) if and only if \( \sum_{s=1}^{k} 1/n_s = m \geq M(A) \). So whether \( n_1, \ldots, n_k \) are the moduli of an exact \( m \)-cover of \( \mathbb{Z} \) only depends on \( \sum_{s=1}^{k} 1/n_s \) and the \( k(k-1)/2 \) numbers \((n_s, n_t), 1 \leq s < t \leq k\). For a given exact \( m \)-cover \((1)\) of \( \mathbb{Z} \), replacing \( \{n_s\}_{s=1}^{k} \) by the unique normal sequence \( \{n'_s\}_{s=1}^{k} \) equivalent to it we find that

\[
\sum_{s=1}^{k} \frac{1}{n'_s} \leq M(A) \leq m = \sum_{s=1}^{k} \frac{1}{n_s}.
\]

As \( n'_s \leq n_s \) for \( s = 1, \ldots, k \), the sequence \( \{n_s\}_{s=1}^{k} \) must be identical with \( \{n'_s\}_{s=1}^{k} \) and hence normal. In the light of the above, the reader should not be surprised by connections between the exact \( m \)-cover \((1)\) of \( \mathbb{Z} \) and the linear form \( \sum_{s=1}^{k} x_s/n_s \).

### 3. Main theorems and their consequences.

In this section we let \((1)\) be an exact \( m \)-cover of \( \mathbb{Z} \); we also let \( I \subseteq \{1, \ldots, k\} \) and \( \bar{I} = \{1, \ldots, k\} \setminus I \). For any rational \( c \), we let \( I^*(c) \) be the number of solutions \( \langle x_s \rangle_{s \in I} \) to the diophantine equation

\[
\sum_{s \in I} \frac{x_s}{n_s} = c \quad \text{with } x_s \in R(n_s) \text{ for all } s \in I,
\]

and \( I_s(c) = |\{J \subseteq I : \sum_{s \in J} 1/n_s = c\}| \) be the number of solutions \( \langle \delta_s \rangle_{s \in I} \) to the equation

\[
\sum_{s \in I} \frac{\delta_s}{n_s} = c \quad \text{with } \delta_s \in R(2) = \{0, 1\} \text{ for all } s \in I.
\]

(When \( I = \emptyset \) and \( c = 0 \) we view each of \((6)\) and \((7)\) as having only the zero solution.) We also set

\[
I_s^{(0)}(c) = \left| \left\{ J \subseteq I : 2 \not| |J| \text{ and } \sum_{s \in J} \frac{1}{n_s} = c \right\} \right|
\]

and

\[
I_s^{(1)}(c) = \left| \left\{ J \subseteq I : 2 \not| |J| \text{ and } \sum_{s \in J} \frac{1}{n_s} = c \right\} \right|.
\]

Let us present our main theorems whose proofs will be given later, and derive a number of interesting corollaries from them.

**Theorem 3.1.** Let \( c \) be a rational number.

(i) When \( |I| \leq m \), if \( I^*(c-n) = 1 \) for a nonnegative integer \( n \) then

\[
\bar{I}_s(c) + \sum_{l=0}^{m-|I|} \binom{m-|I|}{l} I^*(c-l) \geq \binom{m-|I|}{n};
\]
in particular, if \( c \) can be uniquely written in the form \( n + \sum_{s \in I} x_s/n_s \) where \( n \) and \( x_s \) lie in \( \{0, 1, \ldots, m - |I|\} \) and \( \{0, 1, \ldots, n_s - 1\} \) respectively, then
\[
\bar{I}_s(c) \geq \left( \frac{m - |I|}{n} \right).
\]

(ii) When \( |I| \geq m \), if \( \bar{I}_s(c - n) = 1 \) for a nonnegative integer \( n \) then
\[
I^*(c) + \sum_{l \neq n} \left( \frac{|I| - m}{l} \right) \bar{I}_s(c - l) \geq \left( \frac{|I| - m}{n} \right);
\]
in particular, if \( c \) can be uniquely expressed in the form \( n + \sum_{s \in J} 1/n_s \) where \( J \subseteq \bar{I} \) and \( n \in \{0, 1, \ldots, |I| - m\} \), then
\[
I^*(c) \geq \left( \frac{|I| - m}{n} \right).
\]

Below there are corollaries involving the cases \( |I| \leq m \), \( |I| = m \) and \( |I| \geq m \).

**Corollary 3.1.** Assume that those \( n_s \) with \( s \in I \) are pairwise relatively prime. Then \( |I| \leq m \) and
\[
\left\{ J \subseteq \bar{I} : \sum_{s \in I} \frac{1}{n_s} = n + \sum_{s \in J} \frac{x_s}{n_s} \right\} \geq \left( \frac{m - |I|}{n} \right)
\]
for all \( n = 0, 1, 2, \ldots \) and \( x_s \in R(n_s) \) with \( s \in I \); in particular,
\[
\left\{ \sum_{s \in J} \frac{1}{n_s} : J \subseteq \bar{I} \right\} \geq \left\{ \frac{a}{[n_s]_{s \in I}} : a \in \mathbb{Z} \land |I| \leq \frac{a}{[n_s]_{s \in I}} \leq m - |I| \right\}
\]
and
\[
\left\{ J \subseteq \bar{I} : \sum_{s \in J} \frac{1}{n_s} \equiv \frac{a}{\prod_{s \in I} n_s} \ (\text{mod} \ 1) \right\} \geq 2^{m - |I|} \text{ for every } a \in \mathbb{Z}.
\]

**Proof.** By the Chinese Remainder Theorem, \( \bigcap_{s \in I} a_s + n_s \mathbb{Z} \neq \emptyset \) if \( I \neq \emptyset \). Since any integer lies in exactly \( m \) members of (1), \( |I| \) does not exceed \( m \). Let \( N = [n_s]_{s \in I} = \prod_{s \in I} n_s \). By Corollary 2.1, for each \( a \in \mathbb{Z} \) the number \( a/N \) can be expressed uniquely in the form \( q + \sum_{s \in I} x_s/n_s \) with \( q \in \mathbb{Z} \) and \( x_s \in R(n_s) \) for \( s \in I \). Whenever \( x_s \in R(n_s) \) for all \( s \in I \), by Theorem 3.1, (12) holds for every nonnegative integer \( n \). If \( |I|N \leq a \leq (m - |I|)N \) then the corresponding integer \( q = a/N - \sum_{s \in I} x_s/n_s \) lies in the interval \( [0, m - |I|] \) and hence
\[
\left\{ J \subseteq \bar{I} : \sum_{s \in I} \frac{1}{n_s} = \frac{a}{N} = q + \sum_{s \in J} \frac{x_s}{n_s} \right\} \geq \left( \frac{m - |I|}{q} \right) > 0.
\]
This yields (13). For (14) we observe that
\[
\left| \left\{ J \subseteq \bar{I} : \sum_{s \in J} \frac{1}{n_s} \equiv \frac{a}{N} \pmod{1} \right\} \right| \\
\geq \sum_{n=0}^{m-|I|} \left| \left\{ J \subseteq \bar{I} : \sum_{s \in J} \frac{1}{n_s} = n + \sum_{s \in J} x_s \right\} \right| \\
\geq \sum_{n=0}^{m-|I|} \left( m - |I| \right) = 2^{m-|I|}.
\]

This concludes the proof.

Applying Corollary 3.1 with \( I = \emptyset \) we immediately get the theorem of Sun [Su2].

Putting \( I = \{ t \} \quad (1 \leq t \leq k) \) in Corollary 3.1 we then obtain result (I) stated in the first section. In the case \( m = 1 \), result (I) was first observed by the author in [Su4]. When \( m > 1 \), we noted in [Su4] that, providing \( n_1 < \ldots < n_{k-1} < n_{k-1+1} = \ldots = n_k \), for every \( r = 0, 1, \ldots, n_k - 1 \) there exists a \( J \subseteq \{1, \ldots, k-1\} \) with \( \sum_{s \in J} 1/n_s \equiv r/n_k \pmod{1} \). In [Su4] we even conjectured that, if (1) forms an \( m \)-cover of \( \mathbb{Z} \) with \( \sigma(x) = m \) for all \( x \equiv a_t \pmod{n_t} \) where \( 1 \leq t \leq k \), then

\[
\left\{ \sum_{s \in I} \frac{1}{n_s} \right\} : I \subseteq \{1, \ldots, k\} \setminus \{t\} \cap \frac{1}{n_t} \mathbb{Z} \\
= \left\{ \frac{r}{n_t} : r = 0, \ldots, n_t - 1 \right\}.
\]

Result (I) confirms the conjecture for exact \( m \)-covers of \( \mathbb{Z} \). The lower bounds are best possible as is shown by the following example.

**Example.** Let \( k > m > 0 \) be integers. Let \( a_s = 0 \) and \( n_s = 1 \) for \( s = 1, \ldots, m-1 \), \( a_s = 2^{s-m} \) and \( n_s = 2^{s-m+1} \) for \( s = m, \ldots, k-1 \), also \( a_k = 0 \) and \( n_k = 2^{k-m} \). It is clear that \( A = \{ a_s + n_s \mathbb{Z} \}_{s=1}^k \) forms an exact \( m \)-cover of \( \mathbb{Z} \). As each nonnegative integer can be expressed uniquely in the binary form, the reader can easily check that for \( a = 0, 1, 2, \ldots \) and \( t = 1, \ldots, k \) we always have

\[
\left| \left\{ J \subseteq \{1, \ldots, k\} \setminus \{t\} : \sum_{s \in J} \frac{1}{n_s} = \frac{a}{n_t} \right\} \right| = \binom{m-1}{[a/n_t]}.\]

**Corollary 3.2.** Suppose that \( |I| = m \). Then no number occurs exactly once among the \( \frac{2^{k-m}n_1 \ldots n_m}{n_s} \) rationals

\[
\sum_{s \in I} \frac{x_s}{n_s}, \quad x_s \in R(n_s) \text{ for } s \in I; \quad \sum_{s \in J} \frac{1}{n_s}, \quad J \subseteq \bar{I}.
\]
Proof. If $I^*(\sum_{s \in I} x_s/n_s) = 1$ where $x_s \in R(n_s)$ for $s \in I$ then $\bar{I}_s(\sum_{s \in I} x_s/n_s) \geq (m^{-|I|}) = 1$ by Theorem 3.1(i). If $J \subseteq \bar{I}$ and $\bar{I}_{s}(\sum_{s \in J} 1/n_s) = 1$, then $I^*(\sum_{s \in J} 1/n_s) \geq (|I|−m)_0 = 1$ by Theorem 3.1(ii).

We are done.

Corollary 3.3. Assume that $|I| \geq m$. For any $J \subseteq \bar{I}$, if

$$\sum_{s \in J} \frac{1}{n_s} - \sum_{s \in J'} \frac{1}{n_s} \in \{0, 1, \ldots, |I| - m\} \quad \text{for no } J' \subseteq \bar{I} \text{ with } J' \neq J,$$

then

$$I^*(n + \sum_{s \in J} \frac{1}{n_s}) \geq \left(\frac{|I| - m}{n}\right) \quad \text{for } n = 0, 1, 2, \ldots$$

and hence

$$\prod_{s \in I} n_s \geq 2^{|I| - m}[n_s]_{s \in I}.$$

Proof. Let $J$ be a subset of $\bar{I}$ which satisfies (17). Note that $(|I| - m)_0 = 0$ for every integer $n > |I| - m$. For $n \in \mathbb{Z}$ with $0 \leq n \leq |I| - m$, if $J' \subseteq \bar{I}$ and $n' \in \{0, 1, \ldots, |I| - m\}$ then by (17),

$$n + \sum_{s \in J} \frac{1}{n_s} = n' + \sum_{s \in J'} \frac{1}{n_s} \Rightarrow J = J' \text{ and } n = n'.$$

So (18) holds in view of the latter part of Theorem 3.1, and thus by Proposition 2.2,

$$\prod_{s \in I} n_s \geq 2^{|I| - m}[n_s]_{s \in I}.$$

Putting $I = \{1, \ldots, k\}$ and $J = \emptyset$ in Corollary 3.3 we obtain the second half of result (III). When $1 \leq t \leq k$ and $n_t > 1$, Corollary 3.3 in the case $I = \{1, \ldots, k\} \setminus \{t\}$ and $J = \{t\}$ also yields an interesting result.

Let $p(1) = 1$ and $p(n)$ denote the smallest (positive) prime factor of $n$ for $n = 2, 3, \ldots$. For a positive integer $n$ we also put

$$D(n) = \left\{\sum_{p | n} pm_p : \text{all the } m_p \text{ are nonnegative integers}\right\}.$$

Theorem 3.2. Let $c$ be a rational number.
(i) If $|I| \leq m$, then either

$$\bar{I}_s(c) + \sum_{n=0}^{m-|I|} I^*(c-n) \geq p([n_1, \ldots, n_k])$$

or

$$\bar{I}_s^{(0)}(c) - \bar{I}_s^{(1)}(c) = \sum_{n=0}^{m-|I|} (-1)^n \binom{m-|I|}{n} I^*(c-n);$$

moreover

$$\bar{I}_s(c) + \sum_{n=0}^{m-|I|} \binom{m-|I|}{n} I^*(c-n) \in D([n_1, \ldots, n_k])$$

if $|S|, |T| \leq 1$ and $S \cap T = \emptyset$ where

$$S = \{ n \mod 2 : n \in \mathbb{Z}, 0 \leq n \leq m - |I| \text{ and } I^*(c-n) \neq 0 \}$$

and

$$T = \left\{ |J| \mod 2 : J \subseteq \bar{I} \text{ and } \sum_{s \in J} \frac{1}{n_s} = c \right\}.$$

(ii) If $|I| \geq m$, then either

$$I^*(c) + \sum_{n=0}^{|I|-m} \bar{I}_s(c-n) \geq p([n_1, \ldots, n_k])$$

or

$$I^*(c) = \sum_{n=0}^{|I|-m} (-1)^n \binom{|I|-m}{n} (\bar{I}_s^{(0)}(c-n) - \bar{I}_s^{(1)}(c-n));$$

furthermore

$$I^*(c) + \sum_{n=0}^{|I|-m} \binom{|I|-m}{n} \bar{I}_s(c-n) \in D([n_1, \ldots, n_k])$$

if $c \neq n + \sum_{s \in J} 1/n_s$ for any $n = 0, 1, \ldots, |I| - m$ and $J \subseteq \bar{I}$ with $n \equiv |J|$ (mod 2).

**Corollary 3.4.** Let $|I| \leq m$ and $J \subseteq \bar{I}$. Suppose that $\sum_{s \in J} 1/n_s$ cannot be expressed in the form $n + \sum_{s \in I} x_s/n_s$ where $n \in \{0, 1, \ldots, m - |I|\}$ and $x_s \in R(n_s)$ for $s \in I$. Put

$$J = \left\{ J' \subseteq \bar{I} : \sum_{s \in J'} \frac{1}{n_s} = \sum_{s \in J} \frac{1}{n_s} \right\}.$$

Then either $|J| \geq p([n_1, \ldots, n_k])$ or $|J| \equiv 0 \pmod{2}$; either $|J'| \neq |J|$ (mod 2) for some $J' \in J$, or $|J|$ can be expressed as the sum of some (not necessarily distinct) prime divisors of $[n_1, \ldots, n_k]$. 
Proof. Let \( c = \sum_{s \in J} 1/n_s \). As \( \bar{I}_s(c) = \bar{I}_s^{(0)}(c) + \bar{I}_s^{(1)}(c) \), and \( I^*(c - n) = 0 \) for every \( n = 0, 1, \ldots, m - |I| \), the desired results follow from Theorem 3.2(i).

Remark 2. In the case \( I = \emptyset \) Corollary 3.4 was obtained by the author in [Su4].

Corollary 3.5. Assume that \( |I| = m \). Let \( l \) be the total number of ways in which the rational \( c \) is expressed in the form \( \sum_{s \in I} x_s/n_s \) or \( \sum_{s \in I} \delta_s/n_s \) where \( x_s \in R(n_s) \) for \( s \in I \) and \( \delta_s \in \{0, 1\} \) for \( s \in \bar{I} \). Then we have

\[
(27) \quad l \geq p([n_1, \ldots, n_k]) \quad \text{or} \quad l = 2 \left\{|J \subseteq \bar{I} : \sum_{s \in J} 1/n_s = c\}\right|,
\]

and \( l \) can be written as the sum of finitely many (not necessarily distinct) prime divisors of \( n_1, \ldots, n_k \) providing \( \sum_{s \in J} 1/n_s = c \) for no \( J \subseteq \bar{I} \) with \( |J| \equiv 0 \pmod{2} \).

Proof. Obviously \( l = I^*(c) + \bar{I}_s(c) \), and (22) or (25) says that \( I_s^{(0)}(c) - \bar{I}_s^{(1)}(c) = I^*(c) \), i.e. \( l = 2\bar{I}_s^{(0)}(c) \). Therefore Theorem 3.2 yields Corollary 3.5.

Corollary 3.6. Let \( |I| \geq m \). Suppose that \( \sum_{s \in I} m_s/n_s \) cannot be expressed in the form \( n + \sum_{s \in J} 1/n_s \) with \( n \in \{0, 1, \ldots, |I| - m\} \) and \( J \subseteq \bar{I} \), where \( m_s \in R(n_s) \) for each \( s \in I \). Then

\[
(28) \quad \left\{|(x_s)_{s \in I} : x_s \in R(n_s) \text{ for } s \in I \text{ and } \sum_{s \in I} x_s/n_s = \sum_{s \in I} m_s/n_s\}\right|
\]

must be a finite sum of (not necessarily distinct) prime divisors of \([n_1, \ldots, n_k]\).

Proof. Let \( c = \sum_{s \in I} m_s/n_s \). Note that \( \bar{I}_s(c - n) = 0 \) for each \( n = 0, 1, \ldots, |I| - m \). By Theorem 3.2(ii), \( I^*(c) \) belongs to \( D([n_1, \ldots, n_k]) \).

Clearly Corollary 3.6 in the case \( I = \{1, \ldots, k\} \) gives the first half of result (III).

Theorem 3.3. (i) If \( (n_s, n_t) | a_s - a_t \) for all \( s, t \in I \), then

\[
(29) \quad \sum_{n=0}^{m-1} \bar{I}_s \left(n + \frac{r}{[n_s]_{s \in I}}\right) = \left|\left\{J \subseteq \bar{I} : \left\{\sum_{s \in J} \frac{1}{n_s}\right\} = \frac{r}{[n_s]_{s \in I}}\right\}\right| \geq \prod_{s \in I} [n_s]_{s \in I}^{-1}
\]

for each \( r = 0, 1, \ldots, [n_s]_{s \in I} - 1 \).
(ii) Assume \(|I| = m, 0 \leq \theta < 1, \) and \([n_s]_{s \in I} \not\equiv\ Z \) or \((n_i, n_j) \nmid a_i - a_j\) for some \(i, j \in I\). Then either

\[
\sum_{n=0}^{m-1} I_s(n + \theta) = \left| \left\{ J \subseteq \bar{I} : \left\{ \sum_{s \in J} \frac{1}{n_s} \right\} = \theta \right\} \right| \geq p([n_s]_{s \in I})
\]

or

\[
\left| \left\{ J \subseteq \bar{I} : 2 \mid |J| & \left\{ \sum_{s \in J} \frac{1}{n_s} \right\} = \theta \right\} \right| = \left| \left\{ J \subseteq \bar{I} : 2 \mid |J| & \left\{ \sum_{s \in J} \frac{1}{n_s} \right\} = \theta \right\} \right|
\]

and hence

\[
\sum_{n=0}^{m-1} I_s(n + \theta) = \left| \left\{ J \subseteq \bar{I} : \left\{ \sum_{s \in J} \frac{1}{n_s} \right\} = \theta \right\} \right| \equiv 0 \pmod{2};
\]

moreover,

\[
\sum_{n=0}^{m-1} I_s(n + \theta) = \left| \left\{ J \subseteq \bar{I} : \left\{ \sum_{s \in J} \frac{1}{n_s} \right\} = \theta \right\} \right| \in D([n_s]_{s \in I})
\]

if all the \(|J| \pmod{2} \) with \(J \subseteq \bar{I}\) and \(\{\sum_{s \in J} 1/n_s\} = \theta\) are the same.

Remark 3. When those \(n_s\) with \(s \in I\) are pairwise relatively prime, Theorem 3.3(i) yields the lower bound 1 in (29) while (14) gives the bound \(2^{m-|I|}\).

Corollary 3.7. If \(I \neq \emptyset\) and \((n_s, n_t) \mid a_s - a_t\) for all \(s, t \in I\), then

\[
\prod_{s \in I} n_s \leq 2^{k-|I|}, \quad [n_s]_{s \in I} \mid [n_s]_{s \in \bar{I}},
\]

and

\[
\left\{ \left\{ \sum_{s \in J} \frac{1}{n_s} \right\} : J \subseteq \bar{I} \right\} \supseteq \left\{ 0, \frac{1}{[n_s]_{s \in I}}, \ldots, \frac{[n_s]_{s \in I} - 1}{[n_s]_{s \in I}} \right\}.
\]

Proof. (34) follows immediately from Theorem 3.3(i). Since \(\sum_{s \in J} 1/n_s \equiv 1/[n_s]_{s \in I} \pmod{1}\) for some \(J \subseteq \bar{I}\), \([n_s]_{s \in I}\) must divide \([n_s]_{s \in \bar{I}}\). For the inequality in (33) we notice that

\[
2^{k-|I|} \geq \left| \bigcup_{r=0}^{[n_s]_{s \in I} - 1} \left\{ J \subseteq \bar{I} : \left\{ \sum_{s \in I} \frac{1}{n_s} \right\} = \frac{r}{[n_s]_{s \in I}} \right\} \right|
\]

\[
= \sum_{r=0}^{[n_s]_{s \in I} - 1} \left| \left\{ J \subseteq \bar{I} : \left\{ \sum_{s \in I} \frac{1}{n_s} \right\} = \frac{r}{[n_s]_{s \in I}} \right\} \right|
\]

\[
\geq \sum_{r=0}^{[n_s]_{s \in I} - 1} \frac{\prod_{s \in I} n_s}{[n_s]_{s \in I}} = \prod_{s \in I} n_s.
\]
Remark 4. By checking (33) and (34) with $I$ taken to be $K = \{1, \ldots, m-1, k\}$ and $K \cup \{k-1\}$ in the previous example, we find that Corollary 3.7 is sharp. When (1) forms an exact 1-cover of $\mathbb{Z}$ and $I \subseteq \{1, \ldots, k\}$ contains at least two elements, we cannot have $(n_s, n_t) \mid a_s - a_t$ for all $s, t \in I$ with $s \neq t$, and (34) fails to hold because for all $J \subseteq I$ we have

$$\sum_{s \in J} \frac{1}{n_s} \leq \sum_{s \in I} \frac{1}{n_s} = 1 - \sum_{s \in I} \frac{1}{n_s} < 1 - \frac{1}{\left\lfloor \frac{n_s}{n_s} \right\rfloor} = \left\lfloor \frac{n_s}{n_s} \right\rfloor.$$

For any $a, n \in \mathbb{Z}$ with $n > 0$, each integer in $a + n\mathbb{Z}$ belongs to exactly $m$ members of (1) and hence

$$A_{a(n)} = \left\{ b_s + \frac{n_s}{(n, n_s)} \mathbb{Z} \right\}_{s \in J}$$

also forms an exact $m$-cover of $\mathbb{Z}$ where $J = \{1 \leq s \leq k : (n, n_s) \mid a - a_s\}$, $b_s \in \mathbb{Z}$ and $a + b_sn \equiv a_s \pmod{n_s}$ for $s \in J$. Instead of $A = A_0(1)$ we may apply our results to $A_{a(n)}$ so as to get more general ones. See [Su4] for examples of such transformations.

4. Characterizations of exact $m$-covers of $\mathbb{Z}$

Theorem 4.1. Let (1) be a system of arithmetic sequences. Let $I \subseteq \{1, \ldots, k\}$ and $\bar{I} = \{1, \ldots, k\} \setminus I$. If $|I| \leq m$ then (1) is an exact $m$-cover of $\mathbb{Z}$ if and only if

$$\sum_{J \subseteq I} (-1)^{|J|} e^{2\pi i \sum_{s \in J} a_s/n_s} \sum_{s \in I} x_s/n_s = c$$

is valid for all rational $c \geq 0$. If $|I| \geq m$, then (1) forms an exact $m$-cover of $\mathbb{Z}$ if and only if

$$\sum_{x_s \in (\pi(n_s) \cap \mathbb{Z}) \setminus \sum_{s \in I} x_s/n_s = c} e^{2\pi i \sum_{s \in I} a_s x_s/n_s} \sum_{x_s \in I} x_s/n_s = c$$

holds for all rational $c \geq 0$. 


Proof. Put $N = [n_1, \ldots, n_k]$. We assert that (1) forms an exact $m$-cover of $\mathbb{Z}$ if and only if we have the identity

$$\prod_{s=1}^{k} (1 - z^{N/n_s} e^{2\pi i a_s/n_s}) = (1 - z^N)^m.$$  \hspace{1cm} (37)

Apparently any zero of the left hand side of (37) is an $N$th root of unity. Observe that for every integer $x$ the number $e^{-2\pi i x/N}$ is a zero of the left hand side of (37) with multiplicity $m$ if and only if $x$ lies in $a_s + n_s \mathbb{Z}$ for exact $m$ of $s = 1, \ldots, k$. So the assertion follows from Viète’s theorem.

Now consider the case $|I| \leq m$. Clearly the following identities are equivalent:

$$\prod_{s \in I} (1 - z^{N/n_s} e^{2\pi i a_s/n_s}) \cdot \prod_{s \in \overline{I}} (1 - z^{N/n_s} e^{2\pi i a_s/n_s}) = (1 - z^N)^m,$$

$$\prod_{s \in I} (1 - z^{N/n_s} e^{2\pi i a_s/n_s}) \cdot \prod_{s \in \overline{I}} (1 - z^{N/n_s} e^{2\pi i a_s/n_s}) = (1 - z^N)^m - |I| \sum_{s \in I} z^{m_s N/n_s} e^{2\pi i m_s a_s/n_s},$$

$$\sum_{J \subseteq I} (-1)^{|J|} z^{\sum_{s \in J} N/n_s} e^{2\pi i \sum_{s \in J} a_s/n_s} = (1 - z^N)^m \prod_{s \in I} \sum_{m_s=0}^{n_s-1} z^{m_s N/n_s} e^{2\pi i m_s a_s/n_s}.$$  

By the assertion the first one holds if and only if (1) forms an exact $m$-cover of $\mathbb{Z}$. Since the third one is valid if and only if (35) is true for every rational $c \geq 0$, we get the desired result.

For the case $|I| \geq m$, that (1) forms an exact $m$-cover of $\mathbb{Z}$ is equivalent to any of the identities given below:

$$\prod_{s \in I} (1 - z^{N/n_s} e^{2\pi i a_s/n_s}) \cdot \prod_{s \in I} (1 - z^{N/n_s} e^{2\pi i a_s/n_s}) = (1 - z^N)^m,$$

$$\prod_{s \in \overline{I}} (1 - z^{N/n_s} e^{2\pi i a_s/n_s}) \cdot \prod_{s \in I} (1 - z^{N/n_s} e^{2\pi i a_s/n_s}) = (1 - z^N)^m \prod_{s \in \overline{I}} \sum_{m_s=0}^{n_s-1} z^{m_s N/n_s} e^{2\pi i m_s a_s/n_s},$$

$$\sum_{n=0}^{|I|-m} (-1)^n \binom{|I| - m}{n} z^{nN} \sum_{J \subseteq \overline{I}} (-1)^{|J|} z^{\sum_{s \in J} N/n_s} e^{2\pi i \sum_{s \in J} a_s/n_s} = \prod_{s \in \overline{I}} \sum_{m_s=0}^{n_s-1} z^{m_s N/n_s} e^{2\pi i m_s a_s/n_s}. $$
As the last one holds if and only if (36) does for all rational \( c \geq 0 \), we are done.

Remark 5. In the case \( I = \emptyset \) and \( c \in \{1, \ldots, m\} \), that (35) holds for any exact \( m \)-cover (1) of \( \mathbb{Z} \) was first observed by the author in [Su2] with the help of the Riemann zeta function.

The characterization of exact \( m \)-cover (1) of \( \mathbb{Z} \) given in Theorem 4.1 involves a fixed subset \( I \) of \( \{1, \ldots, k\} \). Now we present a new one which depends on all the \( I \subseteq \{1, \ldots, k\} \) with \( |I| = m \).

**Theorem 4.2.** Let (1) be a system of arithmetic sequences. Then (1) forms an exact \( m \)-cover of \( \mathbb{Z} \) if and only if the relation

\[
\sum_{J \subseteq \{1, \ldots, k\} \setminus I_{\{1/n_s\} = \theta}} (-1)^{|J|} e^{2\pi i \sum_{s \in J} a_s / n_s} = \sum_{x_s \in \mathbb{R} / \{\sum_{s \in I} x_s / n_s\} = \theta} e^{2\pi i \sum_{s \in I} a_s x_s / n_s}
\]

holds for all \( \theta \in [0, 1) \) and \( I \subseteq \{1, \ldots, k\} \) with \( |I| = m \).

**Proof.** Let \( N = [n_1, \ldots, n_k] \) and \( I = \{1, \ldots, k\} \setminus I \) for all \( I \subseteq \{1, \ldots, k\} \). First suppose that (1) forms an \( m \)-cover of \( \mathbb{Z} \). Let \( x \) be any integer and \( I \) a subset of \( \{1, \ldots, k\} \) with \( |I| = m \). By taking \( z = r^{1/N} e^{2\pi i x / N} \) in (37), we get the equality

\[
\prod_{s=1}^{k} (1 - r^{1/n_s} e^{2\pi i (x + a_s) / n_s}) = (1 - r)^m
\]

for all \( r \geq 0 \). If \( I = \{1 \leq s \leq k : n_s \mid x + a_s\} \), then

\[
\prod_{s \in I} (1 - r^{1/n_s} e^{2\pi i (x + a_s) / n_s}) / \prod_{s \in I} \sum_{x_s = 0}^{n_s - 1} e^{2\pi i (x + a_s) x_s / n_s}
\]

\[
= \lim_{r \to 1} \prod_{s \in I} (1 - r^{1/n_s} e^{2\pi i (x + a_s) / n_s}) / \prod_{s \in I} \lim_{\bar{r} \to e^{2\pi i (x + a_s) / n_s}} 1 - \bar{r}^{n_s} / 1 - (\bar{r}^{n_s})^{1/n_s}
\]

\[
= \lim_{r \to 1} \prod_{s \in I} (1 - r^{1/n_s} e^{2\pi i (x + a_s) / n_s}) . \prod_{s \in I} \frac{1 - r^{1/n_s}}{1 - r} \cdot \prod_{s \in I} 1 - \bar{r}^{n_s} / 1 - \bar{r}^{n_s}
\]

\[
= \lim_{r \to 1} (1 - r)^{-|I|} \prod_{s=1}^{k} (1 - r^{1/n_s} e^{2\pi i (x + a_s) / n_s})
\]

\[
= \lim_{r \to 1} (1 - r)^{-|I|} (1 - r)^m = 1.
\]
If $I \neq \{1 \leq s \leq k : n_s \mid x + a_s\}$, then $n_s \mid x + a_s$ for some $s \in I$ and $n_t \nmid x + a_t$ for some $t \in I$, therefore

$$\prod_{s \in I} (1 - e^{2\pi i(x + a_s)/n_s}) = 0 = \prod_{t \in I} \sum_{x_t = 0}^{n_t - 1} e^{2\pi i(x + a_t)x_t/n_t}.$$ 

So we always have the identity

$$\prod_{s \in I} (1 - e^{2\pi i(x + a_s)/n_s}) = \prod_{s \in I} \sum_{x_s = 0}^{n_s - 1} e^{2\pi i(x + a_s)x_s/n_s}. \quad (39)$$

Next assume (39) for all $x \in \mathbb{Z}$ and $I \subseteq \{1, \ldots, k\}$ with $|I| = m$. For each integer $x$, if $\{1 \leq s \leq k : n_s \mid x + a_s\} > m$, then we can choose a proper subset $I$ of $\{1 \leq s \leq k : n_s \mid x + a_s\}$ with cardinality $m$ for which the left hand side of (39) is zero but the right hand side of (39) is nonzero; if $\{1 \leq s \leq k : n_s \mid x + a_s\} < m$, then we can select an $I \supset \{1 \leq s \leq k : n_s \mid x + a_s\}$ with $|I| = m$ for which the left hand side of (39) is nonzero while the right hand side of (39) is zero. So (1) forms an exact $m$-cover of $\mathbb{Z}$.

Now let $I$ be any subset of $\{1, \ldots, k\}$ with $|I| = m$. For every $x \in \mathbb{Z}$,

$$\prod_{s \in I} (1 - e^{2\pi i(x + a_s)/n_s}) = \sum_{J \subseteq I} (-1)^{|J|} e^{2\pi i(\sum_{s \in J} a_s/n_s + x \sum_{s \in J} 1/n_s)} = \sum_{r=0}^{N-1} e^{2\pi irx/N} \sum_{\{s \in J : 1/n_s = r/N\}} (-1)^{|J|} e^{2\pi i\sum_{s \in J} a_s/n_s}$$

while $\prod_{s \in I} \sum_{x_s = 0}^{n_s - 1} e^{2\pi i(x + a_s)x_s/n_s}$ coincides with

$$\sum_{x_s \in R(n_s) \text{ for } s \in I} e^{2\pi i(\sum_{s \in I} a_s x_s/n_s + x \sum_{s \in I} x_s/n_s)} = \sum_{r=0}^{N-1} e^{2\pi irx/N} \sum_{x_s \in R(n_s) \text{ for } s \in I \mid \sum_{s \in I} x_s/n_s = r/N} e^{2\pi i\sum_{s \in I} a_s x_s/n_s}.$$

If (38) holds for all $\theta \in (0, 1)$ then (39) follows from the above for each $x \in \mathbb{Z}$. Conversely, providing (39) for all $x \in \mathbb{Z}$, for any $a = 0, 1, \ldots, N - 1$ we have

$$N \sum_{\{s \in J \mid 1/n_s = a/N\}} (-1)^{|J|} e^{2\pi i\sum_{s \in J} a_s/n_s} = \sum_{r=0}^{N-1} \sum_{J \subseteq I \mid \sum_{s \in J} 1/n_s = r/N} (-1)^{|J|} e^{2\pi i\sum_{s \in J} a_s/n_s} \sum_{x = 0}^{N-1} e^{2\pi i(r-a)x/N}$$
Therefore \( n>m \), therefore (38) is valid for every \( \theta \in [0, 1] \).

Combining the above we obtain the desired result.

5. Proofs of Theorems 3.1–3.3

Proof of Theorem 3.1. (i) Assume \( |I| \leq m \) and \( I^*(c-n) = 1 \) where \( n \) is a nonnegative integer. Let \( (m_s)_{s \in I} \) be the unique tuple for which \( \sum_{s \in I} m_s/n_s = c - n \) and \( m_s \in R(n_s) \) for all \( s \in I \). Since \( \binom{m-n}{n} = 0 \) if \( n > m - |I| \), by (35) we have

\[
\sum_{\sum_{s \in J} 1/n_s = c} (-1)^{|J|} e^{2\pi i \sum_{s \in J} a_s/n_s} - (-1)^n \binom{m-n}{n} e^{2\pi i \sum_{s \in I} a_s m_s/n_s} = \sum_{l=0}^{m-|I|} (-1)^l \binom{m-|I|}{l} \sum_{x_s \in R(n_s) \text{ for } s \in I, \sum_{s \in I} x_s/n_s = c-l} e^{2\pi i \sum_{s \in I} a_s x_s/n_s}.
\]

Therefore \( \hat{I}_s(c) + \sum_{l=0}^{m-|I|} \binom{m-|I|}{l} I^*(c-l) \) is greater than or equal to

\[
\left| \sum_{\sum_{s \in J} 1/n_s = c} (-1)^{|J|} e^{2\pi i \sum_{s \in J} a_s/n_s} - \sum_{l=0}^{m-|I|} (-1)^l \binom{m-|I|}{l} \sum_{x_s \in R(n_s) \text{ for } s \in I, \sum_{s \in I} x_s/n_s = c-l} e^{2\pi i \sum_{s \in I} a_s x_s/n_s} \right| = (-1)^n \binom{m-n}{n} e^{2\pi i \sum_{s \in I} a_s m_s/n_s} = \binom{m-n}{n}.
\]
Now we suppose $|I| \geq m$ and $I_*(c-n) = 1$ where $n$ is a nonnegative integer. Let $I'$ be the unique subset of $I$ such that $\sum_{s \in I} 1/n_s = c - n$. By (36) we have

$$
\sum_{x_s \in R(n_s) \text{ for } s \in I} e^{2\pi i \sum_{s \in I} a_s x_s/n_s} = \sum_{|I| - m} (-1)^{|I| - m} \left( \begin{array}{c} m \\ n \end{array} \right) (-1)^{|I|} e^{2\pi i \sum_{s \in I'} a_s/n_s} 
$$

Thus (11) follows.

**Lemma.** Let $c_1, \ldots, c_k$ be nonnegative integers and $d_1, \ldots, d_l$ positive integers. Assume that there exist nonzero numbers $z_1, \ldots, z_k$ for which

$$
\sum_{s \in I} c_s z_s = 0 
$$

for those positive integers $t$ not divisible by any of $d_1, \ldots, d_l$. Then $c_1 + \ldots + c_k$ is the sum of some (not necessarily distinct) numbers among $d_1, \ldots, d_l$.

This is Lemma 9 of [Su4] and the initial idea is due to Y.-G. Chen.

**Proof of Theorem 3.2.** Let $d$ be an integer prime to $N = \prod_{s} n_s$. Since any integer can be written in the form $dx + Ny$ where $x, y \in \mathbb{Z}$, and $dx + Ny \equiv da_s \pmod{n_s}$ if and only if $x \equiv a_s \pmod{n_s}$, it follows that $A_d = \{da_s + n_s \mathbb{Z} \}_{s=1}^k$ also forms an exact $m$-cover of $\mathbb{Z}$. When $|I| \leq m$, by applying Theorem 4.1 to $A_d$ we get

$$
\sum_{J \subseteq I} (-1)^{|J|} e^{2\pi i d \sum_{s \in J} a_s/n_s} 
$$

that is,

$$
\sum_{w \in W_1} B_1(c, w) e^{2\pi i dw} \text{ is zero, where } W_1 \text{ is the union of the sets}
$$

$$
\left\{ \left\{ \sum_{s \in J} \frac{a_s}{n_s} \right\} : J \subseteq I \& \sum_{s \in J} \frac{1}{n_s} = c \right\}
$$

and

$$
\left\{ \left\{ \sum_{s \in I} \frac{x_s}{n_s} \right\} : x_s \in R(n_s) \text{ for } s \in I, \ c - \sum_{s \in I} \frac{x_s}{n_s} \in \{0, 1, \ldots, m - |I|\} \right\}.
$$
Exact $m$-covers

and

$$B_1(c, w) = \sum_{J \subseteq I} (-1)^{|J|} \left( \sum_{s \in J} a_s n_s = c \right) \left( \sum_{s \in J} x_s n_s = w \right)$$

$$- \sum_{n=0}^{m-|I|} (-1)^n \binom{m-|I|}{n} \left( \sum_{s \in I} x_s n_s = c-n \right) \left( \sum_{s \in I} a_s x_s n_s = w \right)$$

for $w \in W_1$. If $|I| \geq m$, then by applying Theorem 4.1 to $A_d$ we obtain the equality

$$\sum_{x_s \in R(n_s) \text{ for } s \in I} \sum_{s \in I} a_s x_s n_s \ e^{2\pi i d \sum_{s \in I} a_s x_s / n_s} = \sum_{n=0}^{|I|-m} (-1)^n \binom{|I|-m}{n} \sum_{J \subseteq I} (-1)^{|J|} \sum_{s \in J} a_s x_s n_s \ e^{2\pi i d \sum_{s \in J} a_s / n_s},$$

i.e., $\sum_{w \in W_2} B_2(c, w) e^{2\pi i d w} = 0$, where $W_2$ is the union of

$$\left\{ \left\{ \sum_{s \in J} a_s / n_s : J \subseteq I \right\} : \sum_{s \in J} x_s / n_s = c-n \text{ for some } n = 0, 1, \ldots, |I|-m \right\}$$

and

$$\left\{ \left\{ \sum_{s \in J} a_s x_s / n_s : x_s \in R(n_s) \text{ for } s \in I \right\} : \sum_{s \in I} x_s / n_s = c \right\},$$

and

$$B_2(c, w) = \left| \left\{ \left( x_s \right)_{s \in I} : x_s \in R(n_s) \text{ for } s \in I, \sum_{s \in I} x_s / n_s = c \text{ & } \sum_{s \in I} a_s x_s / n_s = w \right\} \right|$$

$$- \sum_{n=0}^{|I|-m} (-1)^n \binom{|I|-m}{n} \left( \sum_{J \subseteq I} (-1)^{|J|} \sum_{s \in J} a_s x_s / n_s = w \right)$$

for $w \in W_2$. 
Case 1: $|I| \leq m$. In this case (22) and (23) are obvious if $W_1 = \emptyset$. Suppose that $W_1$ is nonempty. If the inequality

$$
\tilde{I}_s(c) + \sum_{n=0}^{m-|I|} I^*(c-n) \geq |W_1| \geq p(N)
$$

fails or $N = 1$, then $\sum_{w \in W_1} B_1(c, w)e^{2\pi i dw} = 0$ for every $d = 1, \ldots, |W_1|$. Since

$$
|(e^{2\pi i dw})_{1 \leq d \leq |W_1|, w \in W_1}|/ \prod_{w \in W_1} e^{2\pi i w}
$$

is a determinant of Vandermonde’s type, $B_1(c, w) = 0$ for all $w \in W_1$ and hence (22) follows. When $|S|, |T| \leq 1$ and $S \cap T = \emptyset$ where $S$ and $T$ are as in Theorem 3.2, there is an $\varepsilon \in \{1, -1\}$ such that

$$
\varepsilon B_1(c, w) = |B_1(c, w)|
$$

$$
= \left| \left\{ J \subseteq \hat{I} : \sum_{s \in J} \frac{1}{n_s} = c & \left\{ \sum_{s \in J} \frac{a_s}{n_s} \right\} = w \right\} \right|
$$

$$
+ \sum_{n=0}^{m-|I|} \left( \begin{array}{c} m-|I| \\ n \end{array} \right) \left| \left\{ (x_s)_{s \in I} : x_s \in R(n_s) \text{ for } s \in I, \sum_{s \in I} \frac{x_s}{n_s} = c-n & \left\{ \sum_{s \in I} \frac{a_s x_s}{n_s} \right\} = w \right\} \right|
$$

for every $w \in W_1$. If $N \neq 1$ then $\sum_{w \in W_1} |B_1(c, w)|(e^{2\pi i dw}) = 0$ for all positive integers $d$ divisible by none of prime divisors of $N$ and therefore by the Lemma

$$
\tilde{I}_s(c) + \sum_{n=0}^{m-|I|} \left( \begin{array}{c} m-|I| \\ n \end{array} \right) I^*(c-n) = \sum_{w \in W_1} |B_1(c, w)| \in D(N).
$$

If $N = 1$ then the last equality also holds because $B_1(c, w) = 0$ for every $w \in W_1$.

Case 2: $|I| \geq m$. Apparently (25) and (26) are valid if $W_2 = \emptyset$. Now assume $|W_2| \geq 1$. If the equality

$$
I^*(c) + \sum_{n=0}^{|I|-m} \tilde{I}_s(c-n) \geq |W_2| \geq p(N)
$$

fails or $N$ equals one, then $\sum_{w \in W_2} B_2(c, w)e^{2\pi i dw} = 0$ for every $d = 1, \ldots, |W_2|$, hence $B_2(c, w) = 0$ for all $w \in W_2$ and so we have (25). If $c \neq n + \sum_{s \in J} 1/n_s$ for each $n = 0, 1, \ldots, |I| - m$ and $J \subseteq \hat{I}$ with $n \equiv |J|$ (mod 2), then
Exact m-covers

$$B_2(c, w)$$

$$= \left| \left\{ x_s : x_s \in R(n_s) \text{ for } s \in I, \sum_{s \in I} \frac{x_s}{n_s} = c \& \left\{ \sum_{s \in I} \frac{a_s x_s}{n_s} \right\} = w \right\} \right|$$

$$+ \sum_{n=0}^{\lfloor |I| - m \rfloor} \left( |I| - m \right) \left\{ J \subseteq I : \sum_{s \in J} \frac{1}{n_s} = c - n \& \left\{ \sum_{s \in J} \frac{a_s}{n_s} \right\} = w \right\}$$

for all $$w \in W_2$$, so with the help of the Lemma, whether $$N = 1$$ or not, (26) always holds.

**Proof of Theorem 3.3.** (i) First suppose $$|I| = m$$. Let $$r \in R([n_s]_{s \in I})$$.

In the light of Theorem 4.2,

$$\sum_{J \subseteq I} (-1)^{|J|} e^{2\pi i \sum_{s \in J} a_s / n_s}$$

$$= \sum_{x_s \in R(n_s) \text{ for } s \in I} e^{2\pi i \sum_{s \in I} a_s x_s / n_s} \cdot$$

As $$(n_s, n_t) | a_s - a_t$$ for all $$s, t \in I$$, by Corollary 2.3 the absolute value of the right hand side is

$$\prod_{s \in I} n_s / [n_s]_{s \in I}.$$ So

$$\sum_{n=0}^{m-1} \sum_{J \subseteq I} (-1)^{|J|} e^{2\pi i \sum_{s \in J} a_s / n_s}$$

$$\geq \sum_{x_s \in R(n_s) \text{ for } s \in I} \prod_{s \in I} n_s / [n_s]_{s \in I}.$$}

Next we handle the case where $$|I| \neq m$$. Choose an integer $$x$$ such that $$x \in \bigcap_{s \in I} a_s + n_s \mathbb{Z}$$ if $$I \neq \emptyset$$. Let

$$I' = \{ 1 \leq s \leq k : x \equiv a_s \text{ (mod } n_s) \}.$$ Then $$|I'| = m$$ and $$I' \supset I$$. By the previous argument,

$$\left\{ J \subseteq \{1, \ldots, k\} \setminus I' : \left\{ \sum_{s \in J} \frac{1}{n_s} \right\} = \frac{a}{[n_s]_{s \in I'}} \right\} \geq \prod_{s \in I' \setminus I} n_s / [n_s]_{s \in I'}$$

for every $$a \in R([n_s]_{s \in I'})$$. So, for any $$r \in R([n_s]_{s \in I})$$, we have
\[
\left\{ J \subseteq \tilde{I} : \left\{ \sum_{s \in J} \frac{1}{n_s} \right\} = \frac{r}{[n_s]_{s \in I}} \right\} \\
\geq \left\{ J \subseteq \{1, \ldots, k\} \setminus I' : \left\{ \sum_{s \in J} \frac{1}{n_s} \right\} = \frac{r[n_s]_{s \in I'}}{[n_s]_{s \in I}} \right\} \\
\geq \prod_{s \in I'n_s} \frac{\prod_{s \in I'} n_s \cdot \prod_{s \in I' \setminus I} n_s}{\prod_{s \in I} n_s} \\
\geq \prod_{s \in I} n_s.
\]

(ii) If \([n_s]_{s \in I} \not\in \mathbb{Z}, then \{\sum_{s \in I} x_s/n_s\} \neq \emptyset whenever \(x_s \in R(n_s)\) for all \(s \in I\), and thus by Theorem 4.2,

\[
(*) \quad \sum_{w \in W} e^{2\pi iw} \sum_{\left\{ \sum_{s \in J} 1/n_s \right\} = \theta} (-1)^{|J|} = \sum_{\left\{ \sum_{s \in J} 1/n_s \right\} = \theta} (-1)^{|J|} e^{2\pi i \sum_{s \in J} a_s/n_s} = 0
\]

where

\[ W = \left\{ \left\{ \sum_{s \in J} \overline{a_s} \right\} : J \subseteq \tilde{I} \text{ and } \left\{ \sum_{s \in J} 1/n_s \right\} = \emptyset \right\}. \]

If \((n_{s_1}, n_{s_2}) \nmid a_{s_1} - a_{s_2}\) for some \(s_1, s_2 \in I\), then \(\{a_s + n_s \mathbb{Z}\}_{s \in I}\) covers each integer at most \(m-1\) times because \(a_{s_1} + n_{s_1} \mathbb{Z} \cap a_{s_2} + n_{s_2} \mathbb{Z} = \emptyset\), therefore system \(\{a_s + n_s \mathbb{Z}\}_{s \in I}\) forms a cover of \(\mathbb{Z}\) and \((*)\) holds by Theorem 1 of [Su3]. For each integer \(a\) prime to \([n_s]_{s \in I}\), by applying the automorphism \(\sigma_a\) of the cyclotomic field \(\mathbb{Q}(e^{2\pi i/[n_s]_{s \in I}})\) with \(\sigma_a(e^{2\pi i/[n_s]_{s \in I}}) = e^{2\pi ia/[n_s]_{s \in I}}\) we obtain from \((*)\) the equality

\[
(*)_a \quad \sum_{w \in W} (e^{2\pi iw})^a \sum_{\left\{ \sum_{s \in J} 1/n_s \right\} = \theta} (-1)^{|J|} = 0.
\]

Observe that

\[ |W| \leq \left\{ J \subseteq \tilde{I} : \left\{ \sum_{s \in J} 1/n_s \right\} = \theta \right\} = \sum_{n=0}^{m-1} \tilde{I}_s(n + \emptyset). \]

If \(0 < |W| < p([n_s]_{s \in I})\), then \((*)_a\) holds for every \(a = 1, \ldots, |W|\), hence

\[ \sum_{\left\{ \sum_{s \in J} 1/n_s \right\} = \emptyset} (-1)^{|J|} = 0 \text{ for all } w \in W. \]

\[
\left\{ J \subseteq \tilde{I} : \left\{ \sum_{s \in J} 1/n_s \right\} = \emptyset \right\} = \sum_{n=0}^{m-1} \tilde{I}_s(n + \emptyset).
\]
and in particular
\[
\sum_{\substack{J \subseteq I, 2 \mid |J| \\ \{\sum_{s \in J} 1/n_s\} = \theta}} 1 - \sum_{\substack{J \subseteq I, 2 \mid |J| \\ \{\sum_{s \in J} 1/n_s\} = \theta}} 1 = \sum_{\substack{J \subseteq I \\ \{\sum_{s \in J} 1/n_s\} = \theta}} (-1)^{|J|} = 0,
\]
for the determinant of the matrix \((e^{2\pi i w_n})_{1 \leq n \leq |W|})_{w \in W} is nonzero. In the case \(W = 0\) we obviously have the last equality and (32). Assume \(W \neq 0\) below. Provided that all the \(|J| \mod 2\) with \(J \subseteq \bar{I}\) and \(\{\sum_{s \in J} 1/n_s\} = \theta\) are the same, if \([n_s]_{s \in I} = 1\) then \(\theta = 0\) and we must have \(\bar{I} = \emptyset\), i.e. \(k = |I| = m\), which contradicts the fact that \((n_i, n_j) \mid a_i - a_j\) for some \(i, j \in I\); if \([n_s]_{s \in I} > 1\), then (32) follows from the Lemma and the validity of \((*)\) for all integers \(a\) prime to \([n_s]_{s \in I}\). This ends the proof.

Acknowledgments. The author is indebted to Professor R. K. Guy for his sending the related sections from the new edition of his book [Gu]. The paper was written during the author’s visit in Italy; he is grateful to Trento University for financial support. He also thanks the referee for his valuable advice.

References


[GLS] A. Granville, S.-G. Li and Q. Sun, On the number of solutions of the equation \(\sum_{i=1}^n x_i/d_i \equiv 0 \pmod{1}\), and of diagonal equations in finite fields, J. Sichuan Univ. (Nat. Sci. Ed.) 32 (3) (1995), 243–248.


Department of Mathematics
Nanjing University
Nanjing 210093
People’s Republic of China
E-mail: (care of) zhengch@public1.ptt.js.cn

Received on 9.5.1996
and in revised form on 8.1.1997