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On a theorem of Bombieri–Vinogradov type for prime numbers from a thin set

by

D. I. Tolev (Plovdiv)

In 1940 I. M. Vinogradov considered the set

$$S_\lambda = \{p \text{ prime} \mid \{\sqrt{p}\} < p^{-\lambda}\},$$

where $\lambda > 0$ is a fixed number and $\{t\}$ denotes the fractional part of t. Vinogradov proved ([17], Chapter 4) that if $0 < \lambda < 1/10$ then

$$\sum_{p \le x,\, p \in S_\lambda} 1 \sim \frac{x^{1-\lambda}}{(1-\lambda)\log x} \quad \text{as } x \to \infty. \tag{1}$$

A different approach to this problem was developed by Linnik [11] in 1945. In 1979 Kaufman [10] used the method of Linnik and proved the asymptotic formula (1) for $\lambda < 0.1631\ldots$ He also proved that if the Riemann Hypothesis is assumed then (1) holds for $\lambda < 1/4$.

In 1983 Balog [1] and Harman [8] used Vaughan's identity and mean value estimates for Dirichlet polynomials and independently proved without assuming the Riemann Hypothesis that the formula (1) is true for $\lambda < 1/4$. Later Balog [2] generalized his result to prime numbers in arithmetic progressions. We should also mention the works of Schoissengeier [14], [15], Gritsenko [7] and Rivat [13].

In the present paper we use the method of Balog and Harman and we prove a theorem of Bombieri–Vinogradov type for prime numbers from the set S_λ.

Let λ, θ be real numbers such that

$$0 < \lambda < 1/4, \qquad 0 < \theta < 1/4 - \lambda. \tag{2}$$

Let x be a sufficiently large real number, $\mathcal{L} = \log x$; $y, u, v, t, \alpha, \nu, \tau, V, Y, K, M, N, D$ real numbers; a, k, l, m, n integers; A an arbitrarily large positive

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number; ε an arbitrarily small positive number. In formulas which do not involve ε the constants in O-terms and $\ll$-symbols are absolute or depend only on A, λ, θ. In formulas which involve ε the constants also depend on ε. As usual, $[t]$ denotes the integer part of t, $e(t) = e^{2\pi i t}$; $\mu(n)$, $\Lambda(n)$, $\varphi(n)$, $\tau(n)$ denote Möbius' function, von Mangoldt's function, Euler's function and the number of positive divisors of n, respectively.

$\sum_{\chi \bmod k}$ denotes the sum over all characters $\pmod k$, and $\sum^*_{\chi \bmod k}$ the sum over all primitive characters $\pmod k$; finally,

$$\psi_\lambda(y;k,a) = \sum_{\substack{n\le y\\ n\equiv a \,(\mathrm{mod}\, k)\\ \{\sqrt{n}\}<n^{-\lambda}}} \Lambda(n).$$

We prove the following theorem:

Theorem. *If λ and θ satisfy* (2) *and $A>0$ is arbitrarily large then*

$$E = \sum_{k\le x^\theta} \max_{y\le x} \max_{(a,k)=1} \left| \psi_\lambda(y;k,a) - \frac{y^{1-\lambda}}{\varphi(k)(1-\lambda)} \right| \ll x^{1-\lambda}\mathcal{L}^{-A}.$$

Proof. We may suppose that $A>10$. Let $B=10A$. If $k\le x^\theta$ then using only a simple counting argument we find

$$\psi_\lambda(x\mathcal{L}^{-B};k,a) \ll \mathcal{L} \sum_{\substack{n\le x\mathcal{L}^{-B}\\ n\equiv a \,(\mathrm{mod}\, k)\\ \{\sqrt{n}\}<n^{-\lambda}}} 1 \ll k^{-1}x^{1-\lambda}\mathcal{L}^{2-B/2}. \tag{3}$$

Note that to prove the last estimate the upper bound for θ need not be so tight as in (2). The same happens in other places as well. We use the strong restriction $\theta < 1/4-\lambda$ only at the end of the proof to obtain (38) and (39) from (36) and (37).

From (3) we get

$$E \ll E_1 + x^{1-\lambda}\mathcal{L}^{-A}, \tag{4}$$

where

$$E_1 = \sum_{k\le x^\theta} \max_{x\mathcal{L}^{-B}\le y\le x} \max_{(a,k)=1} \left| \psi_\lambda(y;k,a) - \frac{y^{1-\lambda}}{\varphi(k)(1-\lambda)} \right|.$$

Define $u_v = v(1-(\log v)^{-B})$, and

$$S^*_\lambda(v;k,a) = \sum_{\substack{u_v<n\le v\\ n\equiv a \,(\mathrm{mod}\, k)\\ \{\sqrt{n}\}<n^{-\lambda}}} \Lambda(n), \quad S_\lambda(v;k,a) = \sum_{\substack{u_v<n\le v\\ n\equiv a \,(\mathrm{mod}\, k)\\ \{\sqrt{n}\}<\sqrt{n}v^{-1/2-\lambda}}} \Lambda(n),$$

$$P^*_\lambda(v;k,a)=\sum_{\substack{u_v<n\le v\\ n\equiv a\,(\mathrm{mod}\,k)}}\Lambda(n)n^{-\lambda},\quad P_\lambda(v;k,a)=v^{-1/2-\lambda}\sum_{\substack{u_v<n\le v\\ n\equiv a\,(\mathrm{mod}\,k)}}\Lambda(n)n^{1/2}.$$

It is not difficult to see that if $x\mathcal{L}^{-2B}\le v\le x$ and $k\le x^\theta$ then

$$\begin{aligned}S^*_\lambda(v;k,a)-S_\lambda(v;k,a)&\ll k^{-1}x^{1-\lambda}\mathcal{L}^{1-2B},\\ P^*_\lambda(v;k,a)-P_\lambda(v;k,a)&\ll k^{-1}x^{1-\lambda}\mathcal{L}^{1-2B}.\end{aligned}\tag{5}$$

Let us prove, for example, the first of the inequalities above. We have

$$S^*_\lambda(v;k,a)-S_\lambda(v;k,a)\ll\mathcal{L}\sum_{\substack{u_v<n\le v\\ n\equiv a\,(\mathrm{mod}\,k)\\ \sqrt{n}\,v^{-1/2-\lambda}\le\{\sqrt{n}\}<\sqrt{n}\,u_v^{-1/2-\lambda}}}1.$$

If $l^2\le n<(l+1)^2$ and $\sqrt{n}\,v^{-1/2-\lambda}\le\{\sqrt{n}\}<\sqrt{n}\,u_v^{-1/2-\lambda}$ then we have

$$l^2(1-v^{-1/2-\lambda})^{-2}\le n<l^2(1-u_v^{-1/2-\lambda})^{-2}.$$

We use the definition of u_v and the restriction imposed on k and after some calculations we find that the expression being estimated is

$$\begin{aligned}&\ll\mathcal{L}\sum_{\sqrt{u_v}-1<l\le\sqrt{v}}(1+k^{-1}l^2((1-u_v^{-1/2-\lambda})^{-2}-(1-v^{-1/2-\lambda})^{-2}))\\ &\ll k^{-1}x^{1-\lambda}\mathcal{L}^{1-2B}.\end{aligned}$$

For each $y\in[x\mathcal{L}^{-B},x]$ we define the sequence y_i, $0\le i\le i_0$, in the following way:

$$y_0=y,\quad y_{i+1}=y_i(1-(\log y_i)^{-B}),\quad y_{i_0+1}<y(\log y)^{-B}\le y_{i_0}.\tag{6}$$

Clearly

$$i_0\ll\mathcal{L}^{B+1}.\tag{7}$$

If $y\in[x\mathcal{L}^{-B},\,x]$ then using (3), (5)–(7) we get

$$\begin{aligned}&\psi_\lambda(y;k,a)-\sum_{\substack{n\le y\\ n\equiv a\,(\mathrm{mod}\,k)}}\Lambda(n)n^{-\lambda}\\ &\quad\ll\sum_{0\le i\le i_0}(|S^*_\lambda(y_i;k,a)-S_\lambda(y_i;k,a)|+|S_\lambda(y_i;k,a)-P_\lambda(y_i;k,a)|\\ &\qquad+|P_\lambda(y_i;k,a)-P^*_\lambda(y_i;k,a)|)+k^{-1}x^{1-\lambda}\mathcal{L}^{2-B/2}\\ &\quad\ll\sum_{0\le i\le i_0}|S_\lambda(y_i;k,a)-P_\lambda(y_i;k,a)|+k^{-1}x^{1-\lambda}\mathcal{L}^{2-B/2}.\end{aligned}$$

The last inequality and (7) imply

$$E_1 \ll E_2 + \mathcal{L}^{B+1} E_3 + x^{1-\lambda}\mathcal{L}^{-A}, \tag{8}$$

where

$$E_2 = \sum_{k\le x^\theta} \max_{x\mathcal{L}^{-B}\le y\le x} \max_{(a,k)=1} \Bigg| \sum_{\substack{n\le y\\ n\equiv a\,(\mathrm{mod}\,k)}} \Lambda(n) n^{-\lambda} - \frac{y^{1-\lambda}}{\varphi(k)(1-\lambda)} \Bigg|,$$

$$E_3 = \sum_{k\le x^\theta} \max_{x\mathcal{L}^{-2B}\le v\le x} \max_{(a,k)=1} |S_\lambda(v;k,a) - P_\lambda(v;k,a)|.$$

We use the Bombieri–Vinogradov theorem [4] to obtain

$$E_2 \ll x^{1-\lambda}\mathcal{L}^{-A}. \tag{9}$$

Obviously

$$E_3 \ll \mathcal{L} \max_{x\mathcal{L}^{-2B}\le V\le x} E_4, \tag{10}$$

where

$$E_4 = E_4(V) = \sum_{k\le x^\theta} \max_{V/2\le v\le V} \max_{(a,k)=1} |S_\lambda(v;k,a) - P_\lambda(v;k,a)|.$$

Hence, using (4) and (8)–(10) we have

$$E \ll \mathcal{L}^{B+2} \max_{x\mathcal{L}^{-2B}\le V\le x} E_4 + x^{1-\lambda}\mathcal{L}^{-A}. \tag{11}$$

Suppose that

$$\begin{gathered} x\mathcal{L}^{-2B} \le V \le x, \quad V/2 \le v \le V, \quad T = V^{1/2+\lambda}\mathcal{L}^{2B}, \\ T_0 = V^{1/2}/10, \quad k \le x^\theta, \quad (a,k) = 1. \end{gathered} \tag{12}$$

Let χ be a character $\pmod k$. We define

$$F(s) = \sum_{\substack{u_v<n\le v\\ n\equiv a\,(\mathrm{mod}\,k)}} \Lambda(n) n^s, \qquad F_\chi(s) = \sum_{u_v<n\le v} \chi(n)\Lambda(n) n^s,$$

$$L(s) = \sum_{V^{1/2}/10<l\le 10V^{1/2}} l^{-s}, \qquad H(s) = \frac{1}{s}(1-(1-v^{-1/2-\lambda})^s),$$

$$I = \frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT} F(s/2)L(s)H(s)\,ds, \qquad I_0 = \frac{1}{2\pi i}\int_{1/2-iT_0}^{1/2+iT_0} F(s/2)L(s)H(s)\,ds.$$

We use Perron's formula ([6], §17) to get

$$S_\lambda(v;k,a) = \sum_{\substack{u_v<n\le v\\ n\equiv a\,(\mathrm{mod}\,k)}} \Lambda(n)([\sqrt{n}\,] - [\sqrt{n}(1-v^{-1/2-\lambda})]) \tag{13}$$

$$
\begin{aligned}
&= \sum_{\substack{u_v < n \le v \\ n \equiv a \,(\mathrm{mod}\, k)}} \Lambda(n) \\
&\quad \times \sum_{V^{1/2}/10 < l \le 10V^{1/2}} \bigg(\frac{1}{2\pi i} \int_{1/2-iT}^{1/2+iT} n^{s/2} l^{-s} H(s)\, ds \\
&\quad + O\bigg(\min\bigg(1, T^{-1} \bigg| \log \frac{\sqrt{n}}{l} \bigg|^{-1}\bigg) \\
&\quad + \min\bigg(1, T^{-1} \bigg| \log \frac{\sqrt{n}}{l} (1 - v^{-1/2-\lambda}) \bigg|^{-1}\bigg)\bigg)\bigg) \\
&= I + O(\mathcal{L}(\Delta_1 + \Delta_2)),
\end{aligned}
$$

where

$$\Delta_1 = \sum_{\substack{V/4 \le n \le V \\ n \equiv a \,(\mathrm{mod}\, k)}} \sum_{V^{1/2}/10 < l \le 10V^{1/2}} \min\bigg(1, T^{-1} \bigg| \log \frac{\sqrt{n}}{l} \bigg|^{-1}\bigg),$$

$$\Delta_2 = \sum_{\substack{V/4 \le n \le V \\ n \equiv a \,(\mathrm{mod}\, k)}} \sum_{V^{1/2}/10 < l \le 10V^{1/2}} \min\bigg(1, T^{-1} \bigg| \log \frac{\sqrt{n}}{l} (1 - v^{-1/2-\lambda}) \bigg|^{-1}\bigg).$$

We use (12) and after some standard calculations we obtain

$$\Delta_1, \Delta_2 \ll k^{-1} x^{1-\lambda} \mathcal{L}^{2-2B}. \tag{14}$$

If $s = 1/2 + it$, $|t| \le T_0$ then we may approximate the exponential sum $L(s)$ by an integral ([9], Chapter III, §1, Corollary 1) to get

$$L(s) = \frac{(10V^{1/2})^{1-s} - (V^{1/2}/10)^{1-s}}{1-s} + O(V^{-1/4}) = O\bigg(\frac{V^{1/4}}{1+|t|}\bigg).$$

We also have

$$H(s) \ll V^{-1/2-\lambda}, \qquad H(s) = v^{-1/2-\lambda} + O(|s-1| V^{-1-2\lambda}). \tag{15}$$

Hence

$$
\begin{aligned}
I_0 &= \frac{v^{-1/2-\lambda}}{2\pi i} \int_{1/2-iT_0}^{1/2+iT_0} F(s/2) \frac{(10V^{1/2})^{1-s} - (V^{1/2}/10)^{1-s}}{1-s}\, ds \\
&\quad + O\bigg(V^{-3/4-\lambda} \int_{-T_0}^{T_0} \big| F\big(\tfrac14 + \tfrac12 it\big) \big|\, dt \bigg).
\end{aligned} \tag{16}
$$

Using the orthogonality of characters (mod k), Cauchy's inequality and Theorem 6.1 of [12] we get

$$\int_{-T_0}^{T_0} \left|F\left(\tfrac14+\tfrac12 it\right)\right| dt \ll \frac{1}{\varphi(k)} \sum_{\chi \bmod k} \int_{-T_0}^{T_0} \left|F_\chi\left(\tfrac14+\tfrac12 it\right)\right| dt$$
$$\ll \frac{1}{\varphi(k)} \sum_{\chi \bmod k} T_0^{1/2}\Big(\int_{-T_0}^{T_0} \left|F_\chi\left(\tfrac14+\tfrac12 it\right)\right|^2 dt\Big)^{1/2}$$
$$\ll x^{3/2}\mathcal{L}.$$

We substitute the last estimate in (16) and apply Perron's formula again. We get

$$I_0 = P_\lambda(v;k,a) + O(k^{-1}x^{1-\lambda}\mathcal{L}^{-2B}). \tag{17}$$

From (13)–(17) and from the orthogonality of the characters (mod k) we obtain

$$S_\lambda(v;k,a) - P_\lambda(v;k,a)$$
$$\ll k^{-1}x^{1-\lambda}\mathcal{L}^{3-2B} + \frac{V^{-1/2-\lambda}}{\varphi(k)} \sum_{\chi \bmod k} \int_{T_0}^{T} \left|F_\chi\left(\tfrac14+\tfrac12 it\right)\right| \cdot \left|L\left(\tfrac12+it\right)\right| dt.$$

The last estimate and (11) imply

$$E \ll \mathcal{L}^{B+3} \max_{x\mathcal{L}^{-2B}\le V\le x} (V^{-1/2-\lambda}E_5) + x^{1-\lambda}\mathcal{L}^{-A}, \tag{18}$$

where

$$E_5 = E_5(V) = \sum_{k\le x^\theta} \frac1k \sum_{\chi \bmod k} \max_{V/2\le v\le V} \int_{T_0}^{T} \left|F_\chi\left(\tfrac14+\tfrac12 it\right)\right| \cdot \left|L\left(\tfrac12+it\right)\right| dt.$$

We approximate the exponential sum L in the last expression by a shorter one ([9], Chapter III, §1, Theorem 1) and we obtain

$$\left|L\left(\tfrac12+it\right)\right| \ll 1 + \Big|\sum_{tV^{-1/2}/(20\pi)<l\le 5tV^{-1/2}/\pi} l^{-1/2-it}\Big| = 1 + |L_1(t)|,$$

say. Hence we have

$$\begin{aligned} E_5 &\ll \sum_{k\le x^\theta} \frac1k \sum_{\chi \bmod k} \max_{V/2\le v\le V} \int_{T_0}^{T} \left|F_\chi\left(\tfrac14+\tfrac12 it\right)\right|(1+|L_1(t)|)\, dt \\ &\ll \mathcal{L} \max_{x^{1/2}\mathcal{L}^{-1-B}\le Y\le x^{1/2+\lambda}\mathcal{L}^{2B}} E_6, \end{aligned} \tag{19}$$

where

$$E_6 = E_6(V,Y) = \sum_{k\le x^\theta} \frac1k \sum_{\chi \bmod k} \max_{V/2\le v\le V} \int_{Y/2}^{Y} \left|F_\chi\left(\tfrac14+\tfrac12 it\right)\right| (1+|L_1(t)|)\, dt.$$

The interval of summation in $L_1(t)$ depends on t. To get rid of this dependence we apply, for example, Lemma 2.2 of [5] to get

$$|L_1(t)| \ll \int_{-\infty}^{\infty} K(\alpha)|L_2(t,\alpha)|\,d\alpha,$$

where

$$L_2(t,\alpha) = \sum_{YV^{-1/2}/200<l\le 2YV^{-1/2}} e(\alpha l) l^{-1/2-it}$$

and where the kernel $K(\alpha)$ depends only on α, Y, V and satisfies the inequalities

$$K(\alpha)\ge 0, \qquad 1 \ll \int_{-\infty}^{\infty} K(\alpha)\,d\alpha \ll \mathcal{L}.$$

Therefore

$$E_6 \ll \mathcal{L} \max_{0\le\alpha\le 1} E_7, \tag{20}$$

where

$$\begin{aligned} E_7 &= E_7(V,Y,\alpha)\\ &= \sum_{k\le x^\theta} \frac{1}{k} \sum_{\chi \bmod k} \max_{V/2\le v\le V} \int_{Y/2}^{Y} \big|F_\chi\big(\tfrac14+\tfrac12 it\big)\big|(1+|L_2(t,\alpha)|)\,dt. \end{aligned}$$

We use the properties of primitive characters and the inequality

$$\int_{Y/2}^{Y} |L_2(t,\alpha)|\,dt \ll Y, \tag{21}$$

which is a consequence of Cauchy's inequality and Theorem 6.1 of [12]. After some calculations we get

$$E_7 \ll \mathcal{L}(E_8+E_9)+x, \tag{22}$$

where

$$E_8 = E_8(V,Y,\alpha) = \max_{V/2\le v\le V} \int_{Y/2}^{Y} \Big|\sum_{u_v<n\le v} \Lambda(n) n^{1/4+it/2}\Big|(1+|L_2(t,\alpha)|)\,dt,$$

$$\begin{aligned} E_9 &= E_9(V,Y,\alpha)\\ &= \sum_{k\le x^\theta} \frac{1}{k} \sideset{}{^*}\sum_{\chi \bmod k} \max_{V/2\le v\le V} \int_{Y/2}^{Y} \big|F_\chi\big(\tfrac14+\tfrac12 it\big)\big|(1+|L_2(t,\alpha)|)\,dt. \end{aligned}$$

It remains to prove that if V and Y satisfy the conditions imposed in (18), (19) then we have

$$E_8, E_9 \ll x^{3/2-\nu} \tag{23}$$

for some $\nu > 0$. The proof of the theorem follows from (18)–(20), (22), (23).

Let us consider E_9. The estimation of E_8 is similar and, in fact, it was done in [1]. Clearly

$$E_9 \ll \mathcal{L} \max_{K \le x^\theta} (K^{-1} E_{10}), \tag{24}$$

where

$$E_{10} = E_{10}(V, Y, \alpha, K) = \sum_{k \le K} \sideset{}{^*}\sum_{\chi \bmod k} \max_{V/2 \le v \le V} \int_{Y/2}^{Y} \left| F_\chi\left(\tfrac{1}{4} + \tfrac{1}{2} it\right) \right| (1 + |L_2(t, \alpha)|) \, dt.$$

Let

$$D = x^{\lambda + (1 - 4\lambda)/400}. \tag{25}$$

We apply Vaughan's identity [16] to get

$$F_\chi\left(\tfrac{1}{4} + \tfrac{1}{2} it\right) = F_1 - F_2 - F_3 - F_4,$$

where

$$F_1 = \sum_{m \le D} \sum_{u_v/m < n \le v/m} \mu(m)(\log n) \chi(mn)(mn)^{1/4 + it/2},$$

$$F_2 = \sum_{m \le D} \sum_{u_v/m < n \le v/m} c(m) \chi(mn)(mn)^{1/4 + it/2},$$

$$F_3 = \sum_{D < m \le D^2} \sum_{u_v/m < n \le v/m} c(m) \chi(mn)(mn)^{1/4 + it/2},$$

$$F_4 = \sum\sum_{u_v < mn \le v,\, m, n > D} a(m) \Lambda(n) \chi(mn)(mn)^{1/4 + it/2},$$

$$|c(m)| \le \log m, \qquad |a(m)| \le \tau(m).$$

We have

$$E_{10} \ll E_{10}^{(1)} + E_{10}^{(2)} + E_{10}^{(3)} + E_{10}^{(4)}, \tag{26}$$

where $E_{10}^{(i)}$ denotes the contribution to E_{10} arising from F_i.

Let us consider $E_{10}^{(1)}$. We have

$$F_1 = \sum_{m \le D} \mu(m) \chi(m) m^{1/4 + it/2} W_m,$$

where

$$W_m = \sum_{u_v/m<n\le v/m} (\log n)\chi(n)n^{1/4+it/2} = \sum_{1\le l\le k}\chi(l) \sum_{\substack{u_v/m<n\le v/m\\ n\equiv l \,(\mathrm{mod}\, k)}} (\log n)n^{1/4+it/2} = \sum_{1\le l\le k}\chi(l)\Gamma_l,$$

say. We use (2) and (25) to conclude that the exponential sum Γ_l may be approximated by an integral ([9], Chapter III, §1, Corollary 1). More precisely, we have

$$\Gamma_l = \frac{1}{k}\int_{u_v/m}^{v/m} (\log y)y^{1/4+it/2}\,dy + O\bigg(\Big(\frac{x}{m}\Big)^{1/4}\mathcal{L}\bigg).$$

Since the character χ is primitive we have $\sum_{1\le l\le k}\chi(l)=0$. Hence

$$W_m \ll K\Big(\frac{x}{m}\Big)^{1/4}\mathcal{L}, \qquad F_1 \ll DKx^{1/4}\mathcal{L}.$$

The last estimate and (21) imply

$$E_{10}^{(1)} \ll DK^3x^{1/4}Y\mathcal{L}. \tag{27}$$

Using the bounds for Y and K imposed in (19) and (24) and also (2), (25), (27) we get

$$E_{10}^{(1)} \ll Kx^{11/8}. \tag{28}$$

We estimate $E_{10}^{(2)}$ analogously and we obtain

$$E_{10}^{(2)} \ll Kx^{11/8}. \tag{29}$$

Consider now $E_{10}^{(4)}$. We have

$$\begin{aligned} E_{10}^{(4)} &\ll \sum_{k\le K}\sideset{}{^*}\sum_{\chi \bmod k} \max_{V/4\le v\le V} \int_{Y/2}^{Y} \Big|\sum\sum_{\substack{mn\le v\\ m,n>D}} a(m)\Lambda(n)\chi(mn)(mn)^{1/4+it/2}\Big| \\ &\qquad\qquad \times(1+|L_2(t,\alpha)|)\,dt \\ &\ll \mathcal{L}^2 \max_{\substack{D\le M,N\le x/D\\ MN\le x}} E_{11}, \end{aligned} \tag{30}$$

where

$$\begin{aligned} E_{11} &= E_{11}(V,Y,\alpha,K,M,N) \\ &= \sum_{k\le K}\sideset{}{^*}\sum_{\chi \bmod k} \max_{V/4\le v\le V} \int_{Y/2}^{Y} |F^*(t)|(1+|L_2(t,\alpha)|)\,dt, \end{aligned} \tag{31}$$

$$F^*(t) = \sum_{\substack{M<m\le 2M\\ N<n\le 2N\\ mn\le v}}\sum a(m)\Lambda(n)\chi(mn)(mn)^{1/4+it/2}.$$

We may suppose that the maximum in (31) is taken over v of the form $1/2+l$ where l is an integer. Applying again Perron's formula we obtain

$$F^*(t) = \sum_{M<m\le 2M}\sum_{N<n\le 2N} a(m)\Lambda(n)\chi(mn)(mn)^{1/4+it/2} \times\bigg(\frac{1}{2\pi i}\int_{\mathcal{L}^{-1}-ix}^{\mathcal{L}^{-1}+ix}\Big(\frac{v}{mn}\Big)^s\frac{ds}{s} + O\Big(x^{-1}\Big|\log\frac{v}{mn}\Big|^{-1}\Big)\bigg).$$

Hence

$$F^*(t) \ll \mathcal{L}\int_{-x}^{x}|\mathcal{M}|\cdot|\mathcal{N}|\frac{d\tau}{1+|\tau|} + x^{1/3}, \tag{32}$$

where

$$\mathcal{M} = \sum_{M<m\le 2M} a(m)\chi(m)m^{1/4-\mathcal{L}^{-1}+i(t/2-\tau)},$$
$$\mathcal{N} = \sum_{N<n\le 2N} \Lambda(n)\chi(n)n^{1/4-\mathcal{L}^{-1}+i(t/2-\tau)}.$$

Formulas (21), (31) and (32) imply

$$E_{11} \ll \mathcal{L}\int_{-x}^{x} E_{12}\,\frac{d\tau}{1+|\tau|} + Kx^{13/12} \tag{33}$$

where

$$E_{12} = E_{12}(Y,\alpha,K,M,N,\tau) = \sum_{k\le K}\ \sideset{}{^*}\sum_{\chi \bmod k}\int_{Y/2}^{Y}|\mathcal{M}|\cdot|\mathcal{N}|(1+|L_2(t,\alpha)|)\,dt.$$

Suppose, for example, that $M\le N$. Then M,N satisfy

$$D\le M\le x^{1/2},\qquad D\le N\le x/D,\qquad MN\le x. \tag{34}$$

By the Cauchy inequality we have

$$E_{12} \ll (E_{13})^{1/2}(E_{14})^{1/2}, \tag{35}$$

where

$$E_{13} = E_{13}(Y,\alpha,K,N,\tau) = \sum_{k\le K}\ \sideset{}{^*}\sum_{\chi \bmod k}\int_{Y/2}^{Y}|\mathcal{N}|^2\,dt,$$

$$E_{14} = E_{14}(Y,\alpha,K,M,\tau) = \sum_{k\le K}\ \sideset{}{^*}\sum_{\chi \bmod k}\int_{Y/2}^{Y}|\mathcal{M}|^2(1+|L_2(t,\alpha)|^2)\,dt.$$

We estimate E_{13} by Theorem 7.1 of [12] to get

$$E_{13} \ll \mathcal{L}(K^2Y + N)N^{3/2}. \tag{36}$$

To estimate the integral in the expression for E_{14} we use Theorem 6.1 of [12] and also Theorem 1 of [3]. We obtain

$$\int_{Y/2}^{Y} |\mathcal{M}|^2(1 + |L_2(t,\alpha)|^2)\,dt \ll x^{\varepsilon}M^{3/2}Y,$$

hence

$$E_{14} \ll x^{\varepsilon}M^{3/2}K^2Y. \tag{37}$$

We use (2), (24), (25), (30), (33)–(37) to get

$$E_{10}^{(4)} \ll Kx^{3/2-\nu} \tag{38}$$

for some $\nu > 0$. Let us point out that only in this place do we need the tight restriction $\theta < 1/4 - \lambda$.

We proceed with $E_{10}^{(3)}$ analogously to obtain

$$E_{10}^{(3)} \ll Kx^{3/2-\nu} \tag{39}$$

for some $\nu > 0$.

We use (24), (26), (28), (29), (38), (39) to find that $E_9 \ll x^{3/2-\nu}$ for some $\nu > 0$. The estimation of E_8 is similar, so we have proved (23) and the proof of the theorem is complete.

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Department of Mathematics
Plovdiv University "P. Hilendarski"
Plovdiv 4000, Bulgaria
E-mail: dtolev@ulcc.uni-plovdiv.bg

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