On distribution functions of $\xi(3/2)^n \mod 1$

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1. Preliminary remarks. The question about distribution of $(3/2)^n \mod 1$ is most difficult. We present a selection of known conjectures:

(i) $(3/2)^n \mod 1$ is uniformly distributed in $[0,1]$.

(ii) $(3/2)^n \mod 1$ is dense in $[0,1]$.

(iii) (T. Vijayaraghavan [11])

$$\limsup_{n \to \infty} \{ (3/2)^n \} - \liminf_{n \to \infty} \{ (3/2)^n \} > 1/2,$$

where $\{x\}$ is the fractional part of $x$.

(iv) (K. Mahler [6]) There exists no $\xi \in \mathbb{R}^+$ such that $0 \leq \{\xi(3/2)^n\} < 1/2$ for $n = 0, 1, 2, \ldots$

(v) (G. Choquet [2]) There exists no $\xi \in \mathbb{R}^+$ such that the closure of $\{\{\xi(3/2)^n\}; n = 0, 1, 2, \ldots\}$ is nowhere dense in $[0,1]$.

Few positive results are known. For instance, L. Flatto, J. C. Lagarias and A. D. Pollington [3] showed that

$$\limsup_{n \to \infty} \{\xi(3/2)^n\} - \liminf_{n \to \infty} \{\xi(3/2)^n\} \geq 1/3$$

for every $\xi > 0$.

G. Choquet [2] gave infinitely many $\xi \in \mathbb{R}$ for which

$$1/19 \leq \{\xi(3/2)^n\} \leq 1 - 1/19 \quad \text{for } n = 0, 1, 2, \ldots$$

R. Tijdeman [9] showed that for every pair of integers $k$ and $m$ with $k \geq 2$ and $m \geq 1$ there exists $\xi \in [m, m + 1)$ such that

$$0 \leq \{\xi((2k+1)/2)^n\} \leq \frac{1}{2k-1} \quad \text{for } n = 0, 1, 2, \ldots$$

The connection between $(3/2)^n \mod 1$ and Waring’s problem (cf. M. Bennett [1]), and between Mahler’s conjecture (iv) and the $3x + 1$ problem (cf. [3]) is also well known.

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[25]
In this paper we study the set of all distribution functions of sequences \( \xi(3/2)^n \mod 1, \xi \in \mathbb{R} \). It is motivated by the fact that some conjectures involving a distribution function \( g(x) \) of \( \xi(3/2)^n \mod 1 \) may be formulated as in (i)–(iv). For example, the following conjecture implies Mahler’s conjecture: If \( g(x) = \text{constant} \) for all \( x \in I \), where \( I \) is a subinterval of \([0,1] \), then the length \( |I| < 1/2 \).

The study of the set of distribution functions of a sequence, still unsatisfactory today, was initiated by J. G. van der Corput [10]. The one-element set corresponding to the notion of asymptotic distribution function of a sequence \( \mod 1 \) was introduced by I. J. Schoenberg [8]. Many papers have been devoted to the study of the asymptotic distribution function for exponentially increasing sequences. H. Helson and J.-P. Kahane [4] established the existence of uncountably many \( \xi \) such that the sequence \( \xi \theta^n \) does not have an asymptotic distribution function \( \mod 1 \), where \( \theta \) is some fixed real number > 1. I. I. Piatetski-Shapiro [7] characterizes the asymptotic distribution function for the sequence \( \xi q^n \mod 1 \), where \( q > 1 \) is an integer. For a survey, see the monograph by L. Kuipers and H. Niederreiter [5].

In Section 2, we recall the definition of a distribution function \( g \) and we define a mapping \( g \to g_\varphi \) associated with a given measurable function \( \varphi : [0,1] \to [0,1] \). The formula defining \( g \to g_\varphi \) was used implicitly by K. F. Gauss for \( \varphi(x) = 1/x \mod 1 \) in his well-known problem of the metric theory of continued fractions (\( g_\varphi \) is given e.g. in [5, Th. 7.6]). The induced transformation between derivatives \( g' \to g'_\varphi \) is the so-called Frobenius–Perron operator.

In Section 3, choosing \( \varphi(x) \) as \( f(x) = 2x \mod 1 \) and \( h(x) = 3x \mod 1 \), we derive a functional equation of the type \( g_f = g_h \), for any distribution function \( g \) of \( \xi(3/2)^n \mod 1 \). As a consequence we give some sets of uniqueness for \( g \), where \( X \subset [0,1] \) is said to be a set of uniqueness if whenever \( g_1 = g_2 \) on \( X \), then \( g_1 = g_2 \) on \([0,1] \), for any two distribution functions \( g_1, g_2 \) of \( \xi(3/2)^n \mod 1 \) (different values of \( \xi \in \mathbb{R} \), for \( g_1, g_2 \), are also admissible). From this fact we derive an example of a distribution function that is not a distribution function of \( \xi(3/2)^n \mod 1 \) for any \( \xi \in \mathbb{R} \). We also conjecture that every measurable set \( X \subset [0,1] \) with measure \( |X| \geq 2/3 \) is a set of uniqueness. An integral criterion for \( g \) to satisfy \( g_f = g_h \) is also given.

In Section 4, we describe absolutely continuous solutions \( g \) of functional equations of the form \( g_f = g_1 \) and \( g_h = g_2 \) for given absolutely continuous distribution functions \( g_1, g_2 \).

In Section 5, we summarize the examples demonstrating all the above mentioned results.

2. Definitions and basic facts. For the purposes of this paper a distribution function \( g(x) \) will be a real-valued, non-decreasing function of the
Distribution functions of $\xi(3/2)^n \mod 1$ real variable $x$, defined on the unit interval $[0, 1]$, for which $g(0) = 0$ and $g(1) = 1$. Let $x_n \mod 1, n = 1, 2, \ldots$, be a given sequence. According to the terminology introduced in [5], for a positive integer $N$ and a subinterval $I$ of $[0, 1]$, let the counting function $A(I; N; x_n)$ be defined as the number of terms $x_n$, $1 \leq n \leq N$, for which $x_n \in I$.

A distribution function $g$ is called a distribution function of a sequence $x_n \mod 1$, $n = 1, 2, \ldots$, if there exists an increasing sequence of positive integers $N_1, N_2, \ldots$ such that

$$\lim_{k \to \infty} \frac{A([0, x); N_k; x_n)}{N_k} = g(x) \text{ for every } x \in [0, 1].$$

If each term $x_n \mod 1$ is repeated only finitely many times, then the semi-closed interval $[0, x)$ can be replaced by the closed interval $[0, x]$.

Every sequence has a non-empty set of distribution functions (cf. [5, Th. 7.1]). A sequence $x_n \mod 1$ having a singleton set $\{g(x)\}$ satisfies

$$\lim_{N \to \infty} \frac{A([0, x); N; x_n)}{N} = g(x) \text{ for every } x \in [0, 1]$$

and in this case $g(x)$ is called the asymptotic distribution function of a given sequence.

Let $\varphi : [0, 1] \to [0, 1]$ be a function such that, for all $x \in [0, 1]$, $\varphi^{-1}([0, x))$ can be expressed as the union of finitely many pairwise disjoint subintervals $I_i(x)$ of $[0, 1]$ with endpoints $\alpha_i(x) \leq \beta_i(x)$. For any distribution function $g(x)$ we put

$$g_{\varphi}(x) = \sum_i g(\beta_i(x)) - g(\alpha_i(x)).$$

The mapping $g \to g_{\varphi}$ is the main tool of the paper. A basic property is expressed by the following statement:

**Proposition.** Let $x_n \mod 1$ be a sequence having $g(x)$ as a distribution function associated with the sequence of indices $N_1, N_2, \ldots$. Suppose that each term $x_n \mod 1$ is repeated only finitely many times. Then the sequence $\varphi(\{x_n\})$ has the distribution function $g_{\varphi}(x)$ for the same $N_1, N_2, \ldots$, and vice versa every distribution function of $\varphi(\{x_n\})$ has this form.

**Proof.** The form of $g_{\varphi}(x)$ is a consequence of

$$A([0, x); N_k; \varphi(\{x_n\})) = \sum_i A(I_i(x); N_k; x_n)$$

and

$$A(I_i(x); N_k; x_n) = A([0, \beta_i(x)); N_k; x_n) - A([0, \alpha_i(x)); N_k; x_n) + o(N_k).$$

On the other hand, suppose that $\tilde{g}(x)$ is a distribution function of $\varphi(\{x_n\})$ associated with $N_1, N_2, \ldots$. The Helly selection principle guarantees a suit-
able subsequence $N_{n_1}, N_{n_2}, \ldots$ for which some $g(x)$ is a distribution function of $x_n \mod 1$. Thus $\tilde{g}(x) = g_\varphi(x)$. ■

It should be noted that if all of the intervals $I_i(x)$ are of the form $[\alpha_i(x), \beta_i(x))$, then $o(N_k) = 0$ and the assumption of finiteness of repetition is superfluous.

In this paper we take for $\varphi(x)$ the functions

$$f(x) = 2x \mod 1 \quad \text{and} \quad h(x) = 3x \mod 1.$$ 

In this case, for every $x \in [0, 1]$, we have

$$g_f(x) = g(f_1^{-1}(x)) + g(f_2^{-1}(x)) - g(1/2),$$

$$g_h(x) = g(h_1^{-1}(x)) + g(h_2^{-1}(x)) + g(h_3^{-1}(x)) - g(1/3) - g(2/3),$$

with inverse functions

$$f_1^{-1}(x) = x/2, \quad f_2^{-1}(x) = (x + 1)/2,$$

and

$$h_1^{-1}(x) = x/3, \quad h_2^{-1}(x) = (x + 1)/3, \quad h_3^{-1}(x) = (x + 2)/3.$$

3. Properties of distribution functions of $\xi(3/2)^n \mod 1$. Piatetski-Shapiro [7], by means of ergodic theory, proved that a necessary and sufficient condition that the sequence $\xi q^n \mod 1$ with integer $q > 1$ has a distribution function $g(x)$ is that $g_\varphi(x) = g(x)$ for all $x \in [0, 1]$, where $\varphi(x) = qx \mod 1$. For $\xi(3/2)^n \mod 1$ we only prove the following similar property.

**Theorem 1.** Every distribution function $g(x)$ of $\xi(3/2)^n \mod 1$ satisfies

$$g_f(x) = g_h(x) \quad \text{for all} \quad x \in [0, 1].$$

**Proof.** Using $\{q\{x\}\} = \{qx\}$ for any integer $q$, we have $\{2\{\xi(3/2)^n\}\} = \{3\{\xi(3/2)^{n-1}\}\}$. Therefore $f(\{\xi(3/2)^n\})$ and $h(\{\xi(3/2)^{n-1}\})$ form the same sequence and the rest follows from the Proposition. ■

The above theorem yields the following sets of uniqueness for distribution functions of $\xi(3/2)^n \mod 1$.

**Theorem 2.** Let $g_1$, $g_2$ be any two distribution functions satisfying

$$g_i f(x) = (g_i) h(x) \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad x \in [0, 1].$$

Set

$$I_1 = [0, 1/3], \quad I_2 = [1/3, 2/3], \quad I_3 = [2/3, 1].$$

If $g_1(x) = g_2(x)$ for $x \in I_i \cup I_j$, $1 \leq i \neq j \leq 3$, then $g_1(x) = g_2(x)$ for all $x \in [0, 1]$.

**Proof.** Assume that a distribution function $g$ satisfies $g_f = g_h$ on $[0, 1]$ and let $J_i$, $J'_j$, $J''_k$ be the intervals from $[0, 1]$ described in Figure 1.
There are three cases of \( I_i \cup I_j \).

1\(^{\circ} \). Consider first the case \( I_2 \cup I_3 \). Using the values of \( g \) on \( I_2 \cup I_3 \), and the equation \( g_f = g_h \) on \( J_1 \), we can compute \( g(h_1^{-1}(x)) \) for \( x \in J_1 \). Mapping \( x \in J_1 \) to \( x' \in J_2 \) by using \( h_1^{-1}(x) = f_1^{-1}(x') \), we find \( g(f_1^{-1}(x)) \) for \( x \in J_2 \). Then, by the equation \( g_f = g_h \) on \( J_2 \) we can compute \( g(h_1^{-1}(x)) \) for \( x \in J_2 \); hence we have \( g(f_1^{-1}(x)) \) for \( x \in J_3 \), etc. Thus we have \( g(x) \) for \( x \in I_1 \).

2\(^{\circ} \). Similarly for the case \( I_1 \cup I_2 \).

3\(^{\circ} \). In the case \( I_1 \cup I_3 \), first we compute \( g(1/2) \) by using \( g_f(1/2) = g_h(1/2) \), and then we divide the infinite process of computation of \( g(x) \) for \( x \in I_2 \) into two parts:

In the first part, using \( g(y) \), for \( y \in I_1 \cup I_3 \), and \( g_f = g_h \) on \([0,1] \), we compute \( g(h_2^{-1}(x)) \) for \( x \in J'_1 \). Mapping \( x \in J'_1 \rightarrow x' \in J'_2 \) by \( h_2^{-1}(x) = f_1^{-1}(x') \) and employing \( g_f = g_h \) we find \( g(h_2^{-1}(x)) \) for \( x \in J'_2 \). In the same way this leads to \( g(f_2^{-1}) \) on \( J'_3 \), \( g(h_2^{-1}) \) on \( J'_3 \), \( g(f_1^{-1}) \) on \( J'_4 \), \( g(h_2^{-1}) \) on \( J'_4 \), and so on.
Similarly, in the second part, from \( g \) on \( I_1 \cup I_3 \) and \( g_f = g_h \) on \([0, 1]\) we find \( g(h^{-1}) \) on \( J'_1 \), \( g(f_2^{-1}) \) on \( J''_2 \), \( g(h_2^{-1}) \) on \( J''_3 \), etc.

In both parts these infinite processes do not cover the values \( g(2/5) \) and \( g(3/5) \). The rest follows from the equations \( g_f(1/5) = g_h(1/5) \) and \( g_f(4/5) = g_h(4/5) \).

Next we derive an integral formula for testing \( g_f = g_h \). Define
\[
F(x, y) = |\{2x\} - \{3y\}| + |\{2y\} - \{3x\}| - |\{2x\} - \{2y\}| - |\{3x\} - \{3y\}|.
\]

**Theorem 3.** A continuous distribution function \( g \) satisfies \( g_f = g_h \) on \([0, 1]\) if and only if
\[
\int_0^1 F(x, y) \, dx \, dy = 0.
\]

**Proof.** Let \( x_n, n = 1, 2, \ldots, \) be an auxiliary sequence in \([0, 1]\) such that all \((x_m, x_n)\) are points of continuity of \( F(x, y) \), and let \( c_X(x) \) be the characteristic function of a set \( X \). Applying \( c_{[0, x]}(x_n) = c_{(x_n, 1]}(x) \), we can compute
\[
\int_0^1 \left( \frac{1}{N} \sum_{n=1}^N c_{f^{-1}(0, x)}(x_n) - \frac{1}{N} \sum_{n=1}^N c_{h^{-1}(0, x)}(x_n) \right)^2 \, dx \\
= \frac{1}{N^2} \sum_{m,n=1}^N F_{f,h}(x_m, x_n),
\]
where
\[
F_{f,h}(x, y) = \max(f(x), h(y)) + \max(f(y), h(x)) \\
- \max(f(x), f(y)) - \max(h(x), h(y)) \\
= \frac{1}{2}(|f(x) - h(y)| + |f(y) - h(x)| - |f(x) - f(y)| - |h(x) - h(y)|).
\]

Applying the well-known Helly lemma we have
\[
\int_0^1 (g_f(x) - g_h(x))^2 \, dx = \int_0^1 F_{f,h}(x, y) \, dx \, dy
\]
for any continuous distribution function \( g \). Here \( 2F_{f,h}(x, y) = F(x, y) \).

**4. Inverse mapping to** \( g \rightarrow (g_f, g_h) \)

**Theorem 4.** Let \( g_1, g_2 \) be two absolutely continuous distribution functions satisfying \((g_1)_h(x) = (g_2)_f(x)\) for \( x \in [0, 1] \). Then an absolutely continuous distribution function \( g(x) \) satisfies \( g_f(x) = g_1(x) \) and \( g_h(x) = g_2(x) \).
for \( x \in [0, 1] \) if and only if \( g(x) \) has the form

\[
g(x) = \begin{cases} 
\Psi(x) + \Phi(x) - 1/6 & \text{for } x \in [0, 1/6], \\
\Psi(x) + \Phi(x) + g_1(1/3) - 1/6 & \text{for } x \in [1/6, 2/6], \\
\Psi(x) + \Phi(x) - 1/6 - g_1(2x - 1/3) + g_2(3x - 1) & \text{for } x \in [2/6, 3/6], \\
2\Phi(x) + g_1(1/3) - g_2(2/3) + g_2(1/2) - \Psi(x - 3/6) + g_1(2x - 1) & \text{for } x \in [3/6, 4/6], \\
-\Psi(x) + 2\Phi(x) + g_1(1/3) - g_2(2/3) + g_2(1/2) - \Phi(x - 4/6) + g_1(2x - 1) & \text{for } x \in [4/6, 5/6], \\
-\Psi(x) + \Phi(x) + g_1(1/3) + \Psi(x - 5/6) - \Phi(x - 5/6) - g_1(2x - 5/3) + g_2(3x - 2) & \text{for } x \in [5/6, 1],
\end{cases}
\]

where \( \Psi(x) = \int_0^x \psi(t) \, dt \), \( \Phi(x) = \int_0^x \phi(t) \, dt \) for \( x \in [0, 1/6] \), and \( \psi(t), \phi(t) \) are Lebesgue integrable functions on \( [0, 1/6] \) satisfying

\[
0 \leq \psi(t) \leq 2g_1'(2t), \quad 0 \leq \phi(t) \leq 2g_2'(2t + 1/3),
\]

\[
2g_1'(2t) - 3g_2'(3t + 1/2) \leq \psi(t) - \phi(t) \leq -2g_1'(2t + 1/3) + 3g_2'(3t),
\]

for almost all \( t \in [0, 1/6] \).

Proof. We shall use a method which is applicable for any two commuting \( f, h \) having finitely many inverse functions.

The starting point is the set of new variables \( x_i(t) \):

\[
x_1(t) := f_1^{-1} \circ h_1^{-1} \circ h \circ f(t) = h_1^{-1} \circ f_1^{-1} \circ f \circ h(t),
\]

\[
x_2(t) := f_1^{-1} \circ h_2^{-1} \circ h \circ f(t) = h_1^{-1} \circ f_2^{-1} \circ f \circ h(t),
\]

\[
x_3(t) := f_1^{-1} \circ h_3^{-1} \circ h \circ f(t) = h_2^{-1} \circ f_2^{-1} \circ f \circ h(t),
\]

\[
x_4(t) := f_2^{-1} \circ h_1^{-1} \circ h \circ f(t) = h_1^{-1} \circ f_1^{-1} \circ f \circ h(t),
\]

\[
x_5(t) := f_2^{-1} \circ h_2^{-1} \circ h \circ f(t) = h_3^{-1} \circ f_1^{-1} \circ f \circ h(t),
\]

\[
x_6(t) := f_2^{-1} \circ h_3^{-1} \circ h \circ f(t) = h_3^{-1} \circ f_2^{-1} \circ f \circ h(t).
\]

Here the different expressions of \( x_i(t) \) follow from the fact that \( f(h(x)) = h(f(x)) \), \( x \in [0, 1] \). For \( t \in [0, 1/6] \) we have \( x_i(t) = t + (i - 1)/6, i = 1, \ldots, 6 \).

Substituting \( x = h_j^{-1} \circ h \circ f(t), j = 1, 2, 3 \), into \( g_j(x) = g_1(x) \), and \( x = f_1^{-1} \circ f \circ h(t), i = 1, 2 \), into \( g_0(x) = g_2(x) \) we have five linear equations for \( g(x_k(t)), k = 1, \ldots, 6 \). Abbreviating the composition \( f^{-1}_i \circ h_j^{-1} \circ h \circ f(t) \) as \( f_1^{-1}h_2^{-1}h_3f(t) \), and \( x_i(t) \) as \( x_i \), we can write

\[
g(x_1) + g(x_4) - g(1/2) = g_1(h_1^{-1}h_2f(t)),
\]

\[
g(x_2) + g(x_5) - g(1/2) = g_1(h_2^{-1}h_3f(t)),
\]

\[
g(x_3) + g(x_6) - g(1/2) = g_1(h_3^{-1}h_1f(t)),
\]

\[
g(x_1) + g(x_3) + g(x_5) - g(1/3) - g(2/3) = g_2(f_1^{-1}f_2h(t)),
\]

\[
g(x_2) + g(x_4) + g(x_6) - g(1/3) - g(2/3) = g_2(f_2^{-1}f_2h(t)).
\]
Summing up the first three equations and, respectively, the next two equations, we find the necessary condition
\[
g_1(1/3) + g_1(2/3) + 3g(1/2) + (g_1)h(hf(t))
= (g_2)f(fh(t)) + g_2(1/2) + 2(g(1/3) + g(2/3))
\]
for \(t \in [0, 1/6]\), which is equivalent to
\[
g_1(1/3) + g_1(2/3) - g_2(1/2) = 2(g(1/3) + g(2/3)) - 3g(1/2)
\]
and
\[
(g_1)h(x) = (g_2)f(x)
\]
for \(x \in [0, 1]\). Eliminating the fourth equation which depends on the others we can compute \(g(x_1), \ldots, g(x_6)\) by using \(g(x_1), g(x_2), g(1/3), g(1/2),\) and \(g(2/3)\) as follows:
\[
\begin{align*}
g(x_3) &= g(1/3) + g(2/3) - g(1/2) - g(x_1) + g(x_2) \\
&= g_1(h_2^{-1}hf(t)) + g_2(f_1^{-1}fh(t)), \\
g(x_4) &= g(1/2) - g(x_1) + g_1(h_1^{-1}hf(t)), \\
g(x_5) &= g(1/2) - g(x_2) + g_1(h_2^{-1}hf(t)), \\
g(x_6) &= g(1/3) + g(2/3) - g(1/2) + g(x_1) - g(x_2) \\
&= g_1(h_1^{-1}hf(t)) + g_2(f_2^{-1}fh(t)),
\end{align*}
\]
for all \(t \in [0, 1/6]\). Putting \(t = 0\) and \(t = 1/6\), we find
\[
\begin{align*}
g(1/2) &= 2g(1/3) - 2g(1/6) + g_1(1/3) - g_1(2/3) + g_2(1/2), \\
g(2/3) &= 2g(1/3) - 3g(1/6) + 2g_1(1/3) - g_1(2/3) + g_2(1/2).
\end{align*}
\]
These values satisfy the necessary condition \(g_1(1/3) + g_1(2/3) - g_2(1/2) = 2(g(1/3) + g(2/3)) - 3g(1/2)\). Moreover, \(g(1/3) = g(x_2(1/6)), g(1/6) = g(x_2(0)),\) and thus \(g(x_3), \ldots, g(x_6)\) can be expressed by only using \(g(x_1), g(x_2)\). Next, we simplify (1) by using
\[
\begin{align*}
h_1^{-1}hf(t) &= ff_2^{-1}h_1^{-1}hf(t) = f(x_1) & \text{for } g(x_4), \\
h_2^{-1}hf(t) &= ff_1^{-1}h_2^{-1}hf(t) = f(x_5) & \text{for } g(x_5), \\
f_1^{-1}fh(t) &= hh_2^{-1}f_1^{-1}fh(t) = h(x_3) & \text{and} \\
h_2^{-1}hf(t) &= ff_1^{-1}h_2^{-1}hf(t) = f(x_2) & \text{for } g(x_3), \\
f_2^{-1}fh(t) &= hh_3^{-1}f_2^{-1}fh(t) = h(x_6) & \text{and} \\
h_1^{-1}hf(t) &= ff_1^{-1}h_1^{-1}hf(t) = f(x_1) & \text{for } g(x_6).
\end{align*}
\]
Now, each \(g(x_i)\) can be expressed as \(g(x), x \in [(i-1)/6, i/6]\). To do this we use the identity
\[
x_i(x_j(t)) = x_i(t) \quad \text{for } t \in [0, 1] \text{ and } 1 \leq i, j \leq 6,
\]
which immediately follows from the fact that
\[ f_{i}^{-1}h_{j}^{-1}hf_{k}^{-1}h_{l}^{-1}hf(t) = f_{i}^{-1}h_{j}^{-1}hf(t). \]

For example,
\[ g(x_{3}) = g(1/3) + g(2/3) - g(1/2) - g(x_{1}) + g(x_{2}) - g_{1}(f(x_{2})) + g_{2}(h(x_{3})), \]
for \( t \in [0,1/6] \), which is the same as
\[ g(x) = g(1/3) + g(2/3) - g(1/2) - g(x_{1}(x)) + g(x_{2}(x)) - g_{1}(f(x_{2}(x))) + g_{2}(h(x)) \]
for \( x \in [2/6,3/6] \). In our case \( x_{1}(x) = x - i/6 \) and \( x_{2}(x) = x + 1/6 - i/6 \) for \( x \in [i/6,(i + 1)/6] \) and \( i = 0,...,5 \).

Now, assuming the absolute continuity of \( g(x_{1}) \) and \( g(x_{2}) \) we can write
\[ g(x_{1}(t)) = \int_{0}^{t} \psi(u) \, du, \]
\[ g(x_{2}(t)) = \int_{0}^{1/6} \psi(u) \, du + \int_{0}^{t} \phi(u) \, du \]
for \( t \in [0,1/6] \).

Summing up the above we find the expression \( g(x) \) in the theorem. For the monotonicity of \( g(x) \) we can investigate \( g'(x_{i}(t)) \geq 0 \) for \( t \in [0,1/6] \) and \( i = 1,...,6 \), which immediately leads to the inequalities for \( \psi \) and \( \phi \) given in our theorem.

5. Examples and concluding remarks

1. Define a one-jump distribution function \( c_{\alpha} : [0,1] \rightarrow [0,1] \) such that \( c_{\alpha}(0) = 0 \), \( c_{\alpha}(1) = 1 \), and
\[ c_{\alpha}(x) = \begin{cases} 0 & \text{if } x \in [0,\alpha), \\ 1 & \text{if } x \in (\alpha,1]. \end{cases} \]

The distribution functions \( c_{0}(x), c_{1}(x) \), and \( x \) satisfy \( g_{f}(x) = g_{h}(x) \) for every \( x \in [0,1] \).

2. Taking \( g_{1}(x) = g_{2}(x) = x \), further solutions of \( g_{f} = g_{h} \) follow from Theorem 4. In this case
\[ 0 \leq \psi(t) \leq 2, \quad 0 \leq \phi(t) \leq 2, \quad -1 \leq \psi(t) - \phi(t) \leq 1, \]
for all \( t \in [0,1/6] \). Putting \( \psi(t) = \phi(t) = 0 \), the resulting distribution
function is
\[
g_3(x) = \begin{cases} 
0 & \text{for } x \in [0, 2/6], \\
x - 1/3 & \text{for } x \in [2/6, 3/6], \\
2x - 5/6 & \text{for } x \in [3/6, 5/6], \\
x & \text{for } x \in [5/6, 1].
\end{cases}
\]

Taking \( g_1(x) = g_2(x) = g_3(x) \), this \( g_3(x) \) can be used as a starting point for a further application of Theorem 4 which gives another solution of \( g_f = g_h \).

3. Computing
\[
\int_{j/6}^{(j+1)/6} \int_{j/6}^{(j+1)/6} F(x, y) \, dx \, dy
\]
for \( i, j = 1, \ldots, 5 \) directly, we can find
\[
\prod_{j=0}^{11} F(x, y) \, dg_3(x) \, dg_3(y) = 0,
\]
which is also a consequence of Theorem 3 and \( (g_3)_f = (g_3)_h \).

4. Since the mapping \( g \to g_\phi \) is linear, the set of all solutions of \( g_f = g_h \) is convex.

5. Since \( x_f = x_h \), Theorem 2 leads to the fact that the following distribution function \( g_4(x) \) is not a distribution function of \( \xi (3/2)_n \mod 1 \), for any \( \xi \in \mathbb{R} \):
\[
g_4(x) = \begin{cases} 
x & \text{for } x \in [0, 2/3], \\
x^2 - (2/3)x + 2/3 & \text{for } x \in [2/3, 1].
\end{cases}
\]

6. By Figure 1, \( X = [2/9, 1/3] \cup [1/2, 1] \) is also a set of uniqueness. Moreover, \( |X| = 11/18 < 2/3 \). Similarly for \([0, 1/2] \cup [2/3, 7/9]\).

7. Since all the components of \( f^{-1}([0, x]) \) and \( h^{-1}([0, x]) \) are semiclosed the fact that, for fixed \( \xi \neq 0 \) and \( m \), \( \{\xi (3/2)_n^m\} \) only for finitely many \( n \), was not used in the proof of Theorem 1.

References


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