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On distribution functions of $\xi(3/2)^n$ mod 1

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1. Preliminary remarks. The question about distribution of $(3/2)^n$ mod 1 is most difficult. We present a selection of known conjectures:

(i) $(3/2)^n$ mod 1 is uniformly distributed in $[0,1]$.

(ii) $(3/2)^n$ mod 1 is dense in $[0,1]$.

(iii) (T. Vijayaraghavan [11])

$$\limsup_{n\to\infty}\{(3/2)^n\} - \liminf_{n\to\infty}\{(3/2)^n\} > 1/2,$$

where $\{x\}$ is the fractional part of x.

(iv) (K. Mahler [6]) There exists no $\xi \in \mathbb{R}^+$ such that $0 \le \{\xi(3/2)^n\} < 1/2$ for $n = 0, 1, 2, \ldots$

(v) (G. Choquet [2]) There exists no $\xi \in \mathbb{R}^+$ such that the closure of $\{\{\xi(3/2)^n\};\ n = 0, 1, 2, \ldots\}$ is nowhere dense in $[0,1]$.

Few positive results are known. For instance, L. Flatto, J. C. Lagarias and A. D. Pollington [3] showed that

$$\limsup_{n\to\infty}\{\xi(3/2)^n\} - \liminf_{n\to\infty}\{\xi(3/2)^n\} \ge 1/3$$

for every $\xi > 0$.

G. Choquet [2] gave infinitely many $\xi \in \mathbb{R}$ for which

$$1/19 \le \{\xi(3/2)^n\} \le 1 - 1/19 \quad \text{for } n = 0, 1, 2, \ldots$$

R. Tijdeman [9] showed that for every pair of integers k and m with $k \ge 2$ and $m \ge 1$ there exists $\xi \in [m, m+1)$ such that

$$0 \le \{\xi((2k+1)/2)^n\} \le \frac{1}{2k-1} \quad \text{for } n = 0, 1, 2, \ldots$$

The connection between $(3/2)^n$ mod 1 and Waring's problem (cf. M. Bennett [1]), and between Mahler's conjecture (iv) and the $3x+1$ problem (cf. [3]) is also well known.

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In this paper we study the set of all distribution functions of sequences $\xi(3/2)^n \bmod 1$, $\xi \in \mathbb{R}$. It is motivated by the fact that some conjectures involving a distribution function $g(x)$ of $\xi(3/2)^n \bmod 1$ may be formulated as in (i)–(iv). For example, the following conjecture implies Mahler's conjecture: If $g(x) =$ constant for all $x \in I$, where I is a subinterval of $[0,1]$, then the length $|I| < 1/2$.

The study of the set of distribution functions of a sequence, still unsatisfactory today, was initiated by J. G. van der Corput [10]. The one-element set corresponding to the notion of asymptotic distribution function of a sequence mod 1 was introduced by I. J. Schoenberg [8]. Many papers have been devoted to the study of the asymptotic distribution function for exponentially increasing sequences. H. Helson and J.-P. Kahane [4] established the existence of uncountably many ξ such that the sequence $\xi\theta^n$ does not have an asymptotic distribution function mod 1, where θ is some fixed real number > 1. I. I. Piatetski-Shapiro [7] characterizes the asymptotic distribution function for the sequence $\xi q^n \bmod 1$, where $q > 1$ is an integer. For a survey, see the monograph by L. Kuipers and H. Niederreiter [5].

In Section 2, we recall the definition of a distribution function g and we define a mapping $g \to g_\varphi$ associated with a given measurable function $\varphi : [0,1] \to [0,1]$. The formula defining $g \to g_\varphi$ was used implicitly by K. F. Gauss for $\varphi(x) = 1/x \bmod 1$ in his well-known problem of the metric theory of continued fractions (g_φ is given e.g. in [5, Th. 7.6]). The induced transformation between derivatives $g' \to g'_\varphi$ is the so-called Frobenius–Perron operator.

In Section 3, choosing $\varphi(x)$ as $f(x) = 2x \bmod 1$ and $h(x) = 3x \bmod 1$, we derive a functional equation of the type $g_f = g_h$, for any distribution function g of $\xi(3/2)^n \bmod 1$. As a consequence we give some sets of uniqueness for g, where $X \subset [0,1]$ is said to be a *set of uniqueness* if whenever $g_1 = g_2$ on X, then $g_1 = g_2$ on $[0,1]$, for any two distribution functions g_1, g_2 of $\xi(3/2)^n \bmod 1$ (different values of $\xi \in \mathbb{R}$, for g_1, g_2, are also admissible). From this fact we derive an example of a distribution function that is not a distribution function of $\xi(3/2)^n \bmod 1$ for any $\xi \in \mathbb{R}$. We also conjecture that every measurable set $X \subset [0,1]$ with measure $|X| \geq 2/3$ is a set of uniqueness. An integral criterion for g to satisfy $g_f = g_h$ is also given.

In Section 4, we describe absolutely continuous solutions g of functional equations of the form $g_f = g_1$ and $g_h = g_2$ for given absolutely continuous distribution functions g_1, g_2.

In Section 5, we summarize the examples demonstrating all the above mentioned results.

2. Definitions and basic facts. For the purposes of this paper a *distribution function* $g(x)$ will be a real-valued, non-decreasing function of the real variable x, defined on the unit interval $[0,1]$, for which $g(0)=0$ and $g(1)=1$. Let x_n mod 1, $n=1,2,\dots$, be a given sequence. According to the terminology introduced in [5], for a positive integer N and a subinterval I of $[0,1]$, let the *counting function* $A(I;N;x_n)$ be defined as the number of terms x_n, $1\le n\le N$, for which $x_n\in I$.

A distribution function g is called a *distribution function of a sequence* x_n mod 1, $n=1,2,\dots$, if there exists an increasing sequence of positive integers $N_1,N_2,\dots$ such that

$$\lim_{k\to\infty}\frac{A([0,x);N_k;x_n)}{N_k}=g(x)\quad\text{ for every } x\in[0,1].$$

If each term x_n mod 1 is repeated only finitely many times, then the semi-closed interval $[0,x)$ can be replaced by the closed interval $[0,x]$.

Every sequence has a non-empty set of distribution functions (cf. [5, Th. 7.1]). A sequence x_n mod 1 having a singleton set $\{g(x)\}$ satisfies

$$\lim_{N\to\infty}\frac{A([0,x);N;x_n)}{N}=g(x)\quad\text{ for every } x\in[0,1]$$

and in this case $g(x)$ is called the *asymptotic distribution function* of a given sequence.

Let $\varphi:[0,1]\to[0,1]$ be a function such that, for all $x\in[0,1]$, $\varphi^{-1}([0,x))$ can be expressed as the union of finitely many pairwise disjoint subintervals $I_i(x)$ of $[0,1]$ with endpoints $\alpha_i(x)\le\beta_i(x)$. For any distribution function $g(x)$ we put

$$g_\varphi(x)=\sum_i g(\beta_i(x))-g(\alpha_i(x)).$$

The mapping $g\to g_\varphi$ is the main tool of the paper. A basic property is expressed by the following statement:

PROPOSITION. *Let* x_n mod 1 *be a sequence having* $g(x)$ *as a distribution function associated with the sequence of indices* $N_1,N_2,\dots$ *Suppose that each term* x_n mod 1 *is repeated only finitely many times. Then the sequence* $\varphi(\{x_n\})$ *has the distribution function* $g_\varphi(x)$ *for the same* $N_1,N_2,\dots$, *and vice versa every distribution function of* $\varphi(\{x_n\})$ *has this form.*

Proof. The form of $g_\varphi(x)$ is a consequence of

$$A([0,x);N_k;\varphi(\{x_n\}))=\sum_i A(I_i(x);N_k;x_n)$$

and

$$A(I_i(x);N_k;x_n)=A([0,\beta_i(x));N_k;x_n)-A([0,\alpha_i(x));N_k;x_n)+o(N_k).$$

On the other hand, suppose that $\widetilde{g}(x)$ is a distribution function of $\varphi(\{x_n\})$ associated with $N_1,N_2,\dots$ The Helly selection principle guarantees a suitable subsequence $N_{n_1}, N_{n_2}, \ldots$ for which some $g(x)$ is a distribution function of x_n mod 1. Thus $\widetilde{g}(x) = g_\varphi(x)$. ∎

It should be noted that if all of the intervals $I_i(x)$ are of the form $[\alpha_i(x), \beta_i(x))$, then $o(N_k) = 0$ and the assumption of finiteness of repetition is superfluous.

In this paper we take for $\varphi(x)$ the functions

$$f(x) = 2x \bmod 1 \quad \text{and} \quad h(x) = 3x \bmod 1.$$

In this case, for every $x \in [0,1]$, we have

$$g_f(x) = g(f_1^{-1}(x)) + g(f_2^{-1}(x)) - g(1/2),$$
$$g_h(x) = g(h_1^{-1}(x)) + g(h_2^{-1}(x)) + g(h_3^{-1}(x)) - g(1/3) - g(2/3),$$

with inverse functions

$$f_1^{-1}(x) = x/2, \qquad f_2^{-1}(x) = (x+1)/2,$$

and

$$h_1^{-1}(x) = x/3, \quad h_2^{-1}(x) = (x+1)/3, \quad h_3^{-1}(x) = (x+2)/3.$$

3. Properties of distribution functions of $\xi(3/2)^n$ mod 1. Piatetski-Shapiro [7], by means of ergodic theory, proved that a necessary and sufficient condition that the sequence ξq^n mod 1 with integer $q > 1$ has a distribution function $g(x)$ is that $g_\varphi(x) = g(x)$ for all $x \in [0,1]$, where $\varphi(x) = qx$ mod 1. For $\xi(3/2)^n$ mod 1 we only prove the following similar property.

THEOREM 1. *Every distribution function $g(x)$ of $\xi(3/2)^n$ mod 1 satisfies $g_f(x) = g_h(x)$ for all $x \in [0,1]$.*

Proof. Using $\{q\{x\}\} = \{qx\}$ for any integer q, we have $\{2\{\xi(3/2)^n\}\} = \{3\{\xi(3/2)^{n-1}\}\}$. Therefore $f(\{\xi(3/2)^n\})$ and $h(\{\xi(3/2)^{n-1}\})$ form the same sequence and the rest follows from the Proposition. ∎

The above theorem yields the following sets of uniqueness for distribution functions of $\xi(3/2)^n$ mod 1.

THEOREM 2. *Let g_1, g_2 be any two distribution functions satisfying $(g_i)_f(x) = (g_i)_h(x)$ for $i = 1,2$ and $x \in [0,1]$. Set*

$$I_1 = [0, 1/3], \quad I_2 = [1/3, 2/3], \quad I_3 = [2/3, 1].$$

If $g_1(x) = g_2(x)$ for $x \in I_i \cup I_j$, $1 \le i \ne j \le 3$, then $g_1(x) = g_2(x)$ for all $x \in [0,1]$.

Proof. Assume that a distribution function g satisfies $g_f = g_h$ on $[0,1]$ and let J_i, J'_j, J''_k be the intervals from $[0,1]$ described in Figure 1.

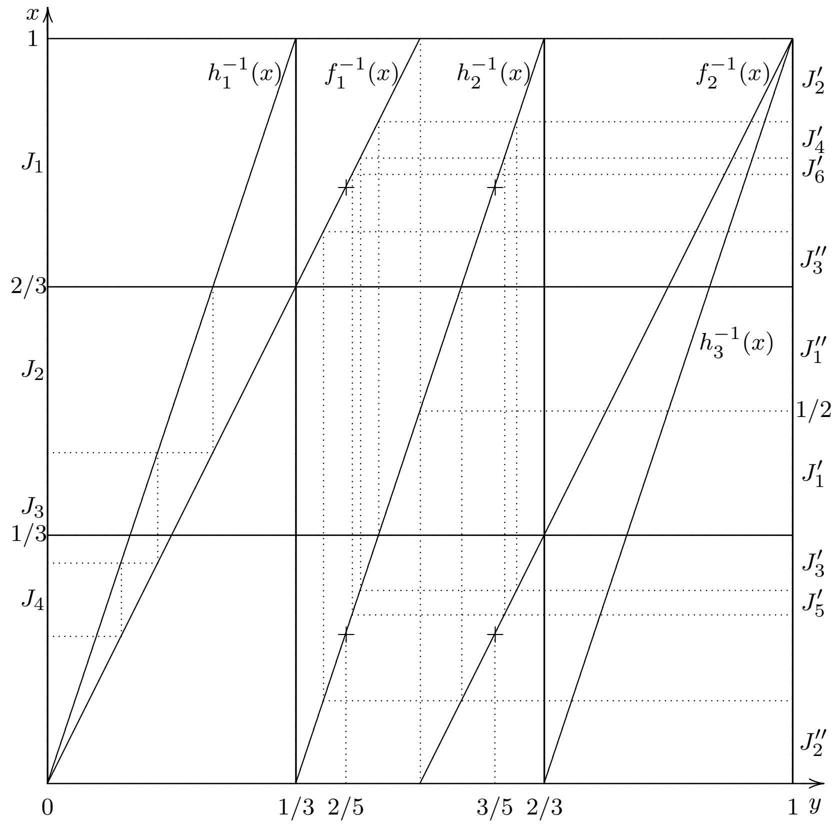

Fig. 1

There are three cases of $I_i \cup I_j$.

1°. Consider first the case $I_2 \cup I_3$. Using the values of g on $I_2 \cup I_3$, and the equation $g_f = g_h$ on J_1, we can compute $g(h_1^{-1}(x))$ for $x \in J_1$. Mapping $x \in J_1$ to $x' \in J_2$ by using $h_1^{-1}(x) = f_1^{-1}(x')$, we find $g(f_1^{-1}(x))$ for $x \in J_2$. Then, by the equation $g_f = g_h$ on J_2 we can compute $g(h_1^{-1}(x))$ for $x \in J_2$; hence we have $g(f_1^{-1}(x))$ for $x \in J_3$, etc. Thus we have $g(x)$ for $x \in I_1$.

2°. Similarly for the case $I_1 \cup I_2$.

3°. In the case $I_1 \cup I_3$, first we compute $g(1/2)$ by using $g_f(1/2) = g_h(1/2)$, and then we divide the infinite process of computation of $g(x)$ for $x \in I_2$ into two parts:

In the first part, using $g(y)$, for $y \in I_1 \cup I_3$, and $g_f = g_h$ on $[0,1]$, we compute $g(h_2^{-1}(x))$ for $x \in J'_1$. Mapping $x \in J'_1 \to x' \in J'_2$ by $h_2^{-1}(x) = f_1^{-1}(x')$ and employing $g_f = g_h$ we find $g(h_2^{-1}(x))$ for $x \in J'_2$. In the same way this leads to $g(f_2^{-1})$ on J'_3, $g(h_2^{-1})$ on J'_3, $g(f_1^{-1})$ on J'_4, $g(h_2^{-1})$ on J'_4, and so on.

Similarly, in the second part, from g on $I_1 \cup I_3$ and $g_f = g_h$ on $[0,1]$ we find $g(h^{-1})$ on J_1'', $g(f_2^{-1})$ on J_2'', $g(h_2^{-1})$ on J_2'', $g(f_1^{-1})$ on J_3'', $g(h_2^{-1})$ on J_3'', etc.

In both parts these infinite processes do not cover the values $g(2/5)$ and $g(3/5)$. The rest follows from the equations $g_f(1/5) = g_h(1/5)$ and $g_f(4/5) = g_h(4/5)$. ■

Next we derive an integral formula for testing $g_f = g_h$. Define

$$F(x,y) = |\{2x\} - \{3y\}| + |\{2y\} - \{3x\}| - |\{2x\} - \{2y\}| - |\{3x\} - \{3y\}|.$$

THEOREM 3. *A continuous distribution function g satisfies $g_f = g_h$ on $[0,1]$ if and only if*

$$\int_0^1\int_0^1 F(x,y)\,dg(x)\,dg(y) = 0.$$

Proof. Let x_n, $n = 1,2,\ldots$, be an auxiliary sequence in $[0,1]$ such that all (x_m, x_n) are points of continuity of $F(x,y)$, and let $c_X(x)$ be the characteristic function of a set X. Applying $c_{[0,x)}(x_n) = c_{(x_n,1]}(x)$, we can compute

$$\int_0^1 \left(\frac{1}{N}\sum_{n=1}^{N} c_{f^{-1}([0,x))}(x_n) - \frac{1}{N}\sum_{n=1}^{N} c_{h^{-1}([0,x))}(x_n)\right)^2 dx = \frac{1}{N^2}\sum_{m,n=1}^{N} F_{f,h}(x_m,x_n),$$

where

$$\begin{aligned} F_{f,h}(x,y) = {} & \max(f(x),h(y)) + \max(f(y),h(x)) \\ & - \max(f(x),f(y)) - \max(h(x),h(y)) \\ = {} & \tfrac{1}{2}(|f(x)-h(y)| + |f(y)-h(x)| - |f(x)-f(y)| - |h(x)-h(y)|). \end{aligned}$$

Applying the well-known Helly lemma we have

$$\int_0^1 (g_f(x) - g_h(x))^2\,dx = \int_0^1\int_0^1 F_{f,h}(x,y)\,dg(x)\,dg(y)$$

for any continuous distribution function g. Here $2F_{f,h}(x,y) = F(x,y)$. ■

4. Inverse mapping to $g \to (g_f, g_h)$

THEOREM 4. *Let g_1, g_2 be two absolutely continuous distribution functions satisfying $(g_1)_h(x) = (g_2)_f(x)$ for $x \in [0,1]$. Then an absolutely continuous distribution function $g(x)$ satisfies $g_f(x) = g_1(x)$ and $g_h(x) = g_2(x)$*

for $x\in[0,1]$ *if and only if* $g(x)$ *has the form*

$$g(x)=\begin{cases}\Psi(x) & \text{for } x\in[0,1/6],\\ \Psi(1/6)+\Phi(x-1/6) & \text{for } x\in[1/6,2/6],\\ \Psi(1/6)+\Phi(1/6)+g_1(1/3)-\Psi(x-2/6) & \\ \qquad +\Phi(x-2/6)-g_1(2x-1/3)+g_2(3x-1) & \text{for } x\in[2/6,3/6],\\ 2\Phi(1/6)+g_1(1/3)-g_1(2/3)+g_2(1/2) & \\ \qquad -\Psi(x-3/6)+g_1(2x-1) & \text{for } x\in[3/6,4/6],\\ -\Psi(1/6)+2\Phi(1/6)+g_1(1/3)-g_1(2/3)+g_2(1/2) & \\ \qquad -\Phi(x-4/6)+g_1(2x-1) & \text{for } x\in[4/6,5/6],\\ -\Psi(1/6)+\Phi(1/6)+g_1(1/3)+\Psi(x-5/6) & \\ \qquad -\Phi(x-5/6)-g_1(2x-5/3)+g_2(3x-2) & \text{for } x\in[5/6,1],\end{cases}$$

where $\Psi(x)=\int_0^x\psi(t)\,dt$, $\Phi(x)=\int_0^x\phi(t)\,dt$ *for* $x\in[0,1/6]$, *and* $\psi(t)$, $\phi(t)$ *are Lebesgue integrable functions on* $[0,1/6]$ *satisfying*

$$0\le\psi(t)\le 2g_1'(2t),\qquad 0\le\phi(t)\le 2g_1'(2t+1/3),$$
$$2g_1'(2t)-3g_2'(3t+1/2)\le\psi(t)-\phi(t)\le-2g_1'(2t+1/3)+3g_2'(3t),$$

for almost all $t\in[0,1/6]$.

Proof. We shall use a method which is applicable for any two commuting f, h having finitely many inverse functions.

The starting point is the set of new variables $x_i(t)$:

$$\begin{aligned}x_1(t)&:=f_1^{-1}\circ h_1^{-1}\circ h\circ f(t)=h_1^{-1}\circ f_1^{-1}\circ f\circ h(t),\\ x_2(t)&:=f_1^{-1}\circ h_2^{-1}\circ h\circ f(t)=h_1^{-1}\circ f_2^{-1}\circ f\circ h(t),\\ x_3(t)&:=f_1^{-1}\circ h_3^{-1}\circ h\circ f(t)=h_2^{-1}\circ f_2^{-1}\circ f\circ h(t),\\ x_4(t)&:=f_2^{-1}\circ h_1^{-1}\circ h\circ f(t)=h_2^{-1}\circ f_1^{-1}\circ f\circ h(t),\\ x_5(t)&:=f_2^{-1}\circ h_2^{-1}\circ h\circ f(t)=h_3^{-1}\circ f_1^{-1}\circ f\circ h(t),\\ x_6(t)&:=f_2^{-1}\circ h_3^{-1}\circ h\circ f(t)=h_3^{-1}\circ f_2^{-1}\circ f\circ h(t).\end{aligned}$$

Here the different expressions of $x_i(t)$ follow from the fact that $f(h(x))=h(f(x))$, $x\in[0,1]$. For $t\in[0,1/6]$ we have $x_i(t)=t+(i-1)/6$, $i=1,\dots,6$.

Substituting $x=h_j^{-1}\circ h\circ f(t)$, $j=1,2,3$, into $g_f(x)=g_1(x)$, and $x=f_i^{-1}\circ f\circ h(t)$, $i=1,2$, into $g_h(x)=g_2(x)$ we have five linear equations for $g(x_k(t))$, $k=1,\dots,6$. Abbreviating the composition $f_i^{-1}\circ h_j^{-1}\circ h\circ f(t)$ as $f_1^{-1}h_2^{-1}hf(t)$, and $x_i(t)$ as x_i, we can write

$$\begin{aligned}&g(x_1)+g(x_4)-g(1/2)=g_1(h_1^{-1}hf(t)),\\ &g(x_2)+g(x_5)-g(1/2)=g_1(h_2^{-1}hf(t)),\\ &g(x_3)+g(x_6)-g(1/2)=g_1(h_3^{-1}hf(t)),\\ &g(x_1)+g(x_3)+g(x_5)-g(1/3)-g(2/3)=g_2(f_1^{-1}fh(t)),\\ &g(x_2)+g(x_4)+g(x_6)-g(1/3)-g(2/3)=g_2(f_2^{-1}fh(t)).\end{aligned}$$

Summing up the first three equations and, respectively, the next two equations, we find the necessary condition

$$g_1(1/3) + g_1(2/3) + 3g(1/2) + (g_1)_h(hf(t)) \\ = (g_2)_f(fh(t)) + g_2(1/2) + 2(g(1/3) + g(2/3))$$

for $t \in [0, 1/6]$, which is equivalent to

$$g_1(1/3) + g_1(2/3) - g_2(1/2) = 2(g(1/3) + g(2/3)) - 3g(1/2)$$

and

$$(g_1)_h(x) = (g_2)_f(x)$$

for $x \in [0, 1]$. Eliminating the fourth equation which depends on the others we can compute $g(x_3), \dots, g(x_6)$ by using $g(x_1)$, $g(x_2)$, $g(1/3)$, $g(1/2)$, and $g(2/3)$ as follows:

$$\begin{aligned} g(x_3) &= g(1/3) + g(2/3) - g(1/2) - g(x_1) + g(x_2) \\ &\quad - g_1(h_2^{-1}hf(t)) + g_2(f_1^{-1}fh(t)), \\ g(x_4) &= g(1/2) - g(x_1) + g_1(h_1^{-1}hf(t)), \\ g(x_5) &= g(1/2) - g(x_2) + g_1(h_2^{-1}hf(t)), \\ g(x_6) &= g(1/3) + g(2/3) - g(1/2) + g(x_1) - g(x_2) \\ &\quad - g_1(h_1^{-1}hf(t)) + g_2(f_2^{-1}fh(t)), \end{aligned} \tag{1}$$

for all $t \in [0, 1/6]$. Putting $t = 0$ and $t = 1/6$, we find

$$\begin{aligned} g(1/2) &= 2g(1/3) - 2g(1/6) + g_1(1/3) - g_1(2/3) + g_2(1/2), \\ g(2/3) &= 2g(1/3) - 3g(1/6) + 2g_1(1/3) - g_1(2/3) + g_2(1/2). \end{aligned}$$

These values satisfy the necessary condition $g_1(1/3) + g_1(2/3) - g_2(1/2) = 2(g(1/3) + g(2/3)) - 3g(1/2)$. Moreover, $g(1/3) = g(x_2(1/6))$, $g(1/6) = g(x_2(0))$, and thus $g(x_3), \dots, g(x_6)$ can be expressed by only using $g(x_1)$, $g(x_2)$. Next, we simplify (1) by using

$$\begin{aligned} h_1^{-1}hf(t) &= ff_2^{-1}h_1^{-1}hf(t) = f(x_4) && \text{for } g(x_4), \\ h_2^{-1}hf(t) &= ff_2^{-1}h_2^{-1}hf(t) = f(x_5) && \text{for } g(x_5), \\ f_1^{-1}fh(t) &= hh_2^{-1}f_1^{-1}fh(t) = h(x_3) && \text{and} \\ h_2^{-1}hf(t) &= ff_1^{-1}h_2^{-1}hf(t) = f(x_2) && \text{for } g(x_3), \\ f_2^{-1}fh(t) &= hh_3^{-1}f_2^{-1}fh(t) = h(x_6) && \text{and} \\ h_1^{-1}hf(t) &= ff_1^{-1}h_1^{-1}hf(t) = f(x_1) && \text{for } g(x_6). \end{aligned}$$

Now, each $g(x_i)$ can be expressed as $g(x)$, $x \in [(i-1)/6, i/6]$. To do this we use the identity

$$x_i(x_j(t)) = x_i(t) \quad \text{for } t \in [0, 1] \text{ and } 1 \le i, j \le 6,$$

which immediately follows from the fact that

$$f_i^{-1}h_j^{-1}hff_k^{-1}h_l^{-1}hf(t) = f_i^{-1}h_j^{-1}hf(t).$$

For example,

$$g(x_3) = g(1/3) + g(2/3) - g(1/2) - g(x_1) + g(x_2) \\ - g_1(f(x_2)) + g_2(h(x_3)),$$

for $t \in [0, 1/6]$, which is the same as

$$g(x) = g(1/3) + g(2/3) - g(1/2) - g(x_1(x)) + g(x_2(x)) \\ - g_1(f(x_2(x))) + g_2(h(x))$$

for $x \in [2/6, 3/6]$. In our case $x_1(x) = x - i/6$ and $x_2(x) = x + 1/6 - i/6$ for $x \in [i/6, (i+1)/6]$ and $i = 0, \ldots, 5$.

Now, assuming the absolute continuity of $g(x_1)$ and $g(x_2)$ we can write

$$g(x_1(t)) = \int_0^t \psi(u)\,du,$$

$$g(x_2(t)) = \int_0^{1/6} \psi(u)\,du + \int_0^t \phi(u)\,du$$

for $t \in [0, 1/6]$.

Summing up the above we find the expression $g(x)$ in the theorem. For the monotonicity of $g(x)$ we can investigate $g'(x_i(t)) \geq 0$ for $t \in [0, 1/6]$ and $i = 1, \ldots, 6$, which immediately leads to the inequalities for ψ and ϕ given in our theorem. ∎

5. Examples and concluding remarks

1. Define a one-jump distribution function $c_\alpha : [0, 1] \to [0, 1]$ such that $c_\alpha(0) = 0$, $c_\alpha(1) = 1$, and

$$c_\alpha(x) = \begin{cases} 0 & \text{if } x \in [0, \alpha), \\ 1 & \text{if } x \in (\alpha, 1]. \end{cases}$$

The distribution functions $c_0(x)$, $c_1(x)$, and x satisfy $g_f(x) = g_h(x)$ for every $x \in [0, 1]$.

2. Taking $g_1(x) = g_2(x) = x$, further solutions of $g_f = g_h$ follow from Theorem 4. In this case

$$0 \leq \psi(t) \leq 2, \quad 0 \leq \phi(t) \leq 2, \quad -1 \leq \psi(t) - \phi(t) \leq 1,$$

for all $t \in [0, 1/6]$. Putting $\psi(t) = \phi(t) = 0$, the resulting distribution function is

$$g_3(x) = \begin{cases} 0 & \text{for } x \in [0, 2/6], \\ x - 1/3 & \text{for } x \in [2/6, 3/6], \\ 2x - 5/6 & \text{for } x \in [3/6, 5/6], \\ x & \text{for } x \in [5/6, 1]. \end{cases}$$

Taking $g_1(x) = g_2(x) = g_3(x)$, this $g_3(x)$ can be used as a starting point for a further application of Theorem 4 which gives another solution of $g_f = g_h$.

3. Computing $\int_{j/6}^{(j+1)/6} (\int_{i/6}^{(i+1)/6} F(x, y)\, dx)\, dy$ for $i, j = 1, \ldots, 5$ directly, we can find

$$\int_0^1\int_0^1 F(x, y)\, dg_3(x)\, dg_3(y) = 0,$$

which is also a consequence of Theorem 3 and $(g_3)_f = (g_3)_h$.

4. Since the mapping $g \to g_\phi$ is linear, the set of all solutions of $g_f = g_h$ is convex.

5. Since $x_f = x_h$, Theorem 2 leads to the fact that the following distribution function $g_4(x)$ is not a distribution function of $\xi(3/2)^n \bmod 1$, for any $\xi \in \mathbb{R}$:

$$g_4(x) = \begin{cases} x & \text{for } x \in [0, 2/3], \\ x^2 - (2/3)x + 2/3 & \text{for } x \in [2/3, 1]. \end{cases}$$

6. By Figure 1, $X = [2/9, 1/3] \cup [1/2, 1]$ is also a set of uniqueness. Moreover, $|X| = 11/18 < 2/3$. Similarly for $[0, 1/2] \cup [2/3, 7/9]$.

7. Since all the components of $f^{-1}([0, x))$ and $h^{-1}([0, x))$ are semiclosed the fact that, for fixed $\xi \neq 0$ and m, $\{\xi(3/2)^m\} = \{\xi(3/2)^n\}$ only for finitely many n, was not used in the proof of Theorem 1.

References

[1] M. Bennett, *Fractional parts of powers of rational numbers*, Math. Proc. Cambridge Philos. Soc. 114 (1993), 191–201.

[2] G. Choquet, *Construction effective de suites* $(k(3/2)^n)$. *Étude des mesures* (3/2)-*stables*, C. R. Acad. Sci. Paris Sér. A-B 291 (1980), A69–A74 (MR 82h:10062a-d).

[3] L. Flatto, J. C. Lagarias and A. D. Pollington, *On the range of fractional parts* $\{\xi(p/q)^n\}$, Acta Arith. 70 (1995), 125–147.

[4] H. Helson and J.-P. Kahane, *A Fourier method in diophantine problems*, J. Analyse Math. 15 (1965), 245–262 (MR 31#5856).

[5] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley, New York, 1974.

[6] K. Mahler, *An unsolved problem on the powers of* 3/2, J. Austral. Math. Soc. 8 (1968), 313–321 (MR 37#2694).

[7] I. I. Piatetski-Shapiro, *On the laws of distribution of the fractional parts of an exponential function*, Izv. Akad. Nauk SSSR Ser. Mat. 15 (1951), 47–52 (MR 13, 213d) (in Russian).

[8] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly 66 (1959), 361–375.
[9] R. Tijdeman, *Note on Mahler's* 3/2*-problem*, Norske Vid. Selsk. Skr. 16 (1972), 1–4.
[10] J. G. van der Corput, *Verteilungsfunktionen I–VIII*, Proc. Akad. Amsterdam 38 (1935), 813–821, 1058–1066; 39 (1936), 10–19, 19–26, 149–153, 339–344, 489–494, 579–590.
[11] T. Vijayaraghavan, *On the fractional parts of the powers of a number*, *I*, J. London Math. Soc. 15 (1940), 159–160.

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