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# Riemann–Hurwitz formula in basic $\mathbb{Z}_S$-extensions

by

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**1. Introduction.** Let $p$ be a prime number and $\mathbb{F}$ be a CM-field. Let $\mathbb{F}_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{F}$. For every $n$, we have a unique subextension $\mathbb{F}_n$ of degree $p^n$ over $\mathbb{F}$ in $\mathbb{F}_\infty$. We denote by $\mathbb{F}^+$ the maximal real subfield of $\mathbb{F}$, and let $h_n^-$ be the relative class number of $\mathbb{F}_n/\mathbb{F}_n^+$. Then we have a well known result:

$$\mathrm{ord}_p(h_n^-) = \mu^- p^n + \lambda^- n + \nu^-,$$

$\mu^- \geq 0$, $\lambda^- \geq 0$, and $\nu^-$ are integers, when $n$ is sufficiently large.

Let $\mathbb{E}$ be a CM-field and a $p$-extension of $\mathbb{F}$. Under the assumption $\mu_{\mathbb{F}}^- = 0$, Y. Kida ([5]) proved a striking analogue of the classical Riemann–Hurwitz genus formula from the theory of compact Riemann surfaces, by describing the behavior of $\lambda^-$ in the $p$-extension. His result can be stated as follows:

Theorem 0 (see [8, Theorem 4.1]). *$\mu_{\mathbb{F}}^- = 0$ if and only if $\mu_{\mathbb{E}}^- = 0$, and when this is the case*,

$$\lambda_{\mathbb{E}}^- - \delta_{\mathbb{E}} = [\mathbb{E}_\infty : \mathbb{F}_\infty](\lambda_{\mathbb{F}}^- - \delta_{\mathbb{F}}) + \sum_{\omega'} (e(\omega'/\nu') - 1) - \sum_{\omega} (e(\omega/\nu) - 1),$$

*where the summation is taken over all places $\omega'$ on $\mathbb{E}_\infty$ (resp. $\omega$ on $\mathbb{E}_\infty^+$) which do not lie above $p$ and $\nu' = \omega'|_{\mathbb{F}_\infty}$ (resp. $\nu = \omega|_{\mathbb{F}_\infty^+}$), $e(\omega/\nu)$ (resp. $e(\omega'/\nu')$) is the ramification index of $\omega$ (resp. $\omega'$) over $\nu$ (resp. $\nu'$) and $\delta_{\mathbb{E}} = 1$ or $0$ (resp. $\delta_{\mathbb{F}} = 1$ or $0$) according as $\mathbb{E}$ (resp. $\mathbb{F}$) contains $\zeta_p$ (or $\zeta_4$ if $p = 2$) or not.*

There are several ways to prove this result. K. Iwasawa ([4]) showed us a proof by using Galois cohomology. W. Sinnott ([8]) gave a proof by using the $p$-adic $L$-function and J. Satoh ([6]) obtained it by using the theory of $\Gamma$-transforms of rational functions. In this paper, we generalize the above result to basic $\mathbb{Z}_S$-extensions when $\mathbb{E}$ and $\mathbb{F}$ are abelian.

Let $S = \{p_1, \ldots, p_s\}$ be a finite set of primes, $\mathbb{Z}_S = \prod_{l \in S} \mathbb{Z}_l$ and $\mathbb{Q}_S$ be the $\mathbb{Z}_S$-extension of $\mathbb{Q}$. Then $\mathbb{F}_S = \mathbb{F}\mathbb{Q}_S$ is called the *basic $\mathbb{Z}_S$-extension* of $\mathbb{F}$. Let $N = p_1^{n_1} \dots p_s^{n_s}$ and $\mathbb{F}_N$ be the unique subextension of degree $N$ of $\mathbb{F}_S$. Let $h_N^-$ denote the relative class number of $\mathbb{F}_N/\mathbb{F}_N^+$. From a theorem of E. Friedman ([2]), when $\mathbb{F}$ is an imaginary abelian number field, we have

$$\mathrm{ord}_{p_i}(h_N^-) = \lambda^-(p_i, S)n_i + \nu^-(p_i, S),$$

where all $n_i$ are sufficiently large and $p_i \in S$.

In this paper, using the relationship between $\lambda^-(p_i, S)$ and the $\lambda$-invariant of the Dirichlet character of $\mathbb{F}$, we obtain the following main result.

THEOREM 1. *For fixed $p \in S$, let $\mathbb{E}$ and $\mathbb{F}$ be imaginary abelian number fields and $\mathbb{E}$ be a $p$-extension of $\mathbb{F}$. We have*

$$\begin{aligned}\lambda_{\mathbb{E}}^-(p,S) - \delta_{\mathbb{E}} = [\mathbb{E}_S : \mathbb{F}_S](\lambda_{\mathbb{F}}^-(p,S) - \delta_{\mathbb{F}}) \\ + \sum_{\omega'}(e(\omega'/\nu') - 1) - \sum_{\omega}(e(\omega/\nu) - 1),\end{aligned}$$

*where the summation is taken over all places $\omega'$ on $\mathbb{E}_S$ (resp. $\omega$ on $\mathbb{E}_S^+$) which do not lie above $p$ and $\nu' = \omega'|_{\mathbb{F}_S}$ (resp. $\nu = \omega|_{\mathbb{F}_S^+}$), and $e(\omega/\nu)$ (resp. $e(\omega'/\nu')$) is the ramification index of $\omega$ (resp. $\omega'$) over $\nu$ (resp. $\nu'$) and $\delta_{\mathbb{E}} = 1$ or $0$ (resp. $\delta_{\mathbb{F}} = 1$ or $0$) according as $\mathbb{E}$ (resp. $\mathbb{F}$) contains $\zeta_p$ (or $\zeta_4$ if $p = 2$) or not.*

**2. Preliminaries.** Let $p \in S$ be a fixed prime number and put

$$q = \begin{cases} 4, & p = 2, \\ p, & p \neq 2. \end{cases}$$

Let $\omega_p$ be the Teichmüller character mod $q$. For every $m \in \mathbb{Z}$ with $(m,p) = 1$ and $m \neq \pm 1$, we have

$$m = \omega_p(m)(1 + m_1 p^{n_m}),$$

with $m_1 \in \mathbb{Z}_p, (m_1, p) = 1$ and $n_m$ being a positive integer. We let $\mathbb{Q}^{(p)}$ denote the basic $\mathbb{Z}_p$-extension on $\mathbb{Q}$ and $T = S - \{p\}$.

Let $\mathcal{O}$ be a ring of integers of a finite extension over $\mathbb{Q}_p$ and let $f(X) = a_0 + a_1 X + \dots \in \mathcal{O}[[X]]$ be a non-zero power series. We define

$$\mu(f) = \min\{\mathrm{ord}_p\, a_i : i \geq 0\}, \qquad \lambda(f) = \min\{i \geq 0 : \mathrm{ord}_p\, a_i = \mu(f)\}.$$

Clearly we have $\mu(fg) = \mu(f) + \mu(g)$ and $\lambda(fg) = \lambda(f) + \lambda(g)$ if $f, g$ are non-zero elements of $\mathcal{O}[[X]]$. So $\mu$ and $\lambda$ can be defined in the quotient field of $\mathcal{O}[[X]]$ in a natural way.

Let $\mathbb{Z}_S^\times$ denote the unit group of $\mathbb{Z}_S$. So

$$\mathbb{Z}_S^\times = U_S \times V_S,$$

where $V_S$ is the torsion part of $\mathbb{Z}_S^\times$ and $U_S = \prod_{l \in S}(1 + 2l\mathbb{Z}_l)$. Let $\langle\ \rangle_S$ and $\omega_S$ denote the projections from $\mathbb{Z}_S^\times$ to $U_S$ and $V_S$ respectively. When $s = 1$, $\omega_S$ is the Teichmüller character. Let $\theta$ be an odd primitive Dirichlet character with values in $\mathbb{C}_p$, where $\mathbb{C}_p$ is a fixed completion of the algebraic closure of $\mathbb{Q}_p$. Any primitive Dirichlet character whose conductor is divisible only by the primes in $S$ can be regarded as a character of $\mathbb{Z}_S^\times$. Such a character is called *of the second kind for $S$* if it is trivial on $V_S$. For a character $\Psi$ of the second kind for $S$, we have the decomposition $\Psi = \Psi^{(p)}\Psi^{(T)}$, where $\Psi^{(p)}$ (resp. $\Psi^{(T)}$) is of the second kind for $p$ (resp. $T$) (see [9]).

Let $\theta$ be an odd primitive Dirichlet character with values in $\mathbb{C}_p$. Fix a generator $u$ of $U_p$. When $\theta\omega_p$ is not of the second kind for $p$, we define

$$\lambda(\theta) = \lambda(g_\theta(X-1)),$$

where

$$g_\theta(X-1) \in 2\mathcal{O}[[X-1]] \quad \text{with} \quad g_\theta(u^s-1) = L_p(s,\theta\omega_p)$$

and $L_p(s,\theta\omega_p)$ is the $p$-adic $L$-function associated with $\theta\omega_p$. When $\theta\omega_p$ is of the second kind for $p$, we define $\lambda(\theta) = -1$. The following proposition is Theorem 1 of [6].

PROPOSITION 1. *Let $\theta$ be an odd primitive Dirichlet character, $\tau$ be an even primitive Dirichlet character and $\mathcal{O}$ be the integer ring of the field generated over $\mathbb{Q}_p$ by the values of $\theta$ and $\tau$. Suppose*

(1) *$\tau$ has a $p$-power order and its conductor $l$ is a prime number*,

(2) *for all $a \in \mathbb{Z}, \theta\tau(a) = \theta(a)\tau(a)$.*

*Then*

(i) *if $\theta \neq \omega_p^{-1}$, we have*

$$\lambda(\theta\tau) = \begin{cases} \lambda(\theta) + p^{n_l}/q & \text{if } \theta(l) \equiv 1 \bmod \wp, \\ \lambda(\theta) & \text{if } \theta(l) \not\equiv 1 \bmod \wp, \end{cases}$$

*where $\wp$ is a prime ideal of $\mathcal{O}$ above $p$,*

(ii) *if $\theta = \omega_p^{-1}$, we have*

$$\lambda(\theta\tau) = \frac{p^{n_l}}{q} - 1.$$

Remark 1. This proposition can also be proved by using the $p$-adic $L$-function (see [8, §2]).

PROPOSITION 2. *Let $\theta$ be an odd primitive Dirichlet character of order prime to $p$, $\tau$ be an even primitive Dirichlet character of $p$-power order and $\theta\tau(a) = \theta(a)\tau(a)$. Suppose the conductor $f(\tau)$ of $\tau$ is prime to $p$. Write $f(\tau) = \prod_l l^{k_l}$, where $k_l \geq 1$ and $l$ are primes. Then*

(i) $k_l = 1$, *for all* $l$,
(ii) *if* $\theta \neq \omega_p^{-1}$, *then*

$$\lambda(\theta\tau) = \lambda(\theta) + \sum_{\substack{l \\ \theta(l)=1}} \frac{p^{n_l}}{q},$$

*if* $\theta = \omega_p^{-1}$, *then*

$$\lambda(\theta\tau) = \Big(\sum_{l} \frac{p^{n_l}}{q}\Big) - 1.$$

Proof. (i) By the Chinese Remainder Theorem, we have $\tau = \prod_l \tau_l$, where $l^{k_l}$ is the conductor of $\tau_l$ and $\tau_l$ has $p$-power order.

If $k_l \neq 1$, consider the natural map

$$i : \mathbb{Z}/(l^{k_l}) \to \mathbb{Z}/(l^{k_l - 1}).$$

For any $x \in \ker i$, $x$ has $l$-power order. Thus $\tau_l(x)$ is an $l$-power root of unity. Note $\tau_l$ has $p$-power order and $(p, l) = 1$, and so we have $\tau_l(x) = 1$. This is a contradiction because $l^{k_l}$ is the conductor of $\tau_l$.

(ii) When $\theta \neq \omega_p^{-1}$, the assertion follows from Proposition 1 and (i) since $\theta\tau(l) \equiv 1 \bmod \wp$ if and only if $\theta(l) = 1$. If $\theta = \omega_p^{-1}$, then $l \equiv 1 \bmod p$ since $\tau_l$ has $p$-power order. Therefore $\theta(l) \equiv 1 \bmod \wp$ and we are done by Proposition 1. ∎

**3. The number of splitting primes.** Let $\mathbf{k}$ be a finite abelian extension of $\mathbb{Q}$. In this section, we compute the number of primes of $\mathbf{k}_S$ above a prime number $l$, which is closely related to the characters of the Galois group. The *character group* of an abelian profinite group $G$ is the set of continuous homomorphisms from $G$ to the roots of unity in $\mathbb{C}_p^\times$ with the induced topology. We denote this character group as $G^\wedge$.

Now we take $\chi \in \mathrm{Gal}(\mathbf{k}_S/\mathbb{Q})^\wedge$. Then $\ker\chi$ is a close subgroup with finite index of $\mathrm{Gal}(\mathbf{k}_S/\mathbb{Q})$ (an open subgroup) and $\chi$ is essentially a usual Dirichlet character. Let $\mathbf{k}^\chi$ be the subfield of $\mathbf{k}_S$ fixed by $\ker\chi$. Then we define

$$\chi(l) = \begin{cases} 0 & \text{if } l \text{ is ramified in } \mathbf{k}^\chi, \\ \chi(\mathrm{Frob}_l) & \text{if } l \text{ is unramified in } \mathbf{k}^\chi. \end{cases}$$

Keeping the above notations, we have the following lemma:

Lemma 1. *For any prime number* $l$,

(i) *there are finitely many primes in* $\mathbf{k}_S$ *above* $l$,
(ii) *the number of primes above* $l$ *in* $\mathbf{k}_S$ *is equal to*

$$\#\{\chi \in \mathrm{Gal}(\mathbf{k}_S/\mathbb{Q})^\wedge : \chi(l) = 1\}.$$

Proof. (i) First consider $S = \{p\}$. Let $\mathcal{Q}$ be a prime in $\mathbf{k}$ above $l$. If $l = p$, the assertion is trivial by [10, Lemma 13.3]. If $l \neq p$, then $\mathcal{Q}$ is unramified in $\mathbf{k}_S/\mathbf{k}$. Write

$$l = \omega_p(l)(1 + p^{n_l} l_1).$$

Then the number of primes of $\mathbf{k}$ above $\mathcal{Q}$ is equal to

$$\begin{aligned}&\#(\mathrm{Gal}(\mathbf{k}_S/\mathbf{k})/\overline{\langle \mathrm{Frob}_{\mathcal{Q}} \rangle}) \\ &\qquad \le \#(\mathrm{Gal}(\mathbb{Q}^{(p)}/\mathbb{Q})/\overline{\langle \mathrm{Frob}_l \rangle}) \cdot [\mathbf{k} : \mathbb{Q}] \le p^{n_l}[\mathbf{k} : \mathbb{Q}] < \infty\end{aligned}$$

and the case $s = 1$ is proved.

If $s > 1$, let $D(\mathcal{Q})$ be the decomposition group of $\mathcal{Q}$. Then $D(\mathcal{Q})$ is a closed subgroup of $\mathbb{Z}_S$ and has the form $p_1^{t_1}\mathbb{Z}_{p_1} \times \ldots \times p_s^{t_s}\mathbb{Z}_{p_s}$, $0 \le t_i \le \infty$, $i = 1, \ldots, s$, where $p_i^{\infty}\mathbb{Z}_{p_i} = 0$. It is sufficient to prove that $t_i < \infty$, $i = 1, \ldots, s$. Suppose $t_i = \infty$. Let $\mathbf{k}^{(p_i)} \subseteq \mathbb{L}$ be a basic $\mathbb{Z}_{p_i}$-extension of $\mathbf{k}$ and $D^{(p_i)}(\mathcal{Q})$ be the decomposition group of $\mathcal{Q}$ over $\mathbf{k}^{(p_i)}$. So we have

$$D^{(p_i)}(\mathcal{Q}) = D(\mathcal{Q})|_{\mathrm{Gal}(\mathbf{k}^{(p_i)}/\mathbf{k})} = 0.$$

This is a contradiction to the case of $s = 1$ and (i) is proved.

(ii) Let $D(l)$ denote the decomposition group of a prime in $\mathbf{k}_S$ above $l$. Then the number of primes in $\mathbf{k}_S$ above $l$ is equal to

$$\begin{aligned}\#(\mathrm{Gal}(\mathbf{k}_S/\mathbb{Q})/D(l)) &= \#((\mathrm{Gal}(\mathbf{k}_S/\mathbb{Q})/D(l))^{\wedge}) \\ &= \#\{\chi \in \mathrm{Gal}(\mathbf{k}_S/\mathbb{Q})^{\wedge} : \chi(l) = 1\}.\end{aligned}$$

This is the desired result. ■

Remark 2. Lemma 1 is not true for arbitrary $\mathbb{Z}_S$-extension (see [10, Ex. 13.2]).

From Lemma 1, we immediately have the following lemma:

LEMMA 2. *Suppose* $\mathbf{k} \cap \mathbb{Q}_S = \mathbb{Q}$, $p \in S$ *with* $p \nmid [\mathbf{k} : \mathbb{Q}]$, $T = S - \{p\}$ *and* $l$ *is a prime number different from* $p$. *Then the number of prime ideals above* $l$ *in* $\mathbf{k}\mathbb{Q}_S$ *is*

$$\begin{aligned}&\#\{\chi \in \mathrm{Gal}(\mathbf{k}\mathbb{Q}_T/\mathbb{Q})^{\wedge} : \chi(l) = 1\}\#\{\chi \in \mathrm{Gal}(\mathbb{Q}^{(p)}/\mathbb{Q})^{\wedge} : \chi(l) = 1\} \\ &\qquad = (p^{n_l}/q)\#\{\chi \in \mathrm{Gal}(\mathbf{k}\mathbb{Q}_T/\mathbb{Q})^{\wedge} : \chi(l) = 1\}.\end{aligned}$$

Proof. By Lemma 1, it is sufficient to prove

$$\begin{aligned}&\#\{\chi \in \mathrm{Gal}(\mathbf{k}\mathbb{Q}_S/\mathbb{Q})^{\wedge} : \chi(l) = 1\} \\ &\qquad = \#\{\chi \in \mathrm{Gal}(\mathbf{k}\mathbb{Q}_T/\mathbb{Q})^{\wedge} : \chi(l) = 1\}\#\{\chi \in \mathrm{Gal}(\mathbb{Q}^{(p)}/\mathbb{Q})^{\wedge} : \chi(l) = 1\}.\end{aligned}$$

Since

$$\mathrm{Gal}(\mathbf{k}\mathbb{Q}_S/\mathbb{Q}) \cong \mathrm{Gal}(\mathbf{k}\mathbb{Q}_T/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}^{(p)}/\mathbb{Q}),$$

we have

$$\mathrm{Gal}(\mathbf{k}\mathbb{Q}_S/\mathbb{Q})^{\wedge} \cong \mathrm{Gal}(\mathbf{k}\mathbb{Q}_T/\mathbb{Q})^{\wedge} \times \mathrm{Gal}(\mathbb{Q}^{(p)}/\mathbb{Q})^{\wedge}.$$

Therefore for any $\chi \in \mathrm{Gal}(\mathbf{k}\mathbb{Q}_S/\mathbb{Q})^\wedge$, we have $\chi = \chi_T \cdot \chi_p$, with $\chi_T \in \mathrm{Gal}(\mathbf{k}\mathbb{Q}_T/\mathbb{Q})^\wedge$, $\chi_p \in \mathrm{Gal}(\mathbb{Q}^{(p)}/\mathbb{Q})^\wedge$ and $\chi(l) = \chi_T(l)\chi_p(l)$. Note $\chi_p(l)$ is a $p$-power root of unity and $\chi_T(l)$ is not, so we have

$$\chi(l) = 1 \Leftrightarrow \chi_T(l) = 1 \text{ and } \chi_p(l) = 1$$

and Lemma 2 is proved. ∎

**4. Proof of Theorem 1.** First let $\mathbf{k}$ be a finite abelian extension of $\mathbb{Q}$ and we use the following notations associated with $\mathbf{k}$:

• $X_{\mathbf{k}}$ (resp. $X_{\mathbf{k}}^-$): the set of all (resp. odd) Dirichlet characters associated with $\mathbf{k}$.

• $X_{\mathbf{k}}(l)$ (resp. $X_{\mathbf{k}}^-(l)$): all the elements of $X_{\mathbf{k}}$ (resp. $X_{\mathbf{k}}^-$) whose conductors are divisible by a prime number $l$.

• $J_{\mathbf{k}}(l)$: all the elements of $X_{\mathbf{k}}$ whose conductors are prime to a prime number $l$.

• We write $\chi_{\mathbf{k}}$ for an element of $X_{\mathbf{k}}$ and $\mathfrak{f}_{\mathbf{k}}$ as the conductor of $\mathbf{k}$. Let $e$, $f$ and $g$ denote the usual meaning as the ramification index, the residue class degree, the number of splitting primes respectively. For a prime number $l$, by [10, Th. 3.7], we have

$$\#J_{\mathbf{k}}(l) = f_{\mathbf{k}}(l)g_{\mathbf{k}}(l) \quad \text{and} \quad \#(X_{\mathbf{k}}/J_{\mathbf{k}}(l)) = e_{\mathbf{k}}(l).$$

Now $\mathbb{E}$, $\mathbb{F}$ are the same as in Section 1. Let $\mathbb{K}$ be the maximal $p$-extension of $\mathbb{Q}$ in $\mathbb{E}$, and $\mathbb{L}$ be the maximal extension of $\mathbb{Q}$ in $\mathbb{E}$ with $p \nmid [\mathbb{L} : \mathbb{Q}]$. $\omega$ (resp. $\omega'$) is a prime of $\mathbb{E}_S^+$ (resp. $\mathbb{E}_S$) which does not lie over the prime $p$, $\nu = \omega|_{\mathbb{F}_S^+}$ (resp. $\nu' = \omega'|_{\mathbb{F}_S}$) and $u = \omega|_{\mathbb{L}_S^+}$ (resp. $u' = \omega|_{\mathbb{L}_S}$).

Suppose $\omega|_{\mathbb{Q}} = l \neq p$. Since the residue field at $u$ or $u'$ has no finite $p$-extensions, it is clear that $f(\omega/u) = f(\omega'/u') = 1$. Furthermore,

$$e_{\mathbb{K}}(l) = e(\omega'/u'), \quad e_{\mathbb{K}^+}(l) = e(\omega/u),$$
$$\#J_{\mathbb{K}} = g(\omega'/u'), \quad \#J_{\mathbb{K}^+} = g(\omega/u).$$

We also note the following:

1) It is easy to check that if Theorem 1 holds for two of $\mathbb{E}/\mathbb{F}, \mathbb{K}/\mathbb{F}$ and $\mathbb{E}/\mathbb{K}$, it holds for the third. This allows us to reduce ourselves to the case where $[\mathbb{F} : \mathbb{Q}]$ is not divisible by $p$ for $p > 2$.

2) We can also assume $\mathbb{E} \cap \mathbb{F}_S = \mathbb{F}$, $\mathbb{F} \cap \mathbb{Q}_S = \mathbb{Q}$ and the conductor of $\mathbb{E}$ is not divisible by $qp$, since any number field between $\mathbb{E}$ and $\mathbb{E}_S$ has the same $\lambda(p, S)$-invariant as that of $\mathbb{E}$.

3) By the above assumption, we have $[\mathbb{E}_S : \mathbb{F}_S] = [\mathbb{E} : \mathbb{F}]$ and $\mathbb{E} \cap \mathbb{Q}_S = \mathbb{Q}$.

With the above notations, we have the following lemma:

LEMMA 3.

$$\sum_{\omega'}(e(\omega'/\nu')-1)-\sum_{\omega}(e(\omega/\nu)-1)$$
$$=\begin{cases}\sum_l p^{n_l-1}\#X_{\mathbb{K}}(l)\#\{\chi_{\mathbb{F}}\Psi^{(T)}:\chi_{\mathbb{F}}\ \textit{odd},\chi_{\mathbb{F}}\Psi^{(T)}(l)=1\} & \textit{if } p>2,\\ \sum_l 2^{n_l-2}\{\#X^-_{\mathbb{K}}(l)-[\mathbb{E}:\mathbb{F}]\#X_{\mathbb{F}\cap\mathbb{K}}(l)^-\}\#\{\chi_{\mathbb{L}}\Psi^{(T)}:\ \chi_{\mathbb{L}}\Psi^{(T)}(l)=1\}, & \\ & \textit{if } p=2,\end{cases}$$

*where $\omega'$ (resp. $\omega$) runs over all the primes in $\mathbb{E}_S$ (resp. $\mathbb{E}_S^+$) which do not lie over $p$, $l$ runs over all the prime numbers different from $p$ and $\Psi^{(T)}$ is taken over the characters of* $\mathrm{Gal}(\mathbb{Q}_T/\mathbb{Q})$.

P r o o f. We have

$$(*)\qquad \sum_{\omega'}(e(\omega'/u')-1)-\sum_{\omega}(e(\omega/u)-1)$$
$$=\sum_{u'}g(\omega'/u')(e(\omega'/u')-1)-\sum_{u}g(\omega/u)(e(\omega/u)-1).$$

If $p>2$, then $\mathbb{F}=\mathbb{L}$, $\nu=u$, $\nu'=u'$ and $\mathbb{K}=\mathbb{K}^+$. By Lemma 2, we have

$$\begin{aligned}(*)&=\sum_{l\neq p}\#X_{\mathbb{K}}(l)\sum_{u'\cap\mathbb{Q}=l}1-\sum_{l\neq p}\#X_{\mathbb{K}}(l)\sum_{u\cap\mathbb{Q}=l}1\\ &=\sum_{l\neq p}\#X_{\mathbb{K}}(l)\#\{\chi_{\mathbb{F}}\Psi^{(T)}:\chi_{\mathbb{F}}\Psi^{(T)}(l)=1\}p^{n_l-1}\\ &\quad-\sum_{l\neq p}\#X_{\mathbb{K}}(l)\#\{\chi_{\mathbb{F}^+}\Psi^{(T)}:\chi_{\mathbb{F}^+}\Psi^{(T)}(l)=1\}p^{n_l-1}\\ &=\sum_{l\neq p}\#X_{\mathbb{K}}(l)p^{n_l-1}\#\{\chi_{\mathbb{F}}\Psi^{(T)}:\chi_{\mathbb{F}}\ \mathrm{odd},\chi_{\mathbb{F}}\Psi^{(T)}(l)=1\}.\end{aligned}$$

For $p=2$, we have $\mathbb{F}\supset\mathbb{L}$ and $\mathbb{L}=\mathbb{L}^+$. So

$$(1)\qquad (*)=\sum_{l\neq p}\#X_{\mathbb{K}}(l)\sum_{u'|_{\mathbb{Q}}=l}1-\sum_{l\neq p}\#X_{\mathbb{K}^+}(l)\sum_{u|_{\mathbb{Q}}=l}1$$
$$=\sum_{l\neq p}\#X^-_{\mathbb{K}}(l)\ 2^{n_l-2}\ \#\{\chi_{\mathbb{L}}\Psi^{(T)}:\chi_{\mathbb{L}}\Psi^{(T)}(l)=1\}.$$

Let $\mathbb{E}=\mathbb{F}$. We have

$$(2)\qquad \sum_{\nu'}(e(\nu'/u')-1)-\sum_{\nu}(e(\nu/u)-1)$$
$$=\sum_{l\neq p}2^{n_l-2}\#X^-_{\mathbb{K}\cap\mathbb{F}}(l)\#\{\chi_{\mathbb{L}}\Psi^{(T)}:\chi_{\mathbb{L}}\Psi^{(T)}(l)=1\}.$$

Since $[\mathbb{E}_S:\mathbb{F}_S]=[\mathbb{E}:\mathbb{F}]$ and $f(\omega'/\nu')=1$, we have

$$e(\omega'/\nu')g(\omega'/\nu') = [\mathbb{E}:\mathbb{F}], \quad e(\omega'/u') = e(\omega'/\nu')e(\nu'/u').$$

Then

$$[\mathbb{E}:\mathbb{F}]\sum_{\nu'}(e(\nu'/u')-1) \\ = \sum_{\nu'} g(\omega'/\nu')(e(\omega'/u') - e(\omega'/\nu')) = \sum_{\omega'}(e(\omega'/u') - e(\omega'/\nu')).$$

The same is true for $\omega, u, \nu$. By (1) and (2), we obtain

$$\sum_{\omega'}(e(\omega'/\nu')-1) - \sum_{\omega}(e(\omega/\nu)-1) \\ = \sum_{l\neq p} 2^{n_l-2}\{\#X^-_{\mathbb{K}}(l) - [\mathbb{E}:\mathbb{F}]\#X^-_{\mathbb{K}\cap\mathbb{F}}(l)\}\#\{\chi_{\mathbb{L}}\Psi^{(T)} : \chi_{\mathbb{L}}\Psi^{(T)}(l) = 1\}. \ ■$$

Now we begin our proof of the main theorem.

Proof of Theorem 1. We know that for any imaginary abelian field $\mathbf{k}$, $\lambda(p,S)$ satisfies the following relation (cf. [9]):

$$\lambda^-_{\mathbf{k}}(p,S) = \delta_{\mathbf{k}} + \sum_{\theta}\sum_{\Psi^{(T)}}\lambda(\theta\Psi^{(T)}),$$

where the outer sum is taken over all odd characters of $\mathbf{k}/\mathbb{Q}$ and the inner sum is taken over all $\Psi^{(T)} \in \mathrm{Gal}(\mathbb{Q}_T/\mathbb{Q})^\wedge$ with $\lambda(\theta\Psi^{(T)}) \neq 0$, and $\delta_{\mathbf{k}} = 1$ if and only if $\omega_p$ is a character of $\mathbf{k}/\mathbb{Q}$. Therefore

$$(**) \qquad \lambda^-_{\mathbb{E}}(p,S) - \delta_{\mathbb{E}} = \sum_{\chi_{\mathbb{E}}\text{ odd}}\sum_{\Psi^{(T)}}\lambda(\chi_{\mathbb{E}}\Psi^{(T)}) = \sum_{\chi_{\mathbb{L}}}\sum_{\chi_{\mathbb{K}}}\sum_{\Psi^{(T)}}\lambda(\chi_{\mathbb{L}}\chi_{\mathbb{K}}\Psi^{(T)})$$

where $\chi_{\mathbb{K}}\chi_{\mathbb{L}}$ is odd.

When $p > 2$, the conductor of $\chi \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})^\wedge$ is not divisible by $p$ since $\mathfrak{f}_{\mathbb{E}}$ is a not divisible by $p^2$ and $[\mathbb{K}:\mathbb{Q}]$ is a $p$-power. Note $\mathbb{L} = \mathbb{F}$ and $\mathbb{K} = \mathbb{K}^+$ in this case. By Propositions 1 and 2, we have

$$\begin{aligned}(**) &= \sum_{\chi_{\mathbb{F}}\text{ odd}}\sum_{\chi_{\mathbb{K}}}\sum_{\Psi^{(T)}}\Big(\lambda(\chi_{\mathbb{F}}\Psi^{(T)}) + \sum_{\substack{l\mid f(\chi_{\mathbb{K}})\\ \chi_{\mathbb{F}}\Psi^{(T)}(l)=1}} p^{n_l-1}\Big) \\ &= [\mathbb{E}:\mathbb{F}]\sum_{\chi_{\mathbb{F}}\text{ odd}}\sum_{\Psi^{(T)}}\lambda(\chi_{\mathbb{F}}\Psi^{(T)}) + \sum_{\chi_{\mathbb{F}}\text{ odd}}\sum_{\Psi^{(T)}}\sum_{\substack{l\neq p\\ \chi_{\mathbb{F}}\Psi^{(T)}(l)=1}}\#X_{\mathbb{K}}(l)p^{n_l-1} \\ &= [\mathbb{E}:\mathbb{F}](\lambda^-_{\mathbb{F}}(p,S) - \delta_{\mathbb{F}}) + \sum_{l\neq p} p^{n_l-1}\#X_{\mathbb{K}}(l)\sum_{\chi_{\mathbb{F}}\text{ odd},\chi_{\mathbb{F}}\Psi^{(T)}(l)=1} 1 \\ &= [\mathbb{E}:\mathbb{F}](\lambda^-_{\mathbb{F}}(p,S) - \delta_{\mathbb{F}})\end{aligned}$$

$$+\sum_{l\neq p} p^{n_l-1}\#X_{\mathbb{K}}(l)\#\{\chi_{\mathbb{F}}\Psi^{(T)} : \chi_{\mathbb{F}} \text{ odd}, \chi_{\mathbb{F}}\Psi^{(T)}(l)=1\}$$
$$= [\mathbb{E}:\mathbb{F}](\lambda_{\mathbb{F}}^-(p,S)-\delta_{\mathbb{F}}) + \sum_{\omega'}(e(\omega'/\nu')-1) - \sum_{\omega}(e(\omega/\nu)-1).$$

If $p=2$, then $\mathbb{L}=\mathbb{L}^+$, $\mathbb{L}\subset\mathbb{F}$ and the conductor of each character of $\mathbb{K}$ is not divisible by 8. By [6, Th. 1],

$$\sum_{\chi_{\mathbb{K}} \text{ odd}} \lambda(\chi_{\mathbb{K}}) = \sum_{l\neq p} 2^{n_l-2}\#X_{\mathbb{K}}^-(l) - [\mathbb{K}^+:\mathbb{Q}].$$

Since $\mathbb{K}\cap\mathbb{F}$ is an imaginary abelian extension of $\mathbb{Q}$, we can choose a primitive odd character $\chi_0$ of $\mathrm{Gal}((\mathbb{F}\cap\mathbb{K})/\mathbb{Q})$ with order 2. Then, for any $\chi\in X_{\mathbb{K}}^-$, we have $\chi=\chi_0\widetilde{\chi}$ with $\widetilde{\chi}\in X_{\mathbb{K}^+}$. By Propositions 1 and 2, we have

$$\sum_{\chi_{\mathbb{K}} \text{ odd}} \sum_{\chi_{\mathbb{L}}\Psi^{(T)}\neq 1} \lambda(\chi_{\mathbb{K}}\chi_{\mathbb{L}}\Psi^{(T)})$$
$$= \sum_{l\neq p} 2^{n_l-2}\#X_{\mathbb{K}}^-(l)\#\{\chi_{\mathbb{L}}\Psi^{(T)}\neq 1 : \chi_{\mathbb{L}}\Psi^{(T)}(l)=1\}$$
$$+ [\mathbb{K}^+:\mathbb{Q}] \sum_{\chi_{\mathbb{L}}\Psi^{(T)}\neq 1} \lambda(\chi_0\chi_{\mathbb{L}}\Psi^{(T)}).$$

Therefore

$$(**) = \sum_{\chi_{\mathbb{L}}} \sum_{\chi_{\mathbb{K}} \text{ odd}} \sum_{\Psi^{(T)}} \lambda(\chi_{\mathbb{K}}\chi_{\mathbb{L}}\Psi^{(T)}) \tag{3}$$
$$= \sum_{\chi_{\mathbb{K}} \text{ odd}} \sum_{\chi_{\mathbb{L}}\Psi^{(T)}\neq 1} \lambda(\chi_{\mathbb{K}}\chi_{\mathbb{L}}\Psi^{(T)}) + \sum_{\chi_{\mathbb{K}} \text{ odd}} \lambda(\chi_{\mathbb{K}})$$
$$= [\mathbb{K}^+:\mathbb{Q}]\Big(\sum_{\chi_{\mathbb{L}}\Psi^{(T)}\neq 1} \lambda(\chi_0\chi_{\mathbb{L}}\Psi^{(T)}) - 1\Big)$$
$$+ \sum_{l\neq p} 2^{n_l-2}\#X_{\mathbb{K}}^-(l)\#\{\chi_{\mathbb{L}}\Psi^{(T)} : \chi_{\mathbb{L}}\Psi^{(T)}(l)=1\}.$$

If we set $\mathbb{E}=\mathbb{F}$ in the above equality, then we obtain

$$\lambda_{\mathbb{F}}^-(2,S)-\delta_{\mathbb{F}} = [\mathbb{K}^+\cap\mathbb{F}:\mathbb{Q}]\Big(\sum_{\chi_{\mathbb{L}}\Psi^{(T)}\neq 1} \lambda(\chi_0\chi_{\mathbb{L}}\Psi^{(T)}) - 1\Big) \tag{4}$$
$$+ \sum_{l\neq p} 2^{n_l-2}\#X_{\mathbb{K}\cap\mathbb{F}}^-(l)\#\{\chi_{\mathbb{L}}\Psi^{(T)} : \chi_{\mathbb{L}}\Psi^{(T)}(l)=1\}.$$

By (3)–$[\mathbb{E}^+:\mathbb{F}^+]$(4) we obtain the desired result since $[\mathbb{E}^+:\mathbb{F}^+][\mathbb{K}^+\cap\mathbb{F}:\mathbb{Q}] = [\mathbb{K}^+:\mathbb{Q}]$ and $[\mathbb{E}:\mathbb{F}]=[\mathbb{E}^+:\mathbb{F}^+]=[\mathbb{E}_S:\mathbb{F}_S]$. ∎

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