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Gauss sums for $O^{-}(2n, q)$

by

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For my father, Chang Hong Kim

1. Introduction. Let λ be a nontrivial additive character of the finite field $\mathbb{F}_q$, and let χ be a multiplicative character of $\mathbb{F}_q$. Unless otherwise stated, in this paper we assume that q is a power of an odd prime. Then we consider the exponential sum

$$\sum_{w \in \mathrm{SO}^{-}(2n,q)} \lambda(\operatorname{tr} w), \tag{1.1}$$

where $\mathrm{SO}^{-}(2n, q)$ is a special orthogonal group over $\mathbb{F}_q$ (cf. (2.8)) and $\operatorname{tr} w$ is the trace of w. Also, we consider

$$\sum_{w \in O^{-}(2n,q)} \chi(\det w)\lambda(\operatorname{tr} w), \tag{1.2}$$

where $O^{-}(2n, q)$ is an orthogonal group over $\mathbb{F}_q$ (cf. (2.2), (2.5), (2.6)) and $\det w$ is the determinant of w.

The purpose of this paper is to find explicit expressions for the sums (1.1) and (1.2). It turns out that both of them are polynomials in q with coefficients involving powers of ordinary Kloosterman sums and the average (over multiplicative characters of all orders) of squares of Gauss sums.

In [5], Hodges expressed certain exponential sums in terms of what we call the "generalized Kloosterman sum over nonsingular symmetric matrices" $K_{\mathrm{sym},t}(A, B)$ (for m even in the main theorem of [5]) and the "signed generalized Kloosterman sum over nonsingular symmetric matrices"

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$L_{\mathrm{sym},t}(A,B)$ (for m odd in the main theorem of [5]), where A,B are $t\times t$ symmetric matrices over $\mathbb{F}_q$ (cf. (7.1) and [9], (7.1)). Some of their general properties were investigated in [5], and, for A or B zero, they were evaluated in [4] (see also [5], Theorem 10). However, they have never been explicitly computed for both A and B nonzero.

From a corollary to the main theorem in [5] and using an explicit expression of a sum similar to (1.2) but over $O(2n+1,q)$, we were able to find, in [9], an expression for $L_{\mathrm{sym},2n+1}\big(\frac{a^2}{4}C^{-1},C\big)$, where C is a nonsingular symmetric matrix of size $2n+1$ over $\mathbb{F}_q$ and $0\neq a\in\mathbb{F}_q$.

In this paper, from the corollary mentioned above and Theorem 6.1, we will be able to find an explicit expression for $K_{\mathrm{sym},2n}\big(\frac{a^2}{4}C^{-1},C\big)$, where C is now a nonsingular symmetric matrix of size $2n$ with $C\sim J^-$ (cf. (2.17) and (4.2) with $r=2n$) and $0\neq a\in\mathbb{F}_q$ as before. $K_{\mathrm{sym},2n}\big(\frac{a^2}{4}C^{-1},C\big)$ for $C\sim J^+$ (cf. (4.1) with $r=2n$) was determined in [11].

Similar sums for other classical groups over a finite field have been considered and the results for these sums will appear elsewhere ([9]–[11]).

Finally, we would like to state the main results of this paper. For some symbols, one is referred to the next section.

THEOREM A. *The sum* $\sum_{w\in\mathrm{SO}^-(2n,q)}\lambda(\operatorname{tr}w)$ *in* (1.1) *equals*

$$
\begin{aligned}
(1.3)\qquad & q^{n^2-n-1}\bigg\{\Big(-\frac{1}{q-1}\sum_{j=1}^{q-1}G(\psi^j,\lambda)^2\Big)\\
&\times\sum_{r=0}^{[(n-1)/2]}q^{r(r+3)}\begin{bmatrix}n-1\\2r\end{bmatrix}_q\prod_{j=1}^{r}(q^{2j-1}-1)\\
&\times\sum_{l=1}^{[(n-2r+1)/2]}q^{l}K(\lambda;1,1)^{n-2r+1-2l}\sum(q^{j_1}-1)\dots(q^{j_{l-1}}-1)\\
&-(q+1)\sum_{r=0}^{[(n-2)/2]}q^{r(r+3)+1}\begin{bmatrix}n-1\\2r+1\end{bmatrix}_q\prod_{j=1}^{r+1}(q^{2j-1}-1)\\
&\times\sum_{l=1}^{[(n-2r)/2]}q^{l}K(\lambda;1,1)^{n-2r-2l}\sum(q^{j_1}-1)\dots(q^{j_{l-1}}-1)\bigg\},
\end{aligned}
$$

where the first and second unspecified sums are respectively over all integers $j_1,\dots,j_{l-1}$ *satisfying* $2l-3\le j_1\le n-2r-2, 2l-5\le j_2\le j_1-2,\dots,1\le j_{l-1}\le j_{l-2}-2$ *and over the same set of integers satisfying* $2l-3\le j_1\le n-2r-3, 2l-5\le j_2\le j_1-2,\dots,1\le j_{l-1}\le j_{l-2}-2$. *Here* $K(\lambda;1,1)$ *is the usual Kloosterman sum* (cf. (2.11)), *and* $G(\psi^j,\lambda)$ *and* $G(\eta,\lambda)$ *are the* *usual Gauss sums* (cf. (2.9)), *where ψ is a multiplicative character of $\mathbb{F}_q$ of order $q-1$ and η is the quadratic character of $\mathbb{F}_q$.*

THEOREM B. *The sum $\sum_{w\in O^-(2n,q)} \chi(\det w)\lambda(\operatorname{tr} w)$ in* (1.2) *is the same as the expression in* (1.3), *except that $-\frac{1}{q-1}\sum_{j=1}^{q-1} G(\psi^j,\lambda)^2$ appearing in the first sum and $q+1$ appearing in the second sum are respectively replaced by $A(\chi,\lambda)$ and $\chi(-1)A(\chi,\lambda)$, where*

$$A(\chi,\lambda) = -\frac{1}{q-1}\sum_{j=1}^{q-1} G(\psi^j,\lambda)^2 + \chi(-1)(q+1).$$

THEOREM C. *Let $0 \neq a \in \mathbb{F}_q$. Then, for any nonsingular symmetric matrix C over $\mathbb{F}_q$ of size $2n$ with $C \sim J^-$* (cf. (2.17) and (4.2)), *the Kloosterman sum over nonsingular symmetric matrices* (cf. (7.1)) *is independent of C, and*

$$K_{\mathrm{sym},2n}\left(\frac{a^2}{4}C^{-1}, C\right) = q^n \sum_{w\in O^-(2n,q)} \lambda_a(\operatorname{tr} w),$$

so that it equals q^n times the expression in Theorem B *with χ trivial, $\lambda = \lambda_a$.*

The above Theorems A, B, and C are respectively stated as Theorem 5.2, Theorem 6.1, and Theorem 7.1.

2. Preliminaries. In this section, we fix some notations that will be used in the sequel, describe some basic groups, recall some classical sums and mention the q-binomial theorem. One may refer to [1], [2] and [12] for some elementary facts of the following.

Let $\mathbb{F}_q$ denote the finite field with q elements, $q = p^d$ ($p > 2$ an odd prime, d a positive integer).

Let λ be an additive character of $\mathbb{F}_q$. Then $\lambda = \lambda_a$ for a unique $a \in \mathbb{F}_q$, where, for $\alpha \in \mathbb{F}_q$,

$$\lambda_a(\alpha) = \exp\left\{\frac{2\pi i}{p}(a\alpha + (a\alpha)^p + \ldots + (a\alpha)^{p^{d-1}})\right\}. \tag{2.1}$$

It is nontrivial if $a \neq 0$.

$\operatorname{tr} A$ and $\det A$ denote respectively the trace of A and the determinant of A for a square matrix A, and ${}^tB$ denotes the transpose of B for any matrix B.

$\mathrm{GL}(n,q)$ is the group of all nonsingular $n\times n$ matrices with entries in $\mathbb{F}_q$. Then

$$O^-(2n,q) = \{w \in \mathrm{GL}(2n,q) \mid {}^twJ^-w = J^-\}, \tag{2.2}$$

where

$$J^- = \begin{bmatrix} 0 & & 1_{n-1} & 0 & 0 \\ & & & \vdots & \vdots \\ 1_{n-1} & & 0 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & -\varepsilon \end{bmatrix}.$$

Here and throughout this paper, ε will denote a fixed element in $\mathbb{F}_q^\times - \mathbb{F}_q^{\times 2}$.

We write $w \in O^-(2n,q)$ as

$$w = \begin{bmatrix} A & B & e \\ C & D & f \\ g & h & i \end{bmatrix}, \tag{2.3}$$

where A, B, C, D are of size $(n-1)\times(n-1)$, e, f of size $(n-1)\times 2$, g, h of size $2\times(n-1)$, and i of size 2×2.

For $\alpha \in \mathbb{F}_q^\times$, δ_α will denote the 2×2 matrix over $\mathbb{F}_q$

$$\delta_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha \end{bmatrix}. \tag{2.4}$$

Then (2.2) is also given by

$$\begin{aligned} &O^-(2n,q) \\ &= \left\{ \begin{bmatrix} A & B & e \\ C & D & f \\ g & h & i \end{bmatrix} \in \mathrm{GL}(2n,q) \;\middle|\; \begin{array}{l} {}^tAC + {}^tCA + {}^tg\delta_\varepsilon g = 0, \\ {}^tBD + {}^tDB + {}^th\delta_\varepsilon h = 0, \\ {}^tAD + {}^tCB + {}^tg\delta_\varepsilon h = 1_{n-1}, \\ {}^tef + {}^tfe + {}^ti\delta_\varepsilon i = \delta_\varepsilon, \\ {}^tAf + {}^tCe + {}^tg\delta_\varepsilon i = 0, \\ {}^tBf + {}^tDe + {}^th\delta_\varepsilon i = 0 \end{array} \right\} \qquad (2.5) \\ &= \left\{ \begin{bmatrix} A & B & e \\ C & D & f \\ g & h & i \end{bmatrix} \in \mathrm{GL}(2n,q) \;\middle|\; \begin{array}{l} A{}^tB + B{}^tA + e\delta_{\varepsilon^{-1}}{}^te = 0, \\ C{}^tD + D{}^tC + f\delta_{\varepsilon^{-1}}{}^tf = 0, \\ A{}^tD + B{}^tC + e\delta_{\varepsilon^{-1}}{}^tf = 1_{n-1}, \\ g{}^th + h{}^tg + i\delta_{\varepsilon^{-1}}{}^ti = \delta_{\varepsilon^{-1}}, \\ A{}^th + B{}^tg + e\delta_{\varepsilon^{-1}}{}^ti = 0, \\ C{}^th + D{}^tg + f\delta_{\varepsilon^{-1}}{}^ti = 0 \end{array} \right\}. \qquad (2.6) \end{aligned}$$

$P(2n,q)$ is the maximal parabolic subgroup of $O^-(2n,q)$ defined by

$$\begin{aligned} &P(2n,q) \qquad (2.7) \\ &= \left\{ \begin{bmatrix} A & 0 & 0 \\ 0 & {}^tA^{-1} & 0 \\ 0 & 0 & i \end{bmatrix} \begin{bmatrix} 1_{n-1} & B & -{}^th\delta_\varepsilon \\ 0 & 1_{n-1} & 0 \\ 0 & h & 1_2 \end{bmatrix} \;\middle|\; \begin{array}{l} A \in \mathrm{GL}(n-1,q), \\ {}^ti\delta_\varepsilon i = \delta_\varepsilon, \\ {}^tB + B + {}^th\delta_\varepsilon h = 0 \end{array} \right\} \end{aligned}$$

$$= \left\{ \begin{bmatrix} A & 0 & 0 \\ 0 & {}^tA^{-1} & 0 \\ 0 & 0 & i \end{bmatrix} \begin{bmatrix} 1_{n-1} & B & -{}^th\delta_\varepsilon \\ 0 & 1_{n-1} & 0 \\ 0 & h & 1_2 \end{bmatrix} \;\middle|\; \begin{array}{l} A \in \mathrm{GL}(n-1,q), \\ i \in O^-(2,q), \\ {}^tB + B + {}^th\delta_\varepsilon h = 0 \end{array} \right\}$$

(cf. (4.9)). Finally,

$$\mathrm{SO}^-(2n,q) = \{w \in O^-(2n,q) \mid \det w = 1\}, \tag{2.8}$$

which is a subgroup of index 2 in $O^-(2n,q)$.

For a multiplicative character χ of $\mathbb{F}_q$ and an additive character λ of $\mathbb{F}_q$, the Gauss sum $G(\chi,\lambda)$ is defined as

$$G(\chi,\lambda) = \sum_{\alpha\in\mathbb{F}_q^\times} \chi(\alpha)\lambda(\alpha). \tag{2.9}$$

In particular, if $\chi = \eta$ is the quadratic character of $\mathbb{F}_q$ and $\lambda = \lambda_a$ is nontrivial, then, as is well known [12, Theorems 5.15 and 5.30],

$$\begin{aligned} G(\eta,\lambda) &= \sum_{\alpha\in\mathbb{F}_q} \lambda(\alpha^2) \\ &= \begin{cases} \eta(a)(-1)^{d-1}\sqrt{q}, & p \equiv 1 \pmod 4, \\ \eta(a)(-1)^{d-1}(\sqrt{-1})^d\sqrt{q}, & p \equiv 3 \pmod 4. \end{cases} \end{aligned} \tag{2.10}$$

For a nontrivial additive character λ of $\mathbb{F}_q$, $a,b \in \mathbb{F}_q$, $K(\lambda;a,b)$ is the Kloosterman sum defined by

$$K(\lambda;a,b) = \sum_{\alpha\in\mathbb{F}_q^\times} \lambda(a\alpha + b\alpha^{-1}). \tag{2.11}$$

For integers n, r with $0 \le r \le n$, we define the q-binomial coefficients as

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \prod_{j=0}^{r-1} (q^{n-j}-1)/(q^{r-j}-1). \tag{2.12}$$

The order of the group $\mathrm{GL}(n,q)$ is denoted by

$$g_n = \prod_{j=0}^{n-1} (q^n - q^j) = q^{\binom{n}{2}} \prod_{j=0}^{n-1} (q^{n-j}-1). \tag{2.13}$$

Then we have

$$\frac{g_n}{g_{n-r}g_r} = q^{r(n-r)} \begin{bmatrix} n \\ r \end{bmatrix}_q, \tag{2.14}$$

for integers n, r with $0 \le r \le n$.

For x an indeterminate, n a nonnegative integer,

$$(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1}). \tag{2.15}$$

Then the q-binomial theorem says

$$\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q (-1)^r q^{\binom{r}{2}} x^r = (x;q)_n. \tag{2.16}$$

$[y]$ denotes the greatest integer $\le y$, for a real number y.

Finally, for $n \times n$ matrices A, B over $\mathbb{F}_q$, we will say that A is equivalent to B and write

$$A \sim B \quad \text{if and only if} \quad B = {}^t w A w \text{ for some } w \in \mathrm{GL}(n,q). \tag{2.17}$$

3. Bruhat decomposition. In this section, we will discuss the Bruhat decomposition of $O^-(2n,q)$ with respect to the maximal parabolic subgroup $P(2n,q)$ of $O^-(2n,q)$ (cf. (2.7)). This decomposition (in fact, its slight variants (3.15) and (3.16)) will play a key role in deriving the main theorems in Sections 5 and 6, and an elementary proof of it will be provided.

As a simple application, we will demonstrate that this decomposition yields the well-known formula for the order of the group $O^-(2n,q)$ when combined with the q-binomial theorem.

Theorem 3.1. (a) *There is a one-to-one correspondence*

$$P(2n,q)\backslash O^-(2n,q) \to P'(n+1,q)\backslash \Lambda$$

given by

$$P(2n,q) \begin{bmatrix} A & B & e \\ C & D & f \\ g & h & i \end{bmatrix} \mapsto P'(n+1,q) \begin{bmatrix} C & D & f \\ g & h & i \end{bmatrix},$$

where

$$\begin{aligned} P'(n+1,q) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(n+1,q) \;\middle|\; \begin{matrix} a \in \mathrm{GL}(n-1,q), \\ b = 0,\ {}^t d \delta_\varepsilon d = \delta_\varepsilon \end{matrix} \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(n+1,q) \;\middle|\; \begin{matrix} a \in \mathrm{GL}(n-1,q), \\ b = 0,\ d \in O^-(2,q) \end{matrix} \right\}, \end{aligned}$$

$$\Lambda = \left\{ \begin{bmatrix} C & D & f \\ g & h & i \end{bmatrix} \;\middle|\; \begin{matrix} C, D, f, g, h, i \text{ are respectively} \\ (n-1)\times(n-1),\ (n-1)\times(n-1),\ (n-1)\times 2, \\ 2\times(n-1),\ 2\times(n-1),\ 2\times 2 \text{ matrices over } \mathbb{F}_q \text{ subject to the} \\ \text{conditions (3.1) below, and the matrix is of full rank } n+1 \end{matrix} \right\},$$

$$\begin{cases} C\,{}^t D + D\,{}^t C + f \delta_{\varepsilon^{-1}}\,{}^t f = 0, \\ g\,{}^t h + h\,{}^t g + i \delta_{\varepsilon^{-1}}\,{}^t i = \delta_{\varepsilon^{-1}}, \\ C\,{}^t h + D\,{}^t g + f \delta_{\varepsilon^{-1}}\,{}^t i = 0, \end{cases} \tag{3.1}$$

and, for $\delta_\varepsilon, \delta_{\varepsilon^{-1}}$, one refers to (2.4).

(b) *For given* $\left[\begin{smallmatrix} C & D & f \\ g & h & i \end{smallmatrix}\right] \in \Lambda$, *there exists a unique* r $(0 \le r \le n-1)$, $p' \in P'(n+1,q)$, $p \in P(2n,q)$ *such that*

$$p'\begin{bmatrix} C & D & f \\ g & h & i \end{bmatrix} p = \begin{bmatrix} 1_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-1-r} & 0 \\ 0 & 0 & 0 & 0 & 1_2 \end{bmatrix}.$$

(c) *We have*

$$O^-(2n,q) = \coprod_{r=0}^{n-1} P\sigma_r P,$$

where $P = P(2n,q)$ *and*

$$\sigma_r = \begin{bmatrix} 0 & 0 & 1_r & 0 & 0 \\ 0 & 1_{n-1-r} & 0 & 0 & 0 \\ 1_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-1-r} & 0 \\ 0 & 0 & 0 & 0 & 1_2 \end{bmatrix} \in O^-(2n,q). \tag{3.2}$$

Proof. It is easy to see that the map in (a) is well defined and injective. For the surjectivity, it is enough to see that, for any given $\left[\begin{smallmatrix} C & D & f \\ g & h & i \end{smallmatrix}\right] \in \Lambda$,

$$\begin{bmatrix} A & B & e \\ C & D & f \\ g & h & i \end{bmatrix} \in O^-(2n,q) \quad \text{for some } A, B, e.$$

Choose $x \in \mathrm{GL}(n-1,q)$ such that $x[C\ D\ f]$ is a row echelon matrix. Let r $(0 \le r \le n-1)$ be the number of pivots in xC. Then, for some $y \in \mathrm{GL}(n-1,q)$,

$$p_1'\begin{bmatrix} C & D & f \\ g & h & i \end{bmatrix} p_1 = \begin{bmatrix} 1_r & 0 & & \\ & & D' & f' \\ 0 & 0 & & \\ & g' & h' & i \end{bmatrix}, \tag{3.3}$$

where

$$p_1' = \begin{bmatrix} x & 0 \\ 0 & 1_2 \end{bmatrix} \in P'(n+1,q), \qquad p_1 = \begin{bmatrix} y & 0 & 0 \\ 0 & {}^t y^{-1} & 0 \\ 0 & 0 & 1_2 \end{bmatrix} \in P(2n,q).$$

Write

$$D' = \begin{bmatrix} D_{11}' & D_{12}' \\ D_{21}' & D_{22}' \end{bmatrix}, \qquad f' = \begin{bmatrix} f_1' \\ f_2' \end{bmatrix},$$

where D_{11}' is of size $r \times r$, D_{22}' of size $(n-1-r)\times(n-1-r)$, and f_1' of size $r \times 2$, etc.

It can be checked directly that if $\left[\begin{smallmatrix} C & D & f \\ g & h & i \end{smallmatrix}\right] \in \Lambda$ then $\widetilde{p}'\left[\begin{smallmatrix} C & D & f \\ g & h & i \end{smallmatrix}\right]\widetilde{p} \in \Lambda$ for any $\widetilde{p}' \in P'(n+1,q)$ and $\widetilde{p} \in P(2n,q)$. Thus the first identity in (3.1) must be satisfied by (3.3). So we get ${}^tD'_{11} + D'_{11} + f'_1\delta_{\varepsilon^{-1}}{}^tf'_1 = 0$, $D'_{21} = 0, f'_2 = 0$.

Put

$$p_2 = \begin{bmatrix} & {}^tD'_{11} & & -D'_{12} & -f'_1 \\ 1_{n-1} & & & & \\ & {}^tD'_{12} & & 0 & 0 \\ 0 & & 1_{n-1} & & 0 \\ 0 & \delta_{\varepsilon^{-1}}{}^tf'_1 & & 0 & 1_2 \end{bmatrix}.$$

Then $p_2 \in P(2n,q)$, and (3.3) right multiplied by p_2 is

$$\begin{bmatrix} 1_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D'_{22} & 0 \\ & g' & h'' & & i' \end{bmatrix}. \tag{3.4}$$

Since (3.4) is of full rank, D'_{22} must be invertible. Hence (3.4) left multiplied by p'_2 is

$$\begin{bmatrix} 1_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-1-r} & 0 \\ & g' & h'' & & i' \end{bmatrix}, \tag{3.5}$$

where

$$p'_2 = \begin{bmatrix} 1_r & 0 & 0 \\ 0 & D'^{-1}_{22} & 0 \\ 0 & 0 & 1_2 \end{bmatrix} \in P'(n+1,q).$$

Put

$$g' = [g'_1\ g'_2], \qquad h'' = [h''_1\ h''_2],$$

where g'_1 and h''_1 are of $2 \times r$. Now, the second and third identities of (3.1) must be satisfied by (3.5). So we get $h''_1 = 0$, $g'_2 = 0$, ${}^ti'\delta_\varepsilon i' = \delta_\varepsilon$.

Let

$$p'_3 = \begin{bmatrix} 1_r & 0 & 0 \\ 0 & 1_{n-1-r} & 0 \\ -i'^{-1}g'_1 & -i'^{-1}h''_2 & i'^{-1} \end{bmatrix}.$$

Then $p'_3 \in P'(n+1,q)$ and (3.5) left multiplied by p'_3 is

$$\begin{bmatrix} 1_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-1-r} & 0 \\ 0 & 0 & 0 & 0 & 1_2 \end{bmatrix}. \tag{3.6}$$

So far we have shown that $p'\left[\begin{smallmatrix} C & D & f \\ g & h & i \end{smallmatrix}\right]p$ equals (3.6) for $p' = p'_3p'_2p'_1 \in P'(n+1,q)$, $p = p_1p_2 \in P(2n,q)$ and for a unique integer r $(0 \le r \le n-1)$. This shows (b).

Write

$$p' = \begin{bmatrix} {}^tA^{-1} & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1_{n-1} & 0 \\ h & 1_2 \end{bmatrix}.$$

Choose any $(n-1)\times(n-1)$ matrix B satisfying ${}^tB+B+{}^th\delta_\varepsilon h=0$. Then ${p''}^{-1}\sigma_r p^{-1}$ is a matrix in $O^-(2n,q)$ whose last $n+1$ rows constitute the matrix $\left[\begin{smallmatrix} C & D & f\\ g & h & i\end{smallmatrix}\right]$, where p'' is given by

$$p''=\begin{bmatrix} A & 0 & 0\\ 0 & {}^tA^{-1} & 0\\ 0 & 0 & i\end{bmatrix}\begin{bmatrix} 1_{n-1} & B & -{}^th\delta_\varepsilon\\ 0 & 1_{n-1} & 0\\ 0 & h & 1_2\end{bmatrix}.$$

This completes the proof for (a).

In view of (a), the Bruhat decomposition in (c) is equivalent to

$$\Lambda=\coprod_{r=0}^{n-1} P'\begin{bmatrix} 1_r & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1_{n-1-r} & 0\\ 0 & 0 & 0 & 0 & 1_2\end{bmatrix}P, \tag{3.7}$$

where $P'=P'(n+1,q)$, $P=P(2n,q)$. (b) says that Λ is a union of double cosets as in (3.7). The disjointness in (3.7) is easy to see. ∎

Put

$$\begin{aligned} Q&=Q(2n,q) \\ &=\left\{\begin{bmatrix} A & 0 & 0\\ 0 & {}^tA^{-1} & 0\\ 0 & 0 & i\end{bmatrix}\begin{bmatrix} 1_{n-1} & B & -{}^th\delta_\varepsilon\\ 0 & 1_{n-1} & 0\\ 0 & h & 1_2\end{bmatrix} \;\middle|\; \begin{array}{l} A\in \mathrm{GL}(n-1,q),\\ i\in \mathrm{SO}^-(2,q),\\ {}^tB+B+{}^th\delta_\varepsilon h=0\end{array}\right\}.\end{aligned} \tag{3.8}$$

Then $Q(2n,q)$ is a subgroup of index 2 in $P(2n,q)$ (cf. (2.7), (4.10)), and

$$O^-(2n,q)=\coprod_{r=0}^{n-1} P\sigma_r Q. \tag{3.9}$$

Write, for each r $(0\le r\le n-1)$,

$$A_r=A_r(q)=\{p\in P(2n,q)\mid \sigma_r p\sigma_r^{-1}\in P(2n,q)\}, \tag{3.10}$$

$$B_r=B_r(q)=\{p\in Q(2n,q)\mid \sigma_r p\sigma_r^{-1}\in P(2n,q)\}. \tag{3.11}$$

Then B_r is a subgroup of A_r of index 2 and

$$|B_r\backslash Q|=|A_r\backslash P|. \tag{3.12}$$

Expressing $O^-(2n,q)$ as a disjoint union of right cosets of $P=P(2n,q)$, the Bruhat decomposition in (c) of Theorem 3.1 and the decomposition in (3.9) can be rewritten as follows.

Corollary 3.2.

$$O^-(2n,q)=\coprod_{r=0}^{n-1} P\sigma_r(A_r\backslash P), \tag{3.13}$$

$$O^-(2n,q)=\coprod_{r=0}^{n-1} P\sigma_r(B_r\backslash Q), \tag{3.14}$$

where $P=P(2n,q)$, *and* Q, A_r, B_r *are respectively as in* (3.8), (3.10), (3.11).

The decomposition in (3.14) can further be modified to give the following decompositions.

COROLLARY 3.3.

$$\begin{aligned}\mathrm{SO}^-(2n,q)=&\Big(\coprod_{\substack{0\le r\le n-1\\ r\ even}} Q\sigma_r(B_r\backslash Q)\Big)\\ &\amalg\Big(\coprod_{\substack{0\le r\le n-1\\ r\ odd}} (\varrho Q)\sigma_r(B_r\backslash Q)\Big),\end{aligned} \tag{3.15}$$

$$\begin{aligned}O^-(2n,q)=&\Big(\coprod_{\substack{0\le r\le n-1\\ r\ even}} Q\sigma_r(B_r\backslash Q)\Big)\\ &\amalg\Big(\coprod_{\substack{0\le r\le n-1\\ r\ even}} (\varrho Q)\sigma_r(B_r\backslash Q)\Big)\\ &\amalg\Big(\coprod_{\substack{0\le r\le n-1\\ r\ odd}} Q\sigma_r(B_r\backslash Q)\Big)\\ &\amalg\Big(\coprod_{\substack{0\le r\le n-1\\ r\ odd}} (\varrho Q)\sigma_r(B_r\backslash Q)\Big),\end{aligned} \tag{3.16}$$

where

$$\varrho=\begin{bmatrix}1_{n-1} & 0 & 0 & 0\\ 0 & 1_{n-1} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}. \tag{3.17}$$

Write $p\in P(2n,q)$ as

$$p=\begin{bmatrix}A & 0 & 0\\ 0 & {}^tA^{-1} & 0\\ 0 & 0 & i\end{bmatrix}\begin{bmatrix}1_{n-1} & B & -{}^th\delta_\varepsilon\\ 0 & 1_{n-1} & 0\\ 0 & h & 1_2\end{bmatrix}, \tag{3.18}$$

with

$$A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix},\quad {}^tA^{-1}=\begin{bmatrix}E_{11} & E_{12}\\ E_{21} & E_{22}\end{bmatrix},\quad B=\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix},$$

and $h=[h_1\ h_2]$. Here A_{11}, A_{12}, A_{21}, and A_{22} are respectively of sizes $r\times r$, $r\times(n-1-r)$, $(n-1-r)\times r$, and $(n-1-r)\times(n-1-r)$, similarly for ${}^tA^{-1}$, B, and h_1 is of size $2\times r$.

Then $\sigma_r p \sigma_r^{-1} \in P$ if and only if $A_{11}B_{11} + A_{12}B_{21} = 0$, $A_{12} = 0$, $E_{21} = 0$, $ih_1 = 0$, $A_{11}{}^t h_1 \delta_\varepsilon + A_{12}{}^t h_2 \delta_\varepsilon = 0$ if and only if $A_{12} = 0$, $B_{11} = 0$, $h_1 = 0$. Recalling the order of $O^-(2,q)$ in (4.12), we have

$$|A_r(q)| = 2(q+1) g_r g_{n-1-r} q^{(n-1)(n+2)/2} q^{r(2n-3r-5)/2}, \tag{3.19}$$

where g_n is as in (2.13). Also,

$$|P(2n,q)| = 2(q+1) g_{n-1} q^{(n-1)(n+2)/2}. \tag{3.20}$$

From (2.14), (3.19) and (3.20), we get

$$|A_r(q) \backslash P(2n,q)| = \begin{bmatrix} n-1 \\ r \end{bmatrix}_q q^{r(r+3)/2}. \tag{3.21}$$

Combining (3.20) and (3.21), we also have

$$|P(2n,q)|^2 |A_r(q)|^{-1} = 2(q+1) q^{n^2-n} \prod_{j=1}^{n-1} (q^j - 1) q^{\binom{r}{2}} q^{2r} \begin{bmatrix} n-1 \\ r \end{bmatrix}_q. \tag{3.22}$$

The decomposition in (3.13) yields

$$|O^-(2n,q)| = \sum_{r=0}^{n-1} |P(2n,q)|^2 |A_r(q)|^{-1}. \tag{3.23}$$

Now, from (3.22) and (3.23) and applying the binomial theorem (2.16) with $x = -q^2$, we have the following theorem. We note here that this result was already shown in [3].

THEOREM 3.4.

$$|O^-(2n,q)| = 2q^{n^2-n}(q^n+1) \prod_{j=1}^{n-1} (q^{2j} - 1). \tag{3.24}$$

Proof.

$$\begin{aligned} |O^-(2n,q)| &= 2(q+1) q^{n^2-n} \prod_{j=1}^{n-1} (q^j - 1) \sum_{r=0}^{n-1} \begin{bmatrix} n-1 \\ r \end{bmatrix}_q q^{\binom{r}{2}} q^{2r} \\ &= 2(q+1) q^{n^2-n} \prod_{j=1}^{n-1} (q^j - 1)(-q^2; q)_{n-1} \\ &= 2q^{n^2-n}(q^n+1) \prod_{j=1}^{n-1} (q^{2j} - 1). \ \blacksquare \end{aligned}$$

4. Some propositions. For r even, every nonsingular symmetric matrix of size r over $\mathbb{F}_q$ is equivalent either to

$$J^+ = \begin{bmatrix} 0 & 1_{r/2} \\ 1_{r/2} & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{bmatrix} \tag{4.1}$$

or to

(4.2)

$$J^- = \begin{bmatrix} 0 & & 1_{r/2-1} & 0 & 0 \\ & & & \vdots & \vdots \\ 1_{r/2-1} & & 0 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & -\varepsilon \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & 0 & \\ & & \ddots & & & & \\ & & & 0 & 1 & & \\ & & & 1 & 0 & & \\ & 0 & & & & 1 & 0 \\ & & & & & 0 & -\varepsilon \end{bmatrix}.$$

On the other hand, for r odd every nonsingular symmetric matrix of size r over $\mathbb{F}_q$ is equivalent either to

$$J = \begin{bmatrix} 0 & 1_{(r-1)/2} & 0 \\ 1_{(r-1)/2} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & & 0 & & \\ & & \ddots & & & \\ & & & 0 & 1 & \\ & 0 & & 1 & 0 & \\ & & & & & 1 \end{bmatrix} \tag{4.3}$$

or to

$$\varepsilon J = \varepsilon \begin{bmatrix} 0 & 1_{(r-1)/2} & 0 \\ 1_{(r-1)/2} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \varepsilon \begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & & 0 & & \\ & & \ddots & & & \\ & & & 0 & 1 & \\ & 0 & & 1 & 0 & \\ & & & & & 1 \end{bmatrix}. \tag{4.4}$$

The following proposition can be proved analogously to the corresponding Proposition 4.1 in [9], so we only sketch the proof.

Proposition 4.1. *Let λ be a nontrivial additive character of $\mathbb{F}_q$, and let B be a nonsingular symmetric matrix of size r with entries in $\mathbb{F}_q$. Then*

$$\sum_{h \in \mathbb{F}_q^{r \times 2}} \lambda(\operatorname{tr} \delta_\varepsilon {}^t h B h) = \begin{cases} q^r & \text{if } r \text{ is even}, \\ -q^r & \text{if } r \text{ is odd}, \end{cases} \tag{4.5}$$

where $\mathbb{F}_q^{r \times 2}$ denotes the set of all $r \times 2$ matrices over $\mathbb{F}_q$, and δ_ε is as in (2.4).

P r o o f. Since the corresponding sums in (4.5) are the same for equivalent matrices B and B', it suffices to consider the cases when B is respectively equal to the matrix on the right hand side of (4.1)–(4.4).

If $B\sim(4.1)$ or $B\sim(4.2)$, then we get exactly the square of the corresponding expressions in Proposition 4.1 of [9].

On the other hand, if $B\sim(4.3)$ or $B\sim(4.4)$, then we get

$$\eta(-\varepsilon)G(\eta,\lambda)^2q^{r-1}=(-1)^{(q+1)/2}G(\eta,\lambda)^2q^{r-1}=-q^r$$

(cf. (2.10)). So in these cases also, up to sign, we get the square of the corresponding expressions in Proposition 4.1 of [9]. ∎

The following can be proved in exactly the same manner as Proposition 4.2 of [9].

PROPOSITION 4.2. *Let λ be a nontrivial additive character of $\mathbb{F}_q$. For a positive integer r, let Ω_r be the set of all $r\times r$ nonsingular symmetric matrices over $\mathbb{F}_q$. Then*

$$\begin{aligned}(4.6)\qquad b_r(\lambda)&=\sum_{B\in\Omega_r}\sum_{h\in\mathbb{F}_q^{r\times 2}}\lambda(\operatorname{tr}\delta_\varepsilon {}^t hBh)\\ &=\begin{cases} q^{r(r+6)/4}\displaystyle\prod_{j=1}^{r/2}(q^{2j-1}-1) & \textit{for } r \textit{ even},\\ -q^{(r^2+4r-1)/4}\displaystyle\prod_{j=1}^{(r+1)/2}(q^{2j-1}-1) & \textit{for } r \textit{ odd},\end{cases}\end{aligned}$$

where δ_ε is as in (2.4).

The next two propositions are well known and will be used in showing Proposition 4.5.

PROPOSITION 4.3 [12, Theorem 5.30]. *Let λ be a nontrivial additive character of $\mathbb{F}_q$ (here $q=p^d$ with p any prime including $p=2$), and let ψ be a multiplicative character of $\mathbb{F}_q$ of order $d=(n,q-1)$. Then*

$$(4.7)\qquad \sum_{\alpha\in\mathbb{F}_q}\lambda(\alpha^n)=\sum_{j=1}^{d-1}G(\psi^j,\lambda),$$

where $G(\psi^j,\lambda)$ is the Gauss sum as in (2.9).

PROPOSITION 4.4 (Davenport–Hasse). *Let λ be an additive character of $\mathbb{F}_q$, and ψ a multiplicative character of $\mathbb{F}_q$, not both of them trivial. Suppose that $\lambda'=\lambda\circ\operatorname{tr}_{\mathbb{F}_{q^s}/\mathbb{F}_q}$ and $\psi'=\psi\circ N_{\mathbb{F}_{q^s}/\mathbb{F}_q}$. Then*

$$(4.8)\qquad G(\psi',\lambda')=(-1)^{s-1}G(\psi,\lambda)^s.$$

For the next proposition, we note the following. We have

$$O^-(2,q) = \{w \in \mathrm{GL}(2,q) \mid {}^t w \delta_\varepsilon w = \delta_\varepsilon\}. \tag{4.9}$$

Now, $\mathrm{SO}^-(2,q) = \{w \in O^-(2,q) \mid \det w = 1\}$ is a subgroup of index 2 in $O^-(2,q)$, and

$$O^-(2,q) = \mathrm{SO}^-(2,q) \amalg \delta_1 \mathrm{SO}^-(2,q). \tag{4.10}$$

Note here that $\delta_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ (cf. (2.4)). Moreover,

$$\mathrm{SO}^-(2,q) = \left\{ \begin{bmatrix} a & b\varepsilon \\ b & a \end{bmatrix} \;\middle|\; a, b \in \mathbb{F}_q,\ a^2 - b^2\varepsilon = 1 \right\}. \tag{4.11}$$

In particular, this says that

$$|\mathrm{SO}^-(2,q)| = q+1, \qquad |O^-(2,q)| = 2(q+1). \tag{4.12}$$

PROPOSITION 4.5. *Let λ be a nontrivial additive character of $\mathbb{F}_q$. Then*

$$\sum_{w \in \mathrm{SO}^-(2,q)} \lambda(\mathrm{tr}\, w) = -\frac{1}{q-1} \sum_{j=1}^{q-1} G(\psi^j, \lambda)^2, \tag{4.13}$$

$$\sum_{w \in \mathrm{SO}^-(2,q)} \lambda(\mathrm{tr}\, \delta_1 w) = q+1, \tag{4.14}$$

where ψ is a multiplicative character of $\mathbb{F}_q$ of order $q-1$.

P r o o f. (4.14) is clear, since, from (4.11), we see that $\lambda(\mathrm{tr}\,\delta_1 w) = \lambda(0) = 1$ for each $w \in \mathrm{SO}^-(2,q)$. Let $K = \mathbb{F}_q(\sqrt{\varepsilon})$ be the quadratic extension field of $\mathbb{F}_q$, and let σ be the Frobenius automorphism of K given by $\sigma\alpha = \alpha^q$. Then, from (4.11), we see that the left hand side of (4.13) equals

$$\begin{aligned}
&\sum_{\alpha \in K,\, N_{K/\mathbb{F}_q}(\alpha)=1} \lambda \circ \mathrm{tr}_{K/\mathbb{F}_q}(\alpha) \\
&\quad = \sum_{\alpha \in \mathbb{F}_q^\times \backslash K^\times} \lambda \circ \mathrm{tr}_{K/\mathbb{F}_q}\left(\frac{\sigma\alpha}{\alpha}\right) \qquad \text{(Hilbert's Theorem 90)} \\
&\quad = \frac{1}{q-1} \sum_{\alpha \in K^\times} \lambda \circ \mathrm{tr}_{K/\mathbb{F}_q}(\alpha^{q-1}) \\
&\quad = \frac{1}{q-1} \Big\{ \sum_{\alpha \in K} \lambda \circ \mathrm{tr}_{K/\mathbb{F}_q}(\alpha^{q-1}) - 1 \Big\}. \qquad (4.15)
\end{aligned}$$

Let ψ be a multiplicative character of $\mathbb{F}_q$ of order $q-1$. Then $\psi \circ N_{K/\mathbb{F}_q}$ is a multiplicative character of K of order $q-1$, and $(\psi \circ N_{K/\mathbb{F}_q})^j = \psi^j \circ N_{K/\mathbb{F}_q}$ for each positive integer j. Thanks to (4.7), the sum in (4.15) can be expressed as

$$\sum_{\alpha\in K}\lambda\circ\mathrm{tr}_{K/\mathbb{F}_q}(\alpha^{q-1})=\sum_{j=1}^{q-2}G(\psi^j\circ N_{K/\mathbb{F}_q},\lambda\circ\mathrm{tr}_{K/\mathbb{F}_q})$$
$$=-\sum_{j=1}^{q-2}G(\psi^j,\lambda)^2\qquad((4.8)).$$

By substituting the last expression into (4.15), we get the desired result. ∎

Remark. For $j=q-1$, ψ^j is trivial and hence $G(\psi^j,\lambda)=-1$. For $j=1,\dots,q-2,\psi^j$ is nontrivial and $G(\psi^j,\lambda)$ is $\sqrt{q}$ in absolute value (cf. [12], Theorem 5.11). Thus, from (4.13), we have

$$\Big|\sum_{w\in\mathrm{SO}^{-}(2,q)}\lambda(\mathrm{tr}\,w)\Big|\le q-1.$$

(4.13) also shows that $\sum_{j=1}^{q-1}G(\psi^j,\lambda)^2$ does not depend on the choice of a multiplicative character ψ of $\mathbb{F}_q$ of order $q-1$.

5. $\mathrm{SO}^{-}(2n,q)$ **case.** In this section, we will consider the sum in (1.1),

$$\sum_{w\in\mathrm{SO}^{-}(2n,q)}\lambda(\mathrm{tr}\,w),$$

for any nontrivial additive character λ of $\mathbb{F}_q$ and find an explicit expression for this by using the decomposition in (3.15).

The sum in (1.1) can be written, using (3.15), as

$$(5.1)\qquad \sum_{\substack{0\le r\le n-1\\ r\text{ even}}}|B_r\backslash Q|\sum_{w\in Q}\lambda(\mathrm{tr}\,w\sigma_r)+\sum_{\substack{0\le r\le n-1\\ r\text{ odd}}}|B_r\backslash Q|\sum_{w\in Q}\lambda(\mathrm{tr}\,\varrho w\sigma_r),$$

where $B_r=B_r(q),Q=Q(2n,q)$ are respectively as in (3.11), (3.8), and ϱ,σ_r are respectively as in (3.17), (3.2). Here one should note that, for each $q\in Q$,

$$\sum_{w\in Q}\lambda(\mathrm{tr}\,w\sigma_r q)=\sum_{w\in Q}\lambda(\mathrm{tr}\,qw\sigma_r)=\sum_{w\in Q}\lambda(\mathrm{tr}\,w\sigma_r),$$

and $\varrho^{-1}q\varrho\in Q$. Write $w\in Q$ as

$$w=\begin{bmatrix}A&0&0\\0&{}^tA^{-1}&0\\0&0&i\end{bmatrix}\begin{bmatrix}1_{n-1}&B&-{}^th\delta_\varepsilon\\0&1_{n-1}&0\\0&h&1_2\end{bmatrix},$$

with

$$A=\begin{bmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{bmatrix},\quad {}^tA^{-1}=\begin{bmatrix}E_{11}&E_{12}\\E_{21}&E_{22}\end{bmatrix},\quad B=\begin{bmatrix}B_{11}&B_{12}\\B_{21}&B_{22}\end{bmatrix},\quad h=[h_1\ h_2],$$

$$
\begin{gathered}
{}^tB_{11}+B_{11}+{}^th_1\delta_\varepsilon h_1=0,\quad {}^tB_{21}+B_{12}+{}^th_1\delta_\varepsilon h_2=0,\\
{}^tB_{22}+B_{22}+{}^th_2\delta_\varepsilon h_2=0.
\end{gathered}\tag{5.2}
$$

Note that the conditions in (5.2) are equivalent to ${}^tB+B+{}^th\delta_\varepsilon h=0$. Here A_{11}, A_{12}, A_{21}, A_{22} are respectively of sizes $r\times r$, $r\times(n-1-r)$, $(n-1-r)\times r$, $(n-1-r)\times(n-1-r)$, similarly for ${}^tA^{-1}, B$, and h_1 is of size $2\times r$.

Now,

$$
w\sigma_r=\begin{bmatrix} M & * & * \\ * & N & * \\ * & * & i \end{bmatrix}
$$

with

$$
M=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} B_{11} & 0 \\ B_{21} & 1_{n-1-r} \end{bmatrix},\qquad N=\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1_{n-1-r} \end{bmatrix},\tag{5.3}
$$

and

$$
\varrho w\sigma_r=\begin{bmatrix} M & * & * \\ * & N & * \\ * & * & \delta_1 i \end{bmatrix}
$$

with M,N as in (5.3). So the sum in (5.1) is

$$
\begin{aligned}
&\sum_{i\in\mathrm{SO}^-(2,q)}\lambda(\operatorname{tr} i)\sum_{\substack{0\le r\le n-1\\ r\text{ even}}}|B_r\backslash Q|\\
&\qquad\times\sum\lambda(\operatorname{tr}A_{11}B_{11}+\operatorname{tr}A_{12}B_{21}+\operatorname{tr}A_{22}+\operatorname{tr}E_{22})\\
&\qquad+\sum_{i\in\mathrm{SO}^-(2,q)}\lambda(\operatorname{tr}\delta_1 i)\sum_{\substack{0\le r\le n-1\\ r\text{ odd}}}|B_r\backslash Q|\\
&\qquad\times\sum\lambda(\operatorname{tr}A_{11}B_{11}+\operatorname{tr}A_{12}B_{21}+\operatorname{tr}A_{22}+\operatorname{tr}E_{22}),
\end{aligned}\tag{5.4}
$$

where the innermost sums are respectively over A, B, h subject to the conditions in (5.2).

Consider, for any r $(0\le r\le n-1)$, the sum

$$
\sum_{A,B,h}\lambda(\operatorname{tr}A_{11}B_{11}+\operatorname{tr}A_{12}B_{21}+\operatorname{tr}A_{22}+\operatorname{tr}E_{22}).\tag{5.5}
$$

For each fixed A, h, the subsum over B in (5.5) is

$$
\sum\lambda(\operatorname{tr}A_{11}B_{11}+\operatorname{tr}A_{12}B_{21}),\tag{5.6}
$$

where the sum is over all B_{11}, B_{21}, B_{22} satisfying ${}^tB_{11}+B_{11}+{}^th_1\delta_\varepsilon h_1=0$, ${}^tB_{22}+B_{22}+{}^th_2\delta_\varepsilon h_2=0$. Since the summand is independent of B_{22}, it equals

$$q^{\binom{n-1-r}{2}}\sum_{B_{11}}\lambda(\operatorname{tr}A_{11}B_{11})\sum_{B_{21}}\lambda(\operatorname{tr}A_{12}B_{21}). \tag{5.7}$$

The sum over B_{21} in (5.7) is nonzero if and only if $A_{12}=0$, in which case it is $q^{r(n-1-r)}$. On the other hand, the sum over B_{11} in (5.7) is nonzero if and only if A_{11} is symmetric, in which case it equals $q^{\binom{r}{2}}\lambda\big(-\frac{1}{2}\operatorname{tr}\delta_\varepsilon h_1 A_{11}{}^t h_1\big)$. To see this, let

$$A_{11}=(\alpha_{ij}),\quad B_{11}=(\beta_{ij}),\quad h_1=\begin{bmatrix}h_{11} & h_{12} & \dots & h_{1r}\\ h_{21} & h_{22} & \dots & h_{2r}\end{bmatrix}.$$

Then the condition ${}^tB_{11}+B_{11}+{}^th_1\delta_\varepsilon h_1=0$ is equivalent to

$$\begin{aligned}\beta_{ii}&=\tfrac{1}{2}(h_{2i}^2\varepsilon-h_{1i}^2) &&\text{for } 1\le i\le r,\\ \beta_{ij}+\beta_{ji}&=h_{2i}h_{2j}\varepsilon-h_{1i}h_{1j} &&\text{for } 1\le i<j\le r.\end{aligned}$$

Using these relations, it is not hard to see that

$$\operatorname{tr}A_{11}B_{11}=-\frac{1}{2}\operatorname{tr}\delta_\varepsilon h_1A_{11}{}^th_1+\sum_{1\le i<j\le r}(\alpha_{ji}-\alpha_{ij})\beta_{ij}.$$

Hence the sum over B_{11} in (5.7) is nonzero if and only if $\alpha_{ji}=\alpha_{ij}$ for $1\le i<j\le r$, i.e., A_{11} is symmetric. Moreover, it is $q^{\binom{r}{2}}\lambda\big(-\frac{1}{2}\operatorname{tr}\delta_\varepsilon h_1A_{11}{}^th_1\big)$ in that case.

We have shown so far that the sum in (5.6) is nonzero if and only if $A=\left[\begin{smallmatrix}A_{11} & 0\\ A_{21} & A_{22}\end{smallmatrix}\right]$ with A_{11} nonsingular symmetric, in which case it equals

$$q^{\binom{n-1-r}{2}+\binom{r}{2}+r(n-1-r)}\lambda\big(-\tfrac{1}{2}\operatorname{tr}\delta_\varepsilon h_1A_{11}{}^th_1\big)=q^{\binom{n-1}{2}}\lambda\big(-\tfrac{1}{2}\operatorname{tr}\delta_\varepsilon h_1A_{11}{}^th_1\big).$$

For such an $A=\left[\begin{smallmatrix}A_{11} & 0\\ A_{21} & A_{22}\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}E_{11} & E_{12}\\ E_{21} & E_{22}\end{smallmatrix}\right]=\left[\begin{smallmatrix}{}^tA_{11}^{-1} & *\\ 0 & {}^tA_{22}^{-1}\end{smallmatrix}\right]$, and hence the sum in (5.5) can be written as

$$\begin{aligned}&q^{\binom{n-1}{2}}\sum_{A_{21},h_2}\sum_{A_{11},h_1}\lambda\big(-\tfrac{1}{2}\operatorname{tr}\delta_\varepsilon h_1A_{11}{}^th_1\big)\sum_{A_{22}}\lambda(\operatorname{tr}A_{22}+\operatorname{tr}A_{22}^{-1})\\ &\quad=q^{(n-1)(n+2)/2+r(n-r-3)}\sum_{A_{11},h_1}\lambda\big(-\tfrac{1}{2}\operatorname{tr}\delta_\varepsilon h_1A_{11}{}^th_1\big)\sum_{A_{22}}\lambda(\operatorname{tr}A_{22}+\operatorname{tr}A_{22}^{-1})\\ &\quad=q^{(n-1)(n+2)/2+r(n-r-3)}b_r(\lambda)K_{\mathrm{GL}(n-1-r,q)}(\lambda;1,1),\end{aligned}$$

where $b_r(\lambda)$ is as in (4.6), and in [10], for $a,b\in\mathbb{F}_q$, $K_{\mathrm{GL}(t,q)}(\lambda;a,b)$ is defined as

$$K_{\mathrm{GL}(t,q)}(\lambda;a,b)=\sum_{w\in\mathrm{GL}(t,q)}\lambda(a\operatorname{tr}w+b\operatorname{tr}w^{-1}). \tag{5.8}$$

Putting everything together, the sum in (5.1) can be written as

$$(5.9)\qquad q^{(n-1)(n+2)/2}\Big\{\sum_{i\in \mathrm{SO}^-(2,q)}\lambda(\operatorname{tr} i)\sum_{\substack{0\le r\le n-1\\ r\,\mathrm{even}}}|B_r\backslash Q|q^{r(n-r-3)} \times b_r(\lambda)K_{\mathrm{GL}(n-1-r,q)}(\lambda;1,1)+\sum_{i\in \mathrm{SO}^-(2,q)}\lambda(\operatorname{tr}\delta_1 i) \times \sum_{\substack{0\le r\le n-1\\ r\,\mathrm{odd}}}|B_r\backslash Q|q^{r(n-r-3)}b_r(\lambda)K_{\mathrm{GL}(n-1-r,q)}(\lambda;1,1)\Big\}.$$

From (3.12), (3.21), (4.6), (4.13), and (4.14), we see that the above expression (5.9) equals

$$(5.10)\qquad q^{(n-1)(n+2)/2} \times\Big\{\Big(-\frac{1}{q-1}\sum_{j=1}^{q-1}G(\psi^j,\lambda)^2\Big)\sum_{\substack{0\le r\le n-1\\ r\,\mathrm{even}}}q^{nr-r^2/4}\begin{bmatrix}n-1\\ r\end{bmatrix}_q \times\prod_{j=1}^{r/2}(q^{2j-1}-1)K_{\mathrm{GL}(n-1-r,q)}(\lambda;1,1) -(q+1)\sum_{\substack{0\le r\le n-1\\ r\,\mathrm{odd}}}q^{(4nr-r^2-2r-1)/4} \times\begin{bmatrix}n-1\\ r\end{bmatrix}_q\prod_{j=1}^{(r+1)/2}(q^{2j-1}-1)K_{\mathrm{GL}(n-1-r,q)}(\lambda;1,1)\Big\}.$$

An explicit expression for (5.8) was obtained in [10].

THEOREM 5.1. *For integers $t\ge 1$ and nonzero elements a,b of $\mathbb{F}_q$, the Kloosterman sum $K_{\mathrm{GL}(t,q)}(\lambda;a,b)$ is given by*

$$(5.11)\qquad K_{\mathrm{GL}(t,q)}(\lambda;a,b) =q^{(t-2)(t+1)/2}\sum_{l=1}^{[(t+2)/2]}q^lK(\lambda;a,b)^{t+2-2l}\sum(q^{j_1}-1)\dots(q^{j_{l-1}}-1),$$

where $K(\lambda;a,b)$ is the usual Kloosterman sum in (2.11) and the inner sum is over all integers $j_1,\dots,j_{l-1}$ satisfying $2l-3\le j_1\le t-1$, $2l-5\le j_2\le j_1-2$, $\dots,$ $1\le j_{l-1}\le j_{l-2}-2$. Here we adopt the convention that the inner sum in (5.11) is 1 for $l=1$, and that $j_0=t+1$ for $l=2$.

Combining (5.10) with the explicit expression of Kloosterman sum in (5.11), and replacing r in the first sum and the second sum in (5.10) respectively by $2r$ and $2r+1$, we obtain the following theorem.

THEOREM 5.2. *Let λ be a nontrivial additive character of $\mathbb{F}_q$. Then the Gauss sum over* $\mathrm{SO}^-(2n,q)$,

$$\sum_{w\in \mathrm{SO}^-(2n,q)} \lambda(\mathrm{tr}\, w),$$

is given by

$$\begin{aligned} q^{n^2-n-1}\Big\{\Big(-\frac{1}{q-1}\sum_{j=1}^{q-1} G(\psi^j,\lambda)^2\Big) &\sum_{r=0}^{[(n-1)/2]} q^{r(r+3)} \begin{bmatrix} n-1 \\ 2r \end{bmatrix}_q \prod_{j=1}^{r}(q^{2j-1}-1) \\ &\times \sum_{l=1}^{[(n-2r+1)/2]} q^l K(\lambda;1,1)^{n-2r+1-2l} \sum (q^{j_1}-1)\ldots(q^{j_{l-1}}-1) \\ &-(q+1)\sum_{r=0}^{[(n-2)/2]} q^{r(r+3)+1} \begin{bmatrix} n-1 \\ 2r+1 \end{bmatrix}_q \prod_{j=1}^{r+1}(q^{2j-1}-1) \\ &\times \sum_{l=1}^{[(n-2r)/2]} q^l K(\lambda;1,1)^{n-2r-2l} \sum (q^{j_1}-1)\ldots(q^{j_{l-1}}-1)\Big\}. \end{aligned}$$

Here $K(\lambda;1,1)$ is the usual Kloosterman sum as in (2.11), *and $G(\psi^j,\lambda)$ and $G(\eta,\lambda)$ are the usual Gauss sums in* (2.9) *with ψ a multiplicative character of $\mathbb{F}_q$ of order $q-1$ and with η the quadratic character of $\mathbb{F}_q$. In addition, the first unspecified sum is over all integers $j_1,\ldots,j_{l-1}$ satisfying $2l-3 \le j_1 \le n-2r-2$, $2l-5 \le j_2 \le j_1-2$, $\ldots$, $1 \le j_{l-1} \le j_{l-2}-2$ and the second one is over all integers $j_1,\ldots,j_{l-1}$ satisfying $2l-3 \le j_1 \le n-2r-3$, $2l-5 \le j_2 \le j_1-2$, $\ldots$, $1 \le j_{l-1} \le j_{l-2}-2$.*

6. $O^-(2n,q)$ case. Let χ be a multiplicative character of $\mathbb{F}_q$, and let λ be a nontrivial additive character of $\mathbb{F}_q$. Then we will consider the Gauss sum in (1.2),

$$\sum_{w\in O^-(2n,q)} \chi(\det w)\lambda(\mathrm{tr}\, w),$$

and find an explicit expression for it.

From the decompositions in (3.15) and (3.16), the above sum is $\sum_{w\in \mathrm{SO}^-(2n,q)} \lambda(\mathrm{tr}\, w)$ plus

$$\begin{aligned} \chi(-1)\Big\{ &\sum_{\substack{0\le r\le n-1 \\ r \text{ even}}} |B_r\backslash Q| \sum_{w\in Q} \lambda(\mathrm{tr}\, \varrho w \sigma_r) \\ &+ \sum_{\substack{0\le r\le n-1 \\ r \text{ odd}}} |B_r\backslash Q| \sum_{w\in Q} \lambda(\mathrm{tr}\, w\sigma_r)\Big\}, \end{aligned} \tag{6.1}$$

where the expression in curly brackets is the same as that in (5.9), except that $\sum_{i\in\mathrm{SO}^-(2,q)}\lambda(\mathrm{tr}\,i)$ and $\sum_{i\in\mathrm{SO}^-(2,q)}\lambda(\mathrm{tr}\,\delta_1 i)$ are interchanged. So the sum in (1.2) equals

$$\begin{aligned}(6.2)\qquad & q^{(n-1)(n+2)/2}A(\chi,\lambda)\\ &\times\Big\{\sum_{\substack{0\le r\le n-1\\ r\,\text{even}}}|B_r\backslash Q|q^{r(n-r-3)}b_r(\lambda)K_{\mathrm{GL}(n-1-r,q)}(\lambda;1,1)\\ &+\chi(-1)\sum_{\substack{0\le r\le n-1\\ r\,\text{odd}}}|B_r\backslash Q|q^{r(n-r-3)}b_r(\lambda)K_{\mathrm{GL}(n-1-r,q)}(\lambda;1,1)\Big\},\end{aligned}$$

where

$$\begin{aligned}(6.3)\qquad A(\chi,\lambda)&=\sum_{i\in\mathrm{SO}^-(2,q)}\lambda(\mathrm{tr}\,i)+\chi(-1)\sum_{i\in\mathrm{SO}^-(2,q)}\lambda(\mathrm{tr}\,\delta_1 i)\\ &=-\frac{1}{q-1}\sum_{j=1}^{q-1}G(\psi^j,\lambda)^2+\chi(-1)(q+1)\end{aligned}$$

with ψ a multiplicative character of $\mathbb{F}_q$ of order $q-1$ (cf. (4.13), (4.14)).

From (3.21) (cf. (3.12)), (4.6), (5.11), and replacing r in the first sum and the second sum in (6.2) respectively by $2r$ and $2r+1$, we get the following theorem.

THEOREM 6.1. *Let χ be a multiplicative character of $\mathbb{F}_q$, and let λ be a nontrivial additive character of $\mathbb{F}_q$. Then the Gauss sum over $O^-(2n,q)$,*

$$\sum_{w\in O^-(2n,q)}\chi(\det w)\lambda(\mathrm{tr}\,w),$$

is given by

$$\begin{aligned}(6.4)\qquad & q^{n^2-n-1}A(\chi,\lambda)\Big\{\sum_{r=0}^{[(n-1)/2]}q^{r(r+3)}\begin{bmatrix}n-1\\2r\end{bmatrix}_q\prod_{j=1}^{r}(q^{2j-1}-1)\\ &\times\sum_{l=1}^{[(n-2r+1)/2]}q^lK(\lambda;1,1)^{n-2r+1-2l}\sum(q^{j_1}-1)\dots(q^{j_{l-1}}-1)\\ &-\chi(-1)\sum_{r=0}^{[(n-2)/2]}q^{r(r+3)+1}\begin{bmatrix}n-1\\2r+1\end{bmatrix}_q\prod_{j=1}^{r+1}(q^{2j-1}-1)\\ &\times\sum_{l=1}^{[(n-2r)/2]}q^lK(\lambda;1,1)^{n-2r-2l}\sum(q^{j_1}-1)\dots(q^{j_{l-1}}-1)\Big\}.\end{aligned}$$

Here $A(\chi,\lambda)$ is as in (6.3), $K(\lambda;1,1)$ *is the usual Kloosterman sum as in* (2.11), *and $G(\eta,\lambda)$ is the usual Gauss sum with η the quadratic character of $\mathbb{F}_q$. Moreover, the first unspecified sum in* (6.4) *is over all integers $j_1,\dots,j_{l-1}$ satisfying $2l-3\le j_1\le n-2r-2$, $2l-5\le j_2\le j_1-2,\dots,$ $1\le j_{l-1}\le j_{l-2}-2$, and the second one is over all integers $j_1,\dots,j_{l-1}$ satisfying $2l-3\le j_1\le n-2r-3$, $2l-5\le j_2\le j_1-2$, $\dots$, $1\le j_{l-1}\le j_{l-2}-2$.*

7. Application to Hodges' Kloosterman sum. In [5], the generalized Kloosterman sum over nonsingular symmetric matrices is defined, for $t\times t$ symmetric matrices A,B over $\mathbb{F}_q$, as

$$K_{\mathrm{sym},t}(A,B)=\sum_{w}\lambda_1(\operatorname{tr}(Aw+Bw^{-1})), \tag{7.1}$$

where w runs over the set Ω_t of all nonsingular symmetric matrices over $\mathbb{F}_q$ of size t.

In contrast to his other papers [6]–[8], Hodges neglected to mention an important special case of the main theorem in [5]. Namely, if $m=t$ and U is a nonsingular matrix in the main theorem, then $s_1=s_2=0$.

Now, take $m=t=2n$, $A=B=J^-$ in (4.2) with $r=2n$, $U=\frac{a}{2}1_{2n}$ with $0\ne a\in\mathbb{F}_q$, in the main theorem of [5]. Then, in view of the above-mentioned observation, we have the identity

$$\sum_{w\in O^-(2n,q)}\lambda_a(\operatorname{tr}w)=q^{-n}K_{\mathrm{sym},2n}\left(\frac{a^2}{4}(J^-)^{-1},J^-\right), \tag{7.2}$$

where $K_{\mathrm{sym},2n}\big(\frac{a^2}{4}(J^-)^{-1},J^-\big)$ is as in (7.1). We state this fact as the following theorem.

THEOREM 7.1. *For $0\ne a\in\mathbb{F}_q$, we have the identity*

$$\begin{aligned}\sum_{w\in O^-(2n,q)}\lambda_a(\operatorname{tr}w)&=q^{-n}K_{\mathrm{sym},2n}\left(\frac{a^2}{4}(J^-)^{-1},J^-\right)\\&=q^{-n}K_{\mathrm{sym},2n}\left(\frac{a^2}{4}C^{-1},C\right),\end{aligned} \tag{7.3}$$

where λ_a is as in (2.1) *and C is any nonsingular symmetric matrix over $\mathbb{F}_q$ of size $2n$ with $C\sim J^-$.*

Remark. The second identity in (7.3) is clear from the definition in (7.1).

Combining Theorems 6.1 and 7.1, we get the following result.

THEOREM 7.2. *Let $0\ne a\in\mathbb{F}_q$, and let C be any nonsingular symmetric matrix over $\mathbb{F}_q$ of size $2n$ with $C\sim J^-$. Then the following generalized*

Kloosterman sum over nonsingular symmetric matrices is the same for every such C, *and*

$$
\begin{aligned}
(7.4)\qquad & K_{\mathrm{sym},2n}\left(\frac{a^2}{4}C^{-1},C\right)\\
&= q^{n^2-1}A(\lambda_a)\Bigg\{\sum_{r=0}^{[(n-1)/2]} q^{r(r+3)}\begin{bmatrix} n-1\\ 2r\end{bmatrix}_q \prod_{j=1}^{r}(q^{2j-1}-1)\\
&\quad\times\sum_{l=1}^{[(n-2r+1)/2]} q^l K(\lambda_a;1,1)^{n-2r+1-2l}\sum (q^{j_1}-1)\dots(q^{j_{l-1}}-1)\\
&\quad-\sum_{r=0}^{[(n-2)/2]} q^{r(r+3)+1}\begin{bmatrix} n-1\\ 2r+1\end{bmatrix}_q \prod_{j=1}^{r+1}(q^{2j-1}-1)\\
&\quad\times\sum_{l=1}^{[(n-2r)/2]} q^l K(\lambda_a;1,1)^{n-2r-2l}\sum (q^{j_1}-1)\dots(q^{j_{l-1}}-1)\Bigg\},
\end{aligned}
$$

where

$$A(\lambda_a) = -\frac{1}{q-1}\sum_{j=1}^{q-1} G(\psi^j,\lambda_a)^2 + q + 1$$

with ψ *a multiplicative character of* $\mathbb{F}_q$ *of order* $q-1$, $K(\lambda_a;1,1)$ *is the usual Kloosterman sum as in* (2.11) (cf. (2.1)), *and* η *is the quadratic character of* $\mathbb{F}_q$. *In addition, the first unspecified sum in* (7.4) *is over all integers* $j_1,\dots,j_{l-1}$ *satisfying* $2l-3 \le j_1 \le n-2r-2$, $2l-5 \le j_2 \le j_1-2$, $\dots$, $1 \le j_{l-1} \le j_{l-2}-2$, *and the second one is over all integers* $j_1,\dots,j_{l-1}$ *satisfying* $2l-3 \le j_1 \le n-2r-3$, $2l-5 \le j_2 \le j_1-2$, $\dots$, $1 \le j_{l-1} \le j_{l-2}-2$.

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