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On the diophantine equation $\binom{n}{k} = x^l$

by

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To the memory of Professor P. Erdős

1. Introduction. Consider the equation

$$(1) \qquad \binom{n}{k} = x^l \quad \text{in integers } n, k, x, l \text{ with } k \geq 2,\ n \geq 2k,\ x > 1,\ l > 1.$$

There is no loss in generality in assuming that $n \geq 2k$, since $\binom{n}{k} = \binom{n}{n-k}$. It is clear that there are infinitely many solutions if $k = l = 2$. For $k = 3$, $l = 2$, equation (1) has only the solution $n = 50$, $x = 140$ (for references see e.g. [4], p. 25 or [7], p. 251). In 1939, P. Erdős [5] proved that no solutions exist if $k \geq 2^l$ or if $l = 3$. Further, he conjectured that (1) has no solution if $l > 3$. R. Obláth [13] confirmed this conjecture for $l = 4$ and $l = 5$.

In 1951, Erdős [6] (see also [7]) proved in an ingenious, elementary way the following.

Theorem A (P. Erdős [6]). *For $k > 3$, equation (1) has no solution.*

There remained the cases $k = 2$ and $k = 3$. In what follows, we consider the equations

$$(2) \qquad \binom{n}{2} = x^l \quad \text{in integers } n, x, l \text{ with } n > 2,\ x > 1,\ l > 2,$$

and

$$(3) \qquad \binom{n}{3} = x^l \quad \text{in integers } n, x, l \text{ with } n > 3,\ x > 1,\ l > 2.$$

It follows from results of P. Dénes [3] that for certain regular primes l, equations (2) and (3) have no solutions in n and x.

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In [10] (see also [8]) I proved the following. Assume that for given prime $l > 5$, equation (2) is not solvable. Then equation (3) has at most one solution. Further, if in addition

$$3^{l-1} \not\equiv 1 \pmod{l^2} \tag{4}$$

holds, then (3) has no solution.

An important contribution was made by R. Tijdeman [16] who proved that equations (2) and (3) have only finitely many solutions, and all these solutions can be effectively determined. In his proof Tijdeman used a profound effective inequality of A. Baker concerning linear forms in logarithms. Recently, N. Terai [15] utilized a recent estimate of linear forms in logarithms to show that if (2) or (3) is solvable then $l < 4250$.

H. Darmon and L. Merel [2] have recently proved that for given integer $l \geq 3$, the equation

$$x^l + y^l = 2 \cdot z^l \quad \text{in relatively prime integers } x, y, z$$

has only trivial solutions for which $xyz = 0$ or ± 1. In their proof the authors combined various recent powerful results in number theory, including Wiles' proof of most cases of the Shimura–Taniyama conjecture. If now equation (2) is solvable then $n = y^l$, $n - 1 = 2z^l$ or $n = 2z^l$, $n - 1 = y^l$ with some coprime positive integers y, z, whence $y^l \pm 1 = 2z^l$. Thus, the next theorem immediately follows from the above theorem of Darmon and Merel.

Theorem B (H. Darmon and L. Merel). *Equation* (2) *has no solution.*

In the present paper we shall prove the following.

Theorem 1. *Equation* (3) *has no solution.*

To prove Theorem 1, we combine the arguments of [10] with Theorem B, a result on (4) and a recent theorem of M. A. Bennett and B. M. M. de Weger [1]. As will be shown after the proof of Theorem 1, this latter theorem can be replaced in our proof by the results of Dénes [3] and Terai [15].

Together with Theorems A and B, Theorem 1 proves Erdős' conjecture. The following theorem provides a complete solution of equation (1). It may be regarded as a joint result of Erdős, Darmon, Merel and the present author.

Theorem 2 (P. Erdős, case $k > 3$; H. Darmon and L. Merel, case $k = 2$; K. Győry). *Apart from the case $k = l = 2$, equation* (1) *has only the solution $n = 50$, $k = 3$, $x = 140$, $l = 2$.*

As was quoted above, for $k = 3$, $l = 2$ the assertion of Theorem 2 had been proved a long time ago. For other values of k and l, Theorem 2 is an immediate consequence of Theorems A, B and Theorem 1.

2. Proof of Theorem 1. We first show that we can make some restrictions concerning l. The number 140^2 is the only square which can be represented in the form $\binom{n}{3}$ with $n > 3$. Since 140 is not a full power, in equation (3) the exponent l cannot be even. Further, the results of Erdős [5] and Obláth [13] imply that l must be greater than 5. Therefore it suffices to prove that (3) has no solution for any prime $l > 5$.

For primes $l > 5$ satisfying (4), the proof of a result of my thesis [10] can be adapted. A similar result was earlier published in my paper ([8], Thm. 2). However, [10] and [8] were written in Hungarian, and the proof of the theorem in question in [8] is not complete. Hence we shall give here a detailed proof of our Theorem 1.

We shall need five lemmas. In Lemmas 1 to 3, l denotes a prime greater than 3.

LEMMA 1. *Let a, b be relatively prime integers with $a + b \neq 0$. Then*

$$\left(a + b, \frac{a^l + b^l}{a + b}\right) = 1 \text{ or } l.$$

Further, $l^2 \nmid \frac{a^l+b^l}{a+b}$ and each prime divisor $\neq l$ of $\frac{a^l+b^l}{a+b}$ is of the form $lt + 1$.

Proof. See e.g. [11]. ∎

The following two lemmas have been proved by means of Eisenstein's reciprocity theorem.

LEMMA 2 (S. Lubelski [12]). *Let a, b, c be integers such that*

$$\frac{a^l + b^l}{a + b} = c^l, \quad (a, b) = 1, \quad (a^2 - b^2, l) = 1. \tag{5}$$

Then for each prime r with $r \neq l$ and $r \mid a - b$, we have

$$r^{l-1} \equiv 1 \pmod{l^2}. \tag{6}$$

Proof. See [12], Satz 2. ∎

LEMMA 3 (K. Győry [9]). *Let a, b, c be integers satisfying* (5). *Then we have* (6) *for each prime r with $r \neq l$ and $r \mid a + b$.*

Proof. This was proved in [9] (cf. the Lemma in the proof of Satz 1). ∎

LEMMA 4. *If l is an odd prime with $l < 2^{30}$ and*

$$3^{l-1} \equiv 1 \pmod{l^2} \tag{7}$$

then $l = 11$ or $l = 1006003$.

Proof. See e.g. [14], pp. 169–170, and the references given there. ∎

The next result has recently been proved by means of rational approximation to hypergeometric functions, the theory of linear forms in logarithms and some recent computational methods.

Lemma 5 (M. A. Bennett and B. M. M. de Weger [1]). *Let a, b and l be integers with $b > a > 1$ and $3 \le l < 17$ or $l > 347$. Then the equation*

$$|ax^l - by^l| = 1$$

has at most one solution in positive integers x, y.

Proof. This is an immediate consequence of Theorem 1.1 in [1]. ∎

Proof of Theorem 1. Suppose that equation (3) has a solution n, x, l with $n > 3$, $l > 2$. As was remarked above, we may assume that l is a prime greater than 5.

We first show that, for l, (7) must hold. To prove this, we follow the arguments of [10] (cf. also [8]). It follows from (3) that

$$n(n-1)(n-2) = 6x^l.$$

We distinguish three cases according as n, $n-1$ or $n-2$ is divisible by 3. Among the numbers n, $n-1$ and $n-2$ at most one is divisible by 2^2. Hence, apart from the prime factors 2 and 3, each of the numbers n, $n-1$ and $n-2$ must be an lth power.

First, we consider the case when n is divisible by 3. We have the following three subcases. In what follows, u, v and w denote positive integers.

(i,1) $n = 3u^l$, $n-1 = 2v^l$, $n-2 = w^l$, whence $\binom{n-1}{2} = (vw)^l$ and $n-1 > 2$. However, by Theorem B this is not possible.

(i,2) $n = 3 \cdot 2^l u^l$, $n-1 = v^l$, $n-2 = 2w^l$, whence $\binom{n-1}{2} = (vw)^l$, which is also impossible.

(i,3) $n = 6u^l$, $n-1 = v^l$, $n-2 = 2^l w^l$, which gives $v^l - 1 = (2w)^l$. This is, however, not solvable in positive integers v, w because $l > 5$.

When $n-2$ is divisible by 3, we have the following three subcases.

(ii,1) $n = u^l$, $n-1 = 2v^l$, $n-2 = 3w^l$, whence $\binom{n}{2} = (uv)^l$ and $n > 2$. But this is impossible by Theorem B.

(ii,2) $n = 2u^l$, $n-1 = v^l$, $n-2 = 3 \cdot 2^l w^l$, whence $\binom{n}{2} = (uv)^l$, which is again impossible.

(ii,3) $n = 2^l u^l$, $n-1 = v^l$, $n-2 = 6w^l$, whence $v^l + 1 = (2u)^l$, which has no solution in positive integers v, u.

Finally, consider those subcases when $n-1$ is divisible by 3.

(iii,1) $n = u^l$, $n-1 = 6v^l$, $n-2 = w^l$, whence $u^l - w^l = 2$, which is impossible.

We have showed that the above cases cannot hold. It remains to deal with the following two subcases:

(iii,2) $n = 2w^l$, $n-1 = 3v^l$, $n-2 = 2^l u^l$,

(iii,3) $n = 2^l u^l$, $n-1 = 3v^l$, $n-2 = 2w^l$.

It is easily seen that, in both cases, v and w must be greater than 1. In the cases (iii,2) and (iii,3) we obtain the systems of equations

$$2(w^l - 1) = 3v^l - 1 = 2^l u^l \quad \text{in integers } u, v, w \text{ with } u \geq 1,\ v, w > 1, \tag{8}$$

and

$$2(w^l + 1) = 3v^l + 1 = 2^l u^l \quad \text{in integers } u, v, w \text{ with } u \geq 1,\ v, w > 1, \tag{9}$$

respectively.

It is sufficient to prove that none of the systems (8) and (9) is solvable. Consider (8) and (9) simultaneously. It follows from (8) and (9) that

$$(2u)^l \pm 1 = 3v^l. \tag{10}$$

Here and in the sequel the upper and lower sign must be taken according as the case under consideration is a consequence of (8) or (9), respectively. First assume that $l \nmid v$ and $l \nmid (2u)^2 - 1$. Then, by Lemma 1, we infer from (10) that

$$\frac{(2u)^l \pm 1}{2u \pm 1} = c^l$$

with some non-zero integer c, and that $3 \mid 2u+1$ in the first case and $3 \mid 2u-1$ in the second case. Hence, by Lemma 3, we deduce in both cases that

$$3^{l-1} \equiv 1 \pmod{l^2}. \tag{7}$$

Next assume that $l \mid (2u)^2 - 1$ or $l \mid v$. If $l \mid v$, then it follows from (10) that $2u \pm 1 \equiv 0 \pmod{l}$, which implies that $l \mid (2u)^2 - 1$. Hence it is sufficient to deal with the systems of equations (8) and (9) under the assumption that

$$2u \equiv 1 \pmod{l} \tag{11}$$

or

$$2u \equiv -1 \pmod{l}. \tag{12}$$

Consider again (8) and (9) simultaneously. It follows from (8) and (9) that

$$w^l \mp 1 = 2^{l-1} u^l. \tag{13}$$

By (11) and (12), we have $l \nmid u$. Thus, by Lemma 1, we infer from (13) that

$$\frac{w^l \mp 1}{w \mp 1} = d^l \tag{14}$$

with some non-zero integer d. To apply Lemma 2 to (14), we have to show that $l \nmid w^2 - 1$. Together with (11) or (12), (13) implies in both cases that $2w$ can be congruent only to 1, -1, 3 or $-3 \pmod{l}$. But $l > 5$, hence it follows indeed in both cases that $w \not\equiv 1 \pmod{l}$ and $w \not\equiv -1 \pmod{l}$. In view of $w^l \equiv w \pmod{3}$, (8), resp. (9), implies that $3 \mid w+1$, resp. $3 \mid w-1$

holds. Thus, by applying Lemma 2 to (14) we conclude again that (7) must hold.

It follows both from (8) and from (9) that v, w satisfy the equation

$$|2w^l - 3v^l| = 1 \quad \text{in positive integers } v, w. \tag{15}$$

This equation has the solution $v = w = 1$. Hence, if $l < 17$ or $l > 347$, it follows by Lemma 5 that equation (15) has no solution in positive integers with $v > 1$ or $w > 1$. Thus there remains the case $17 \le l \le 347$. However, in view of Lemma 4, (7) does not hold for these values of l. This completes the proof of Theorem 1. ∎

Remark. After having proved above that (7) must hold, the results of Terai [15] and Dénes [3] can also be used in place of Lemma 5. Indeed, Terai's theorem implies that no solutions of (3) exist if $l \ge 4250$. By Lemma 4, $l = 11$ is the only prime l for which both (7) and $l < 4250$ hold. Finally, it follows from Satz 8 of Dénes [3] that equation (10) is not solvable for $l = 11$, and the proof of Theorem 1 is again complete. ∎

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