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Averages of twisted elliptic L-functions

by

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1. Introduction. Let E be a modular elliptic curve over $\mathbb{Q}$ with conductor N defined by the Weierstrass equation $y^2 = \omega(x)$, d be a fundamental discriminant with $(d,N)=1$ and E_d be the twisted elliptic curve defined by $dy^2 = \omega(x)$. Let $L(s,E)$ and $L(s,E_d)$ denote the Hasse–Weil L-functions associated with E and E_d, respectively. Then $L(s,E_d)$ is obtained from $L(s,E)$ by twisting by a real primitive character $\chi_d \pmod d$, and both $L(s,E)$ and $L(s,E_d)$ are entire functions with good analytic properties (see Section 2).

It is well known that many interesting arithmetical problems about elliptic curves can be translated, at least conjecturally, into analytic problems about the associated L-functions. In particular, due to the Birch and Swinnerton-Dyer conjecture and to Kolyvagin's theorem, the order of vanishing of $L(s,E)$ at $s=1$, called the *analytic rank* of E and denoted by $\operatorname{rank}(E)$, and the non-vanishing of $L'(s,E_d)$ at $s=1$ have attracted much attention in recent years. In this context, the techniques from analytic number theory have been proved to be particularly effective when dealing with averaging problems over suitable families of elliptic curves. In this paper we consider two such problems.

Let

$$N(D) = \#\{d \le D : d \text{ fundamental discriminant with } (d,N)=1 \\ \text{and } L'(1,E_d) \ne 0\}.$$

Our first result is

Theorem 1. *Let $\varepsilon > 0$. Then $N(D) \gg_\varepsilon D^{1-\varepsilon}$.*

This improves on Iwaniec's $N(D) \gg_\varepsilon D^{2/3-\varepsilon}$ in [8]. The second result is

Theorem 2. $\sum^*_{d\le D} \operatorname{rank}(E_d) = o(D\log D)$.

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Throughout the paper * means that the summation is over fundamental discriminants with $(d, N) = 1$. Theorem 2 is only slightly better than the trivial bound

$$\sum_{d\le D}^{*} \operatorname{rank}(E_d) \ll D\log D.$$

Stronger results can be obtained under the assumption of the Riemann Hypothesis for the functions $L(s, E_d)$ (see, e.g., Goldfeld [5], Mestre [10], Brumer [1], Fouvry–Pomykała [4], Duke [2], Murty [13], Michel [11] and Fermigier [3] for related results). These results should be compared with the remark after Theorem 5 in Section 2.

Our results are based on the recent large sieve type estimates over fundamental discriminants obtained by Heath-Brown [6]. The quality of such estimates determines the quality of our results above. In particular, if the factor D^{ε} appearing in Theorems 3 and 4 below, which comes from the application of Heath-Brown's estimates, could be replaced by some power of $\log D$, then we would get a corresponding improvement of the type

$$N(D) \gg D\log^{-c} D \quad \text{with some } c \ge 0$$

and

$$\sum_{d\le D}^{*} \operatorname{rank}(E_d) \ll D\log\log D,$$

as will be clear from the arguments in Section 2.

2. Outline of the proofs. In this section we outline the basic ingredients of the proofs. The main tool is

THEOREM 3. *Let* $\varepsilon > 0$ *and* $\tau = |t| + 1$. *Then*

$$\sum_{d\le D}^{*} |L'(1, E_d)|^2 \ll_{\varepsilon} D^{1+\varepsilon}$$

and

$$\sum_{d\le D}^{*} |L(\sigma + it, E_d)|^2 \ll_{\varepsilon} (D + (D\tau)^{3-2\sigma})(D\tau)^{\varepsilon}$$

uniformly for $1 \le \sigma \le 3/2$ *and* $t \in \mathbb{R}$.

The proof of Theorem 3, which follows the proof of Theorem 2 of [6], will be sketched in Section 3.

Assume, more generally, that

$$\sum_{d\le D}^{*} |L'(1, E_d)|^2 \ll DG(D) \tag{1}$$

with a non-decreasing function $G(D) \gg \log^2 D$. By a slight variant of the arguments in Jutila [9] and Murty–Murty [14] we can get

$$\sum_{d\le D}^{*} L'(1,E_d) \sim CD\log D \tag{2}$$

with a certain constant $C \neq 0$. Hence from (1), (2) and the Cauchy–Schwarz inequality we deduce that

$$N(D) \gg D\frac{\log^2 D}{G(D)},$$

and hence Theorem 1 follows at once from the first estimate of Theorem 3.

The proof of Theorem 2 is based on the use of Weil's explicit formula and of a suitable average density estimate for the zeros of the functions $L(s,E_d)$. Writing

$$N(\sigma,T,d) = \#\{\varrho = \beta + i\gamma : L(\varrho,E_d) = 0,\ \beta \ge \sigma \text{ and } |\gamma| \le T\}$$

we have

Theorem 4. *Let $\varepsilon > 0$. Then*

$$\sum_{d\le D}^{*} N(\sigma,T,d) \ll_\varepsilon D^{(3-2\sigma)/(2-\sigma)}(T+1)^{(7-4\sigma)/(4-2\sigma)}(DT)^\varepsilon$$

uniformly for $1 \le \sigma \le 3/2$.

The proof of Theorem 4, which is based on Theorem 3 and on Heath-Brown's estimates, follows the lines of Montgomery's zero-detecting method and will be sketched in Section 4.

In Section 5 we will use the method of Weil's explicit formula together with Theorem 4 to prove Theorem 2. In fact, we will prove the following general result:

Theorem 5. *Assume that there exist constants $c, A > 0$ and a non-decreasing function $L(D) \ge 2$ such that*

$$\sum_{d\le D}^{*} N(1+\delta,T,d) \ll (T+1)^A D^{1-c\delta} L(D)$$

uniformly for $1/\log D \le \delta \le 1/2$. *Then*

$$\sum_{d\le D}^{*} \operatorname{rank}(E_d) \ll_{c,A} D\log L(D).$$

Theorem 5 appears to be the limit of our method, and Theorem 2 follows at once from Theorems 4 and 5, since we can choose $L(D) = D^\varepsilon$. Observe that the Riemann Hypothesis for the functions $L(s,E_d)$ allows, in particular, the choice $L(D) = 2$.

We recall here some basic facts about the functions $L(s,E_d)$ which will be needed later on (see, e.g., [8]). For $\sigma>3/2$, $L(s,E_d)$ has an Euler product expansion of degree 2 satisfying the Ramanujan conjecture and

$$L(s,E_d)=\sum_{n=1}^{\infty}a(n)\chi_d(n)n^{-s},\qquad \frac{1}{L(s,E_d)}=\sum_{n=1}^{\infty}b(n)\chi_d(n)n^{-s},$$
$$\frac{L'}{L}(s,E_d)=\sum_{n=1}^{\infty}e_d(n)n^{-s}$$

with $|a(n)|,|b(n)|\le n^{1/2}\tau(n)$ and $|e_d(n)|\le n^{1/2}\Lambda(n)\tau(n)$, where τ is the divisor function. Moreover, the functions $L(s,E_d)$ are entire, of finite order on every right half-plane and satisfy the functional equation

$$\Lambda(s,E_d)=w_d\Lambda(2-s,E_d)$$

where

$$\Lambda(s,E_d)=\left(\frac{d\sqrt{N}}{2\pi}\right)^s\Gamma(s)L(s,E_d)\quad\text{and}\quad|w_d|=1.$$

We finally remark that all the constants may depend on the data of the fixed elliptic curve E.

3. Proof of Theorem 3. We first state the basic tool of our paper, i.e., Corollary 3 of Heath-Brown [6]. Denoting by $S(Q)$ the set of all real primitive characters of modulus at most Q, we state Corollary 3 of [6] as

Proposition 1. *Let Q,N be positive integers and $a_1,\ldots,a_N\in\mathbb{C}$. Then for every $\varepsilon>0$,*

$$\sum_{\chi\in S(Q)}\Big|\sum_{n\le N}a_n\chi(n)\Big|^2\ll_\varepsilon Q^\varepsilon N^{1+\varepsilon}(Q+N)\max_{n\le N}|a_n|^2.$$

As we have already remarked, we follow the proof of Theorem 2 of [6], due to the similarity between $L(s,E_d)$ and $L(s,\chi_d)^2$, where $L(s,\chi)$ denotes the Dirichlet L-series formed with the character χ. Write

$$S(D,s)=\sideset{}{^*}\sum_{D<d\le 2D}|L(s,E_d)|^2$$

and denote by $\nu(\sigma)$ the infimum of the $\nu\in\mathbb{R}$ for which

$$S(D,s)\ll(D+(D\tau)^{3-2\sigma})(D\tau)^\nu \tag{3}$$

uniformly in D and t.

Using the Mellin transform and the properties of the Γ-function as on p. 268 of [6], we see that for $1/2 \le \alpha < \sigma$ and $X > 1$,

$$S(D,s) \ll \sideset{}{^*}\sum_{D<d\le 2D} \Big|\sum_{n=1}^{\infty} a(n)\chi_d(n)n^{-s}e^{-n/X}\Big|^2 + X^{2(\alpha-\sigma)} \int_{-\infty}^{\infty} S(D,\alpha+iu)e^{-|u-t|}\,du.$$

From the functional equation of the functions $L(s,E_d)$ and (3) we get $S(D,\alpha+iu) \ll (D(|u|+1))^{4(1-\alpha)}(D+(D(|u|+1))^{2\alpha-1})(D(|u|+1))^{\nu(2-\alpha)+\varepsilon}$ and hence

$$S(D,s) \ll \sideset{}{^*}\sum_{D<d\le 2D} \Big|\sum_{n=1}^{\infty} a(n)\chi_d(n)n^{-s}e^{-n/X}\Big|^2 + X^{2(\alpha-\sigma)}(D\tau)^{4(1-\alpha)}(D+(D\tau)^{2\alpha-1})(D\tau)^{\nu(2-\alpha)+\varepsilon}. \tag{4}$$

Due to the decay of $e^{-n/X}$ and the bound for the coefficients $a(n)$, we see that the contribution of the terms with $n > X\log^2 D\tau$ in the inner sum on the right hand side of (4) is negligible. We split the remaining part of that sum into sub-sums where n runs over intervals of the type $M < n \le 2M$, and applying Proposition 1 to each sub-sum we see that

$$\sideset{}{^*}\sum_{D<d\le 2D} \Big|\sum_{n=1}^{\infty} a(n)\chi_d(n)n^{-s}e^{-n/X}\Big|^2 \ll (D+X^{3-2\sigma})(D\tau X)^{\varepsilon}. \tag{5}$$

Let $\delta > 0$ be sufficiently small. If $1+\delta \le \sigma \le 3/2$ we choose $\alpha = 2-\sigma$ and $X = (D\tau)^{1+\delta}$. Then $1/2 \le \alpha < \sigma$, and from (4) and (5) we get

$$\begin{aligned} S(D,s) &\ll (D+(D\tau)^{3-2\sigma})(D\tau)^{\varepsilon(2+\delta)+\delta} \\ &\quad + (D+(D\tau)^{3-2\sigma})(D\tau)^{-4\delta(\sigma-1)+\nu(\sigma)+\varepsilon} \\ &\ll (D+(D\tau)^{3-2\sigma})(D\tau)^{3\varepsilon}\{(D\tau)^{\delta} + (D\tau)^{\nu(\sigma)-4\delta(\sigma-1)}\}. \end{aligned}$$

Hence

$$\nu(\sigma) \le \max(\delta, \nu(\sigma) - 4\delta(\sigma-1)) + 3\varepsilon,$$

and choosing $\delta = \sqrt{\varepsilon}$ we see that this implies that $\nu(\sigma) \le \delta + 3\varepsilon$ in this case.

If $1 \le \sigma \le 1+\delta$, where $\delta = \sqrt{\varepsilon}$, we choose $\alpha = 1-\delta$ and still $X = (D\tau)^{1+\delta}$. Then $1/2 \le \alpha < \sigma$, and from (4) and (5) we get

$$\begin{aligned} S(D,s) &\ll (D+(D\tau)^{3-2\sigma})(D\tau)^{(2+\delta)\varepsilon+\delta} \\ &\quad + (D\tau)^{2(1+\delta)(1-\delta-\sigma)+4\delta}(D+(D\tau)^{1-2\delta})(D\tau)^{\nu(1+\delta)+\varepsilon} \\ &\ll (D+(D\tau)^{3-2\sigma})\{(D\tau)^{3\varepsilon+\delta} + (D\tau)^{\nu(1+\delta)+4\delta+\varepsilon}\}, \end{aligned}$$

since $\delta \ge \sigma - 1$ and $2(1+\delta)(1-\delta-\sigma) < 0$. But $\nu(1+\delta) \le \delta + 3\varepsilon$, hence $\nu(\sigma) \le 5\delta + 4\varepsilon$ in this case.

Since ε is arbitrarily small and $\delta = \sqrt{\varepsilon}$, the second assertion of Theorem 3 follows. Moreover, using in addition the functional equation of the functions $L(s, E_d)$, we see that

$$\sum_{d \le D}^{*} |L(s, E_d)|^2 \ll D^{1+\varepsilon} \tag{6}$$

uniformly for $|s-1| \le 2(\log D)^{-1}$. Hence the first assertion of Theorem 3 follows from (6), using Cauchy's integral formula and choosing the circle $|s-1| = (\log D)^{-1}$ as path of integration.

4. Proof of Theorem 4. Here we follow the zero-detecting method of Chapter 12 of Montgomery [12], as presented in the proof of Theorem 3 of [6]. Let $R = R(D, T, \sigma, t)$ be the number of $d \le D$, d fundamental discriminant with $(d, N) = 1$, for which $L(s, E_d)$ has a zero in the square

$$\sigma \le \operatorname{Re} s < \sigma + (\log DT)^{-1}, \qquad t \le \operatorname{Im} s < t + (\log DT)^{-1} \tag{7}$$

with $1 \le \sigma \le 3/2$ and $|t| \le T$, and let $Y \gg X \gg 1$, $T \ge 2$,

$$M_X(s, E_d) = \sum_{n \le X} b(n)\chi_d(n) n^{-s}$$

and, for $\sigma > 3/2$,

$$L(s, E_d) M_X(s, E_d) = \sum_{n > X} c(n)\chi_d(n) n^{-s}.$$

Following the procedure in Chapter 12 of [12], two cases arise:

(i) there are $\gg R$ values of d as above, with corresponding zeros $\varrho = \beta + i\gamma$ in the square (7), for which

$$\Big| \int_{-c_1 \log DT}^{c_1 \log DT} L(1 + i\gamma + iu, E_d) M_X(1 + i\gamma + iu, E_d) Y^{1-\beta+iu} \times \Gamma(1 - \beta + iu)\, du \Big| \gg 1$$

where $c_1 > 0$ is a suitable constant, and

(ii) there are $U \in [X, Y^2]$ and $\gg R(\log Y)^{-1}$ values of d as above, with corresponding zeros $\varrho = \beta + i\gamma$ in the square (7), for which

$$\Big| \sum_{U < n \le 2U} c(n)\chi_d(n) n^{-\varrho} e^{-n/Y} \Big| \gg (\log Y)^{-1}.$$

In the first case we have

$$\int_{t - c_1 \log DT}^{t + 1 + c_1 \log DT} |L(1 + iu, E_d) M_X(1 + iu, E_d)|\, du \gg Y^{\sigma - 1} (\log DT)^{-1}.$$

Summing over d and applying twice the Cauchy–Schwarz inequality we get

$$RY^{\sigma-1}(\log DT)^{-1} \ll \Big(\int\limits_{t-c_1\log DT}^{t+1+c_1\log DT} \sideset{}{^*}\sum_{d\le D} |L(1+iu,E_d)|^2\,du\Big)^{1/2} \times \Big(\int\limits_{t-c_1\log DT}^{t+1+c_1\log DT} \sideset{}{^*}\sum_{d\le D} |M_X(1+iu,E_d)|^2\,du\Big)^{1/2}. \tag{8}$$

The second factor in (8) can be dealt with by means of Proposition 1. Splitting the interval $[1,X]$ into ranges of the form $V<n\le 2V$ we get

$$\begin{aligned}\int\limits_{t-c_1\log DT}^{t+1+c_1\log DT} &\sideset{}{^*}\sum_{d\le D} |M_X(1+iu,E_d)|^2\,du \\ &\ll \log DT\log X \\ &\quad\times \max_{t-c_1\log DT\le u\le t+1+c_1\log DT}\sum_V \sideset{}{^*}\sum_{d\le D}\Big|\sum_{V<n\le 2V} b(n)\chi_d(n)n^{-1-iu}\Big|^2 \\ &\ll (D+X)(DX)^{\varepsilon}\log T.\end{aligned} \tag{9}$$

From Theorem 3 we get

$$\int\limits_{t-c_1\log DT}^{t+1+c_1\log DT} \sideset{}{^*}\sum_{d\le D} |L(1+iu,E_d)|^2\,du \ll (DT)^{1+\varepsilon} \tag{10}$$

and hence from (8)–(10) we obtain

$$R \ll Y^{1-\sigma}(DT)^{1/2}(D+X)^{1/2}(DTX)^{\varepsilon} \tag{11}$$

in the first case.

Consider now the second case. Assume that $Y\le (DT)^{c_2}$ for some constant c_2 and write $s=\sigma+it$. Since ϱ is in the square (7), by partial summation and the Cauchy–Schwarz inequality we get

$$\begin{aligned}(\log DT)^{-2} &\ll \Big|\sum_{U<n\le 2U} c(n)\chi_d(n)n^{-\varrho}e^{-n/Y}\Big|^2 \\ &\ll \Big|\sum_{U<n\le 2U} c(n)\chi_d(n)n^{-s}e^{-n/Y}\Big|^2 \\ &\quad+ \int\limits_U^{2U}\Big|\sum_{U<n\le V} c(n)\chi_d(n)n^{-s}e^{-n/Y}\Big|^2\frac{dV}{V}\end{aligned}$$

for $\gg R(\log DT)^{-1}$ values of d. Since $|c(n)|\le n^{1/2}\tau_4(n)$, summing over d and using Proposition 1 we obtain

$$R \ll (DTU)^{\varepsilon}(D+U)U^{2(1-\sigma)}e^{-U/Y}$$

for some $U \in [X, Y^2]$, and hence

$$R \ll (DTY)^{\varepsilon}(DX^{2(1-\sigma)} + Y^{3-2\sigma}) \tag{12}$$

in the second case.

A comparison of (11) and (12) together with the choice

$$X = D \quad \text{and} \quad Y = (DT^{1/2})^{1/(2-\sigma)}$$

shows that the conditions on X and Y are satisfied and

$$R \ll (DT^{1/2})^{(3-2\sigma)/(2-\sigma)}(DT)^{\varepsilon}, \tag{13}$$

uniformly for $1 \le \sigma \le 3/2$ and $|t| \le T$. Since the exponent in (13) is a decreasing function of σ and the number of zeros of each function $L(s, E_d)$ in the square (7) is uniformly $\ll \log DT$, Theorem 4 follows at once by summation over squares of the type (7).

5. Proof of Theorem 5. Let us first establish some notation. Given an integrable function $f : \mathbb{R} \to \mathbb{C}$ with compact support, define

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{sx}\,dx$$

to be its Laplace transform. For a function f of real or complex variable and for $\lambda > 1$ we define $f_\lambda(z) = f(z/\lambda)$. Moreover,

$$\log^+ x = \begin{cases} \log x & \text{if } x \ge e, \\ 1 & \text{if } 0 \le x \le e. \end{cases}$$

In our application of Weil's explicit formula, we will need to use a *test function* $\phi_\lambda(s)$ satisfying (14)–(16) below. These requirements prevent us from using the classical test functions (see, e.g., Mestre [10] and Fermigier [3]). We summarize the properties of our test function in the following Proposition 2, which will be proved at the end of the paper.

Proposition 2. *There exists an even, non-negative function $f \in C^\infty(\mathbb{R})$ with $f(0) = 1$ and support contained in $[-B, B]$ for some $B \ge 2$ such that*

$$F(s) \ll \exp(c_3|\sigma| - c_4|s|^{3/4}) \tag{14}$$

and

$$\operatorname{Re} F(s) \ge 0 \quad \textit{if } |\sigma| < 1, \tag{15}$$

where $c_3, c_4 > 0$ are certain constants.

For $\lambda > 1$ we consider the test function

$$\phi_\lambda(s) = F_\lambda(s-1),$$

where F_λ is the Laplace transform of f_λ and f is as in Proposition 2, which satisfies

$$\phi_\lambda(s) = \lambda F(\lambda(s-1)) \quad \text{and} \quad \phi_\lambda(s) = \phi_\lambda(2-s). \tag{16}$$

Weil's explicit formula. We follow the approach by Mestre [10], based on Weil [16]. Let $R(\alpha, T)$ denote the rectangle with vertices $\alpha - iT, \alpha + iT$, $2-\alpha-iT$ and $2-\alpha+iT$. Choose $\alpha \in (3/2, 7/4)$ and, for each fundamental discriminant d with $(d, N) = 1$, a real number $T_d \geq 2$ such that the boundary $\partial R(\alpha, T_d)$ of $R(\alpha, T_d)$ omits the poles of $\frac{\Lambda'}{\Lambda}(s, E_d)$, and such that the estimate

$$\frac{\Lambda'}{\Lambda}(s, E_d) \ll \log^2(dT_d) \tag{17}$$

holds for $s \in \partial R(\alpha, T_d)$. Moreover, given $T = T(D) \geq 2$ to be determined later on, the T_d's can be chosen to satisfy $T - 1 \leq T_d \leq T$. This is easily done by adapting a classical argument in the theory of Dirichlet L-series.

Choosing $\phi_\lambda(s)$ as above, by Cauchy's theorem we have

$$\frac{1}{2\pi i} \int\limits_{\partial R(\alpha, T_d)} \frac{\Lambda'}{\Lambda}(s, E_d)\phi_\lambda(s)\, ds = \sum_{|\gamma| \leq T_d} \phi_\lambda(\varrho) \tag{18}$$

where $\varrho = \beta + i\gamma$ runs over the non-trivial zeros of $L(s, E_d)$, counted with multiplicity. Due to the functional equation

$$\frac{\Lambda'}{\Lambda}(s, E_d) = -\frac{\Lambda'}{\Lambda}(2-s, E_d)$$

and (16), the contribution to (18) of the vertical sides of $R(\alpha, T_d)$ is

$$\frac{1}{\pi i} \int\limits_{\alpha - iT_d}^{\alpha + iT_d} \frac{\Lambda'}{\Lambda}(s, E_d)\phi_\lambda(s)\, ds.$$

From (14), (16) and (17) we see that the contribution to (18) of the horizontal sides is $\ll \lambda \log^2(dT) \exp\big(\frac{1}{2}c_5\lambda - c_6(\lambda T)^{1/2}\big)$ and

$$\begin{aligned}\frac{1}{\pi i} \int\limits_{\alpha - iT_d}^{\alpha + iT_d} \frac{\Lambda'}{\Lambda}(s, E_d)\phi_\lambda(s)\, ds = \frac{1}{\pi i} \int\limits_{\alpha - i\infty}^{\alpha + i\infty} \frac{\Lambda'}{\Lambda}(s, E_d)\phi_\lambda(s)\, ds \\ + O\big(\lambda \log^2(dT) \exp\big(\tfrac{1}{2}c_5\lambda - c_6(\lambda T)^{1/2}\big)\big)\end{aligned}$$

with suitable constants $c_5, c_6 > 0$. Hence (18) becomes

$$\begin{aligned}&\sum_{|\gamma| \leq T_d} \phi_\lambda(\varrho) \\ &= \frac{1}{\pi i} \int\limits_{\alpha - i\infty}^{\alpha + i\infty} \frac{\Lambda'}{\Lambda}(s, E_d)\phi_\lambda(s)\, ds + O(\lambda \log^2(DT) \exp(c_5\lambda - c_6(\lambda T)^{1/2}))\end{aligned} \tag{19}$$

uniformly for $d \leq D$.

We evaluate the integral in (19) following Mestre [10]. We get

$$\frac{1}{\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\Lambda'}{\Lambda}(s,E_d)\phi_\lambda(s)\,ds = 2\log d - 2I(\lambda) - 2S(\lambda,d) + O(1) \tag{20}$$

where

$$I(\lambda) = \int_0^\infty \left(\frac{f_\lambda(x)e^{-x}}{1-e^{-x}} - \frac{e^{-x}}{x}\right)dx$$

and

$$S(\lambda,d) = \sum_{p,m\ge 1} e_d(p^m)\frac{\log p}{p^m} f_\lambda(\log p^m).$$

Observe that the integral $I(\lambda)$ is uniformly bounded for $\lambda > 1$, since

$$I(\lambda) \ll \frac{1}{\lambda}\max_{0\le x\le 2/\lambda}|f'(x)| + 1.$$

Summing over d, from (19) and (20) we get

$$\begin{aligned}\sum_{d\le D}{}^{*}\sum_{|\gamma|\le T_d}\phi_\lambda(\varrho) = 2\sum_{d\le D}{}^{*}\log d - 2\sum_{d\le D}{}^{*} S(\lambda,d) + O(D)\\ + O(D\log^2(TD)\exp(c_5\lambda - c_6(\lambda T)^{1/2})).\end{aligned} \tag{21}$$

Estimation of $\sum^*_{d\le D} S(\lambda,d)$. From the bound for the coefficients $e_d(n)$ we immediately get

$$\sum_{d\le D}{}^{*}\sum_{p,m\ge 2} e_d(p^m)\frac{\log p}{p^m} f_\lambda(\log p^m) \ll \lambda D. \tag{22}$$

In order to deal with the remaining part of $S(\lambda,d)$ we recall that

$$e_d(p) = \begin{cases} e(p)\left(\frac{d}{p}\right) & \text{if } p\nmid dN,\\ O(1) & \text{otherwise,}\end{cases}$$

where $e(p) = e_1(p)$ and $\left(\frac{d}{p}\right)$ is the Legendre symbol. Hence

$$\begin{aligned}\sum_{d\le D}{}^{*}\sum_{p} e_d(p)\frac{\log p}{p} f_\lambda(\log p)\\ = \sum_{p\nmid 2N} e(p)\frac{\log p}{p} f_\lambda(\log p)\sum_{\substack{d\le D\\ (d,p)=1}}{}^{*}\left(\frac{d}{p}\right) + O(D).\end{aligned} \tag{23}$$

We treat the inner sum on right hand side of (23) by means of the Pólya–Vinogradov inequality. Since the summation is not over consecutive integers, we use the arithmetic structure of fundamental discriminants to transform it in a suitable way. Writing $d = ed'$ with

$$e = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ 4 & \text{if } d \equiv 0 \pmod 4 \end{cases}$$

and d' square-free, from the characterization of the fundamental discriminants we see that

$$\sum_{\substack{d \le D \\ (d,p)=1}}^{*} \left(\frac{d}{p}\right) = \sum_{e=1,4} \sum_{\substack{l \mid N \\ (l,p)=1}} \sum_{\substack{a \le \sqrt{D/e} \\ (a,p)=1}} \mu(l)\mu(a) \sum_{\substack{d' \equiv 0 \pmod{[a^2,l]} \\ (d',p)=1}}^{**} \left(\frac{d'}{p}\right), \tag{24}$$

where $[a^2, l]$ is the least common multiple of a^2 and l, and ** means that the summation is over $d' \le D/e$ with $d' \equiv 1 \pmod 4$ if $e = 1$ and $d' \equiv 2$ or $3 \pmod 4$ if $e = 4$. Recalling that $p > 2$ and using the characters to detect the progressions (mod 4), from the Pólya–Vinogradov inequality and (24) we get

$$\sum_{\substack{d \le D \\ (d,p)=1}}^{*} \left(\frac{d}{p}\right) \ll D^{1/2} p^{1/2+\varepsilon}. \tag{25}$$

Hence from (22)–(25) we obtain

$$\sum_{d \le D}^{*} S(\lambda, d) \ll \lambda D + D^{1/2} \exp(2B\lambda). \tag{26}$$

Application of the density estimate. From (15), (16), (21) and (26), taking real parts we deduce that

$$\begin{aligned} \lambda \sum_{d \le D}^{*} \operatorname{rank}(E_d) \ll\ & D \log D + \lambda D + D^{1/2} \exp(2B\lambda) \\ & + D \log^2(DT) \exp(c_5 \lambda - c_6 (\lambda T)^{1/2}) \\ & + \sum_{d \le D}^{*} \sum_{\substack{|\gamma| \le T \\ |\beta - 1| \ge 1/\lambda}} |\phi_\lambda(\varrho)|. \end{aligned}$$

From now on we assume that $2 < \lambda \le c_7 \log D$ with a suitably small constant $c_7 > 0$, and choose, e.g., $T = \log^3 D$. Hence

$$\lambda \sum_{d \le D}^{*} \operatorname{rank}(E_d) \ll D \log D + \sum_{d \le D}^{*} \sum_{\substack{|\gamma| \le T \\ |\beta - 1| \ge 1/\lambda}} |\phi_\lambda(\varrho)|. \tag{27}$$

We split the region $\left[\frac{1}{\lambda}, \frac{1}{2}\right] \times [0, T]$ into rectangles of the type $\left[\frac{m}{\lambda}, \frac{m+1}{\lambda}\right] \times [n, n+1]$ with $1 \le m \le \lambda$ and $0 \le n \le T$, and analogously for the other similar regions where the zeros counted in (27) lie. Hence from (14), (16) and the assumption of Theorem 5 we get, for suitable constants $c_i > 0$ with $i = 8, 9, 10, 11$,

$$(28)\quad \sum_{d\le D}^{*} \sum_{\substack{|\gamma|\le T\\ |\beta-1|\ge 1/\lambda}} |\phi_\lambda(\varrho)|$$
$$\ll \lambda D \sum_{1\le m\le\lambda} \sum_{0\le n\le T} \exp(c_8 m - c_9(\lambda n)^{3/4}) D^{-cm/\lambda}(n+1)^A L(D)$$
$$\ll \lambda D \sum_{1\le m\le\lambda} \exp\Big(c_8 m - c\frac{m}{\lambda}\log D + \log L(D)\Big)$$
$$\times \sum_{0\le n\le T} (n+1)^A \exp(-c_9 n^{3/4})$$
$$\ll \lambda D \sum_{1\le m\le\lambda} \exp(-c_{10} m) \max_{1\le m\le\lambda} \exp\Big(\log L(D) - c_{11}\frac{m}{\lambda}\log D\Big)$$
$$\ll \lambda D \exp\Big(\log L(D) - c_{11}\frac{\log D}{\lambda}\Big).$$

We choose

$$(29)\qquad \lambda = c_{11}\frac{\log D}{\log L(D)}$$

and the result follows at once from (27)–(29).

Proof of Proposition 2. Here we give a sketch of the proof of Proposition 2. Let $\omega \in (1, 2]$ and

$$\varphi(x) = \begin{cases} 1 - |x| & \text{if } x \in [-1, 1],\\ 0 & \text{otherwise.}\end{cases}$$

Note that the Fourier transform $\widehat{\varphi}$ of φ is non-negative on the real axis. For any integer $n \ge 1$ we define

$$g_n(x) = a_n\varphi(xa_n) \quad \text{with } a_n = n(\log^+ n)^\omega,$$

we consider the convolution $\psi_n = g_1 * g_2 * \ldots * g_n$ and let $\psi(x) = \lim_{n\to\infty}\psi_n(x)$. An argument similar to the one used in the proof of Theorem 1.3.5 of Hörmander [7] shows that ψ is the uniform limit of the ψ_n, $\psi \in C^\infty(\mathbb{R})$ and has support contained in $[-B, B]$, with $B = \sum_{n=1}^\infty a_n^{-1}$. It is easy to see that ψ is even, non-negative, $\widehat{\psi}$ is positive on the real axis and $\psi(0) \ne 0$. By an obvious normalization we may assume that $\psi(0) = 1$. Taking the Fourier transform, from the properties of the convolution we see that $\widehat{\psi}(0) = 1$.

For any integer $k > 1$ we have, taking the kth derivative,

$$|\psi^{(k)}(x)| \le 8^k k!(\log k)^{k\omega}. \tag{30}$$

In fact, for $n > k$ we see that

$$\begin{aligned}|\psi_n^{(k)}(x)| &\le \int_{-\infty}^{\infty} |\psi_n^{(k+1)}(t)|\,dt \le \prod_{1\le j\le k+1} \int_{-\infty}^{\infty} |g_j'(t)|\,dt \prod_{k+2\le j\le n} \int_{-\infty}^{\infty} g_j(t)\,dt \\ &\le \prod_{j\le k+1} \int_{-\infty}^{\infty} |g_j'(t)|\,dt.\end{aligned}$$

But $g_j'(x) \le a_j^2$ almost everywhere and the support of g_j' is contained in $[-a_j^{-1}, a_j^{-1}]$, hence

$$|\psi_n^{(k)}(x)| \le \prod_{j\le k+1} a_j^2 2a_j^{-1} \le 2^k(k+1)!(\log(k+1))^{k\omega} \le 8^k k!(\log k)^{k\omega},$$

and (30) follows by uniform convergence.

From (30) we deduce that the Laplace transform Ψ of ψ satisfies

$$\Psi(s) \ll \exp\left(c_{12}|\mathrm{Re}\, s| - c_{13}\left|\frac{s}{8}\right|\left(\log^+\left|\frac{s}{8}\right|\right)^{-\omega}\right), \tag{31}$$

where $c_{12} = c_{12}(\omega)$. In fact, for any integer $k > 1$ we get

$$|s^k\Psi(s)| = \left|\int_{-\infty}^{\infty} \psi^{(k)}(x)e^{sx}\,dx\right| \le 2B8^k k!(\log k)^{k\omega}\exp(2B|\mathrm{Re}\, s|), \tag{32}$$

and (31) follows by a simple computation based on a suitable choice of k which minimizes (32), i.e., $k = [C|s/8|(\log^+|s/8|)^{-\omega}]$ with a suitable constant C.

Let now

$$h(s) = \mathrm{ch}(s)^{-1} = \left(\frac{e^s + e^{-s}}{2}\right)^{-1}$$

and, finally, write

$$f(x) = \psi(x)h(x).$$

Hence f is even, non-negative, $f(0) = 1$, $f \in C^\infty(\mathbb{R})$ and its support is contained in $[-B, B]$. It remains to prove (14) and (15). By Cauchy's integral formula we see that

$$|h^{(k)}(x)| \le c_{14}^k k!$$

with some constant $c_{14} > 0$, hence (30) holds for h with $\omega = 0$ and c_{14} in place of 8. Hence (14) follows, in an even stronger form, from Leibniz's rule and the same argument leading to (31).

Let H denote the Laplace transform of h. It is well known that $H(s) = \pi h\left(\frac{\pi s}{2i}\right)$. Moreover, by Proposition 3 on p. 59 of Szmydt–Ziemian [15] we find that F can be expressed as convolution of Ψ and H,

$$F(\sigma+it) = 2\pi \int_{-\infty}^{\infty} \widehat{\psi}(t-x)H(\sigma+ix)\,dx \tag{33}$$
$$= 2\pi^2 \int_{-\infty}^{\infty} \widehat{\psi}(t-x)h\left(\frac{\pi}{2i}(\sigma+ix)\right)dx$$

where $|\sigma|<1$. Writing $\mathrm{sh}(s) = (e^s - e^{-s})/2$, we have

$$\mathrm{Re}\, h(s) = \mathrm{Re}(\mathrm{ch}(s))/|\mathrm{ch}(s)|^2$$

and

$$\mathrm{ch}(s) = \cos(\mathrm{Im}\, s)\mathrm{ch}(\mathrm{Re}\, s) + i\sin(\mathrm{Im}\, s)\mathrm{sh}(\mathrm{Re}\, s).$$

Hence, since $\widehat{\psi}$ is positive on the real axis, (15) follows from (33), and Proposition 2 is proved.

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References

[1] A. Brumer, *The average rank of elliptic curves I*, Invent. Math. 109 (1992), 445–472.

[2] W. Duke, *The critical order of vanishing of automorphic L-functions with large level*, ibid. 119 (1995), 165–174.

[3] S. Fermigier, *Annulation de la cohomologie cuspidale de sous-groupes de congruence de $GL_n(\mathbb{Z})$*, Math. Ann. 306 (1996), 247–256.

[4] E. Fouvry et J. Pomykała, *Rang des courbes elliptiques et sommes d'exponentielles*, Monatsh. Math. 116 (1993), 111–125.

[5] D. Goldfeld, *Conjectures on elliptic curves over quadratic fields*, in: Proc. Southern Illinois Number Theory Conference, Lecture Notes in Math. 751, Springer, 1979, 108–118.

[6] D. R. Heath-Brown, *A mean value estimate for real character sums*, Acta Arith. 72 (1995), 235–275.

[7] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer, 1983.

[8] H. Iwaniec, *On the order of vanishing of modular L-functions at the critical point*, Sém. Théor. Nombres Bordeaux 2 (1990), 365–376.

[9] M. Jutila, *On the mean value of $L(\frac{1}{2},\chi)$ for real characters*, Analysis 1 (1981), 149–161.

[10] J.-F. Mestre, *Formules explicites et minorations de conducteurs de variétés algébriques*, Compositio Math. 58 (1986), 209–232.

[11] P. Michel, *Rang moyen de familles de courbes elliptiques et lois de Sato–Tate*, Monatsh. Math. 120 (1995), 127–136.
[12] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Lecture Notes in Math. 227, Springer, 1971.
[13] M. R. Murty, *The analytic rank of* $J_0(N)(\mathbb{Q})$, in: Number Theory, K. Dilcher (ed.), Amer. Math. Soc., 1995, 263–279.
[14] M. R. Murty and V. K. Murty, *Mean values of derivatives of modular L-series*, Ann. of Math. 133 (1991), 447–475.
[15] Z. Szmydt and B. Ziemian, *The Mellin Transformation and Fuchsian Type Partial Differential Equations*, Kluwer, 1992.
[16] A. Weil, *Sur les "formules explicites" de la théorie des nombres premiers*, Comm. Lund (vol. dédié à M. Riesz) (1952), 252–265.

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