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On the lattice point problem for ellipsoids

by

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1. Introduction and results. Let $\mathbb{R}^d$, $1 \le d < \infty$, denote the real d-dimensional Euclidean space with scalar product $\langle \cdot, \cdot \rangle$ and norm

$$|x|^2 = \langle x, x\rangle = x_1^2 + \ldots + x_d^2 \quad \text{for } x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Denote by $|x|_\infty = \max\{|x_j| : 1 \le j \le d\}$ the maximum-norm. Let $\mathbb{Z}^d$ be the standard lattice of points with integer coordinates in $\mathbb{R}^d$.

For a (measurable) set $B \subset \mathbb{R}^d$ let $\operatorname{vol} B$ denote the Lebesgue measure of B, and let $\operatorname{vol}_{\mathbb{Z}} B$ denote the lattice volume of B, that is, the number of points in B with integer coordinates.

Consider a quadratic form

$$Q[x] = \langle Qx, x\rangle \quad \text{for } x \in \mathbb{R}^d,$$

where $Q : \mathbb{R}^d \to \mathbb{R}^d$ denotes a symmetric positive linear bounded operator, that is, $Q[x] > 0$ for $x \ne 0$. Define the ellipsoid

$$E_s = \{x \in \mathbb{R}^d : Q[x] \le s\} \quad \text{for } s \ge 0.$$

Let

$$0 < q_1^2 \le \ldots \le q_d^2 = q^2$$

denote the eigenvalues of the operator Q.

We shall prove the following

Theorem 1.1. *For $d \ge 9$,*

$$\Delta(s, Q) := \sup_{a \in \mathbb{R}^d} \left| \frac{\operatorname{vol}_{\mathbb{Z}}(E_s + a) - \operatorname{vol} E_s}{\operatorname{vol} E_s} \right| \le c q_1^2 (q/q_1)^{2d+4} s^{-1} \quad \textit{for } s \ge q_1^2,$$

where the constant c can depend on the dimension d only.

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Remark. For the dimension $d = 8$ we can prove that

$$\Delta(s,Q) \le c q_8^{20} q_1^{-18} s^{-1} \ln^2(s+1) \quad \text{for } s \ge q_1^2,$$

with an absolute constant c.

For *general* ellipsoids Landau (1915) obtained the estimate $\mathcal{O}(s^{-1+1/(1+d)})$. This result has been extended by Hlawka (1950) to convex bodies with smooth boundary and strictly positive Gaussian curvature. Hlawka's estimate has been recently improved by Krätzel and Nowak (1991, 1992) to $\mathcal{O}(s^{-1+\lambda})$, where $\lambda = 5/(6d+2)$, for $d \ge 8$, and $\lambda = 12/(14d+8)$, for $3 \le d \le 7$. An abstract of results of the present paper appeared as Bentkus and Götze (1995a), based on the preprint by Bentkus and Götze (1994b).

Since the lower bound $\Delta(s,Q) = \Omega(s^{-1})$ holds for spheres ($Q = \text{Identity}$) (see, e.g. Fricker (1982)), Theorem 1.1 solves the problem of *uniform* error bounds for ellipsoids with arbitrary center provided that the dimension d is sufficiently large, i.e., $d \ge 9$.

The bound of Theorem 1.1 shows that the number of lattice points in an ellipsoid depends asymptotically only on the size of ellipsoid, i.e., only on radii of the largest inscribed and the smallest circumscribed spheres, and does not depend on assumptions like "rationality" or "orientation" of the ellipsoid, that is, on the conjugation class of Q under the action of $\mathrm{SL}(d,\mathbb{Z})$.

For special ellipsoids a number of particular results are available. For example, the error bound $\mathcal{O}(s^{-1})$ holds for $d \ge 5$ for a fixed rational form Q (see Walfisz (1924, 1927), $d \ge 9$, and Landau (1924), $d \ge 5$). Jarník (1928) proved the same bound for diagonal forms Q with arbitrary (nonzero) real coefficients. Related results are due to Novák (1968), Diviš and Novák (1974). For a discussion see the monographs by Walfisz (1957), Landau and Walfisz (1962), Fricker (1982) and Krätzel (1988).

Our results were obtained by extending the methods for proving optimal rates of convergence in the Central Limit Theorem (CLT) for ellipsoids (Bentkus and Götze (1994a)). Bounds for rates of convergence in the multivariate CLT for convex bodies seem to correspond to bounds in the lattice point problem for these bodies interpreting s as the number, say N, of random vector summands in the CLT. This fact was mentioned by Esseen (1945), who proved the rate $\mathcal{O}(s^{-1+1/(1+d)})$ for balls around the origin and random vectors with identity covariance, a result similar to the result of Landau (1915). For sums taking values in a *lattice* and *special* ellipsoids the relation of these error bounds for the lattice point problem and the CLT was made explicit in Yarnold (1972).

The results of Esseen were extended to convex bodies by Matthes (1970), yielding a result similar to that of Hlawka (1950).

The bound $\mathcal{O}(N^{-1})$, for $d \geq 5$, of Bentkus and Götze (1994a) for ellipsoids with diagonal Q and random vectors with independent components (and with arbitrary distribution) is comparable to the results of Jarník (1928). The bound $\mathcal{O}(N^{-1})$, for $d \geq 9$, for arbitrary ellipsoids and random vectors—an analogue of the results of the present paper—is obtained in Bentkus and Götze (1995b). The proofs of these probabilistic results are more involved since we have to deal with a general class of distributions instead of uniform bounded ones in number theory.

The basic steps of the proof consist of:

(1) rewriting in Section 2 the lattice point approximation error as a difference of measures of the ellipsoid, which are defined as convolutions of uniform measures on cubes in $\mathbb{Z}^d$ resp. $\mathbb{R}^d$;

(2) using regularization (see Lemma 8.1) and Fourier transforms to bound the error by integrals over Fourier transforms of the distributions of $Q[x]$ under these measures;

(3) using double large sieve bounds to estimate the Fourier transforms;

(4) estimating the size and separating the location of maxima of the Fourier transforms, for $d \geq 9$, by means of inequality (5.3) in Section 5. (See as well (1.2) below.)

Once the problem has been reformulated in terms of measures with finite support in step (1), it is sufficient to assume that the quadratic form Q is nondegenerate (see the Remark in Section 2).

The inequality (5.3) in step (4) represents the essential tool of our proof. For trigonometric sums defined as

$$S(t) = \sum_{x \in \mathbb{Z}^d} p_x \mathrm{e}\{tQ[x]\} \tag{1.1}$$

with

$$p_x = (2A+1)^{-2d} \sum_{|m|_\infty \leq A} \sum_{|n|_\infty \leq A} \mathbf{I}\{m+n=x\},$$

where the sums are taken over $m, n \in \mathbb{Z}^d$, and $\mathrm{e}\{x\} := \exp\{ix\}$, it yields the bound

$$|S(t)S(t+\varepsilon)| \leq c(q/q_1)^d \mathcal{M}(\varepsilon) \quad \text{for all } t \in \mathbb{R} \text{ and } \varepsilon \geq 0, \tag{1.2}$$

where

$$\mathcal{M}(t) := \begin{cases} (|t|A^2)^{-d/2} & \text{for } |t| \leq A^{-1}, \\ |t|^{d/2} & \text{for } |t| \geq A^{-1}. \end{cases} \tag{1.3}$$

Taking $t = 0$ the inequality (1.2) yields a "double large sieve" estimate for distributions on the lattice (see Bombieri and Iwaniec (1986)). In the present paper we derive (1.2) from the double large sieve bound—an alternative proof to the original proof in Bentkus and Götze (1994b), which depended on symmetrization arguments like Weyl's inequality and its generalizations (see Lemma 7.1), which had been quite useful in the investigation of the convergence rates and Edgeworth expansions in the CLT in Hilbert and Banach spaces (see Bentkus and Götze (1994a, 1995b)). Using this symmetrization method a refined version of (1.2) is proved in (5.13) for weights p_x which are convolutions of more than two uniform distributions.

We shall use the following notation. By c with or without indices we shall denote generic constants which may depend on the dimension d only. We shall write $B \ll D$ instead of $B \le cD$. By $[B]$ we denote the integer part of a real number B. We shall write $r = \sqrt{s}$ and $A = [r]$. If $r \ge 1$ then the natural number A is at least 1 and $A \le r \le 2A$.

2. A reduction to the Fourier transforms. Assuming that

$$|a|_\infty \le 1/2 \quad \text{and} \quad q_1 = 1, \tag{2.1}$$

we shall prove the following bound for the error $\varDelta(s,Q)$ in Theorem 1.1:

$$\varDelta(s,Q) \ll q^{2d+4} s^{-1} \quad \text{for } s \ge 1. \tag{2.2}$$

This result implies Theorem 1.1. Indeed, the assumption $|a|_\infty \le 1/2$ does not restrict the generality since

$$\operatorname{vol}_{\mathbb{Z}}(E_s + a) = \operatorname{vol}_{\mathbb{Z}}(E_s + a - m) \quad \text{for any } m \in \mathbb{Z}^d,$$

and we can replace a in (2.2) by $a - m$, with some $m \in \mathbb{Z}^d$ such that $|a - m|_\infty \le 1/2$. The condition $q_1 = 1$ does not restrict the generality either since we can derive Theorem 1.1 from (2.2) replacing Q in (2.2) by Q/q_1^2 and s by s/q_1^2.

Let us recall some definition and properties related to (Borel) measures. Let $\mathcal{B}^d$ denote the class of Borel subsets of $\mathbb{R}^d$. In this paper we shall consider only nonnegative normalized measures, that is, σ-additive set functions $\mu : \mathcal{B}^d \to \mathbb{R}$ such that $\mu(\mathbb{R}^d) = 1$ and $\mu(C) \ge 0$ for any $C \in \mathcal{B}^d$. We shall write $\int f(x)\,\mu(dx)$ for the (Lebesgue) integral of a measurable function $f : \mathbb{R}^d \to \mathbb{C}$ with respect to the measure μ, and denote as usual by $\mu * \nu(C) = \int \mu(C - x)\,\nu(dx)$, for $C \in \mathcal{B}^d$, the convolution of the measures μ and ν. Equivalently, $\mu * \nu$ is defined as the measure such that $\int f(x)\,\mu * \nu(dx) = \int f(x + y)\,\mu(dx)\,\nu(dy)$, for any integrable function f.

For $r \ge 0$ consider the cube $B(r) = \{x \in \mathbb{R}^d : |x|_\infty \le r\}$. The *uniform lattice measure* μ_r concentrated on the lattice points in $B(r)$ is defined by

$$\mu_r(C) = \frac{\operatorname{vol}_{\mathbb{Z}} C \cap B(r)}{\operatorname{vol}_{\mathbb{Z}} B(r)} \quad \text{for } C \in \mathcal{B}^d.$$

We define the *uniform measure* ν_r in $B(r) \subset \mathbb{R}^d$ by

$$\nu_r(C) = \frac{\operatorname{vol} C \cap B(r)}{\operatorname{vol} B(r)} \quad \text{for } C \in \mathcal{B}^d.$$

Introduce the measures

$$\mu = \mu_{6r} = \mu_{[6r]+1/2}, \qquad \nu = \nu_{[6r]+1/2}$$

and

$$\chi = \mu_r = \mu_{[r]+1/2}, \qquad \omega = \chi * \chi * \chi * \chi.$$

Notice that for $C \subset E_r$, $\mu(C)$ and $\chi(C)$ are proportional to the number of lattice points in C. Similarly, $\nu(C)$ is proportional to the volume of $C \subset E_r$.

The use of measures will simplify the notation. For example, we can rewrite the trigonometric sum (1.1) as

$$S(t) = \int \mathrm{e}\{tQ[x]\}\, \chi * \chi(dx).$$

The following Lemma 2.1 reduces the proof of (2.2) to an estimation of the Fourier transforms.

Lemma 2.1. *Write*

$$f(t) = \int \mathrm{e}\{tQ[x-a]\}\, \mu * \omega(dx), \qquad g(t) = \int \mathrm{e}\{tQ[x-a]\}\, \nu * \omega(dx).$$

We have

$$\Delta(s,Q) \ll q^d\,(I_1 + I_2 + I_3 + I_4),$$

where

$$I_1 = \int\limits_{|t| \le s^{-1+2/d}} (|f(t)| + |g(t)|)\, dt, \qquad I_2 = \int\limits_{|t| \le s^{-1+2/d}} |f(t) - g(t)|\, \frac{dt}{|t|},$$

and

$$I_3 = \int\limits_{s^{-1+2/d} \le |t| \le s^{-2/d}} (|f(t)| + |g(t)|)\, \frac{dt}{|t|}, \qquad I_4 = \int\limits_{s^{-2/d} \le |t| \le 1} (|f(t)| + |g(t)|)\, \frac{dt}{|t|}.$$

Using Lemma 2.1 we reduce the proof of (2.2) to the proof that $I_j \ll q^{d+4} s^{-1}$, for $1 \le j \le 4$. That is done in Sections 3 and 4.

Proof of Lemma 2.1. Let us start with the proof of

$$\Delta(s,Q) \ll q^d\, |\mu * \omega(E_s + a) - \nu * \omega(E_s + a)|. \tag{2.3}$$

Recall that

$$1 = q_1^2 \le \ldots \le q_d^2 = q^2$$

denote the eigenvalues of the operator Q, or in other words,

$$1 = 1/q_1 \ge \ldots \ge 1/q_d = 1/q > 0$$

denote the lengths of half-axes of the ellipsoid E_1. Due to the assumption $q_1 = 1$ the longest half-axis of E_1 has length 1. Therefore $E_1 \subset B(1)$ and $E_s \subset B(r)$. Consequently, $E_s + a \subset B(r + 1/2)$ since $|a|_\infty \le 1/2$.

Observe that, for any set $C \subset B(r + 1/2)$,

$$\mu(C) = \mu * \omega(C), \qquad \nu(C) = \nu * \omega(C) \tag{2.4}$$

since the measure $\mu*\omega$ (resp. $\nu*\omega$) is still a uniform lattice (resp. uniform) measure on $B(r+1/2)$. For example, let us verify that $\mu(C)=\mu*\omega(C)$. The measure ω is concentrated in the cube $B(4r)$, that is, $\omega(B(4r))=1$. Therefore

$$\mu*\omega(C)=\int\mu(C-x)\,\omega(dx)=\int\limits_{B(4r)}\mu(C-x)\,\omega(dx).$$

For $y\in C$ and $x\in B(4r)$, the triangle inequality implies

$$|y-x|_\infty\le r+1/2+4r\le[6r]+1/2,$$

which means that both C and the shifted set $C-x$ are subsets of $B([6r]+1/2)$. The shift x assumes integer values only (with ω-measure 1). Therefore, due to the invariance of μ under shifts by *integer vectors* inside the cube $B([6r]+1/2)$,

$$\int\limits_{B(4r)}\mu(C-x)\,\omega(dx)=\int\limits_{B(4r)}\mu(C)\,\omega(dx)=\mu(C),$$

which implies (2.4).

Clearly

$$V:=\operatorname{vol}_{\mathbb Z}B([6r]+1/2)=\operatorname{vol}B([6r]+1/2).$$

For $C\subset B([6r]+1/2)$, we have

$$\mu(C)=V^{-1}\operatorname{vol}_{\mathbb Z}C,\qquad \nu(C)=V^{-1}\operatorname{vol}C.$$

Therefore, in the case of $C=E_s+a$, we obtain

$$|\mu(C)-\nu(C)|=V^{-1}|\operatorname{vol}_{\mathbb Z}C-\operatorname{vol}C|=\varDelta(s,Q)\,V^{-1}\operatorname{vol}E_s. \tag{2.5}$$

Notice that $1\ll q^dV^{-1}\operatorname{vol}E_s$ since the ellipsoid E_1 contains the cube $B(1/q)$ as a subset. Thus (2.4) and (2.5) imply (2.3).

Consider the functions

$$F(z)=\mu*\omega(\{x\in\mathbb R^d:Q[x-a]\le z\}),$$
$$G(z)=\nu*\omega(\{x\in\mathbb R^d:Q[x-a]\le z\})$$

for $z\in\mathbb R$. The functions F,G are distribution functions since they are nondecreasing, $F(-\infty)=G(-\infty)=0$ and $F(\infty)=G(\infty)=1$. We can write

$$\begin{aligned}F(s)-G(s)&=\mu*\omega(E_s+a)-\nu*\omega(E_s+a)\\&=\mu*\omega(\{x\in\mathbb R^d:Q[x-a]\le s\})\\&\quad-\nu*\omega(\{x\in\mathbb R^d:Q[x-a]\le s\}).\end{aligned}\tag{2.6}$$

Notice that $f(t)$ (resp. $g(t)$) is equal to the Fourier–Stieltjes transform $\widehat F(t)=\int \mathrm e\{tz\}\,dF(z)$ of the distribution function F (resp. of G), by a change of variable. Applying the smoothing Lemma 8.1, we have

$$(2.7)\qquad |F(s)-G(s)|\le \int_{-1}^{1}|f(t)-g(t)|\,\frac{dt}{|t|}+\int_{-1}^{1}|f(t)|\,dt+\int_{-1}^{1}|g(t)|\,dt.$$

Splitting the integrals in (2.7) and estimating $1\le 1/|t|$ for $|t|\le 1$, we obtain

$$(2.8)\qquad |F(s)-G(s)|\ll I_1+I_2+I_3+I_4.$$

Now the relations (2.3), (2.6) and (2.8) together imply the result of the lemma. ∎

Remark. Once the problem has been reformulated in (2.3) and (2.6) as a problem of the estimation of distribution functions, we may consider the general case of conic sections instead of ellipsoids since from now on only the assumption that the operator Q is invertible will be used.

3. Bounds for the integrals I_1, I_3 and I_4 of Lemma 2.1. Throughout we shall write

$$(3.1)\qquad \varphi(t)=\varphi_{b,L}(t):=\Big|\int \mathrm{e}\{tQ[x-b]+t\langle L,x\rangle\}\,\chi*\chi(dx)\Big|,\qquad b,L\in\mathbb{R}^d,$$

and

$$\psi(t)=\sup_{b,L}\varphi_{b,L}(t).$$

Notice that φ and ψ are even continuous functions such that $\varphi(0)=\psi(0)=1$ and $0\le\varphi,\psi\le 1$.

LEMMA 3.1. *We have*

$$(3.2)\qquad |f(t)|\le\psi(t),\qquad |g(t)|\le\psi(t),$$

and

$$(3.3)\qquad \int_C|f(t)|\,\frac{dt}{|t|}\le\sup_{b,L}\int_C\varphi(t)\,\frac{dt}{|t|},\qquad \int_C|g(t)|\frac{dt}{|t|}\le\sup_{b,L}\int_C\varphi(t)\frac{dt}{|t|},$$

for $t\in\mathbb{R}$ and $C\in\mathcal{B}$.

Proof. For example, let us verify the first inequality in (3.2). Using Fubini's Theorem, and the definition of ψ, we have

$$\begin{aligned}|f(t)|&=\Big|\int \mathrm{e}\{tQ[x-a]\}\,\mu*\omega(dx)\Big|\\&\le\int\Big|\int \mathrm{e}\{tQ[x+y-a]\}\,\chi*\chi(dx)\Big|\,\mu*\chi*\chi(dy)\\&\le\int\sup_{z\in\mathbb{R}^d}\Big|\int \mathrm{e}\{tQ[x+z]\}\,\chi*\chi(dx)\Big|\,\mu*\chi*\chi(dy)\\&=\sup_{z\in\mathbb{R}^d}\Big|\int \mathrm{e}\{tQ[x+z]\}\,\chi*\chi(dx)\Big|\le\psi(t).\end{aligned}$$

The proof of (3.3) is similar to the proof of (3.2). For example, let us prove the second inequality in (3.3). We have

$$\begin{aligned}\int_C |g(t)|\frac{dt}{|t|} &= \int_C \Big|\int \mathrm{e}\{tQ[x-a]\}\,\nu*\omega(dx)\Big|\frac{dt}{|t|}\\ &\le \int\int_C \Big|\int \mathrm{e}\{tQ[x+y-a]\}\,\chi*\chi(dx)\Big|\,\frac{dt}{|t|}\,\mu*\chi*\chi(dy)\\ &\le \sup_{z\in\mathbb{R}^d}\int_C \Big|\int \mathrm{e}\{tQ[x+z]\}\,\chi*\chi(dx)\Big|\,\frac{dt}{|t|}\\ &\le \sup_{b,L\in\mathbb{R}^d}\int_C \varphi(t)\,\frac{dt}{|t|}. \ \blacksquare\end{aligned}$$

Recall that $A=[r]$ and $s=r^2$. Since we assume that $r\ge 1$, the inequalities $A^2\ll r^2\ll s\ll A^2$ hold, and the function

$$\mathcal{N}(t):=\begin{cases}(|t|s)^{-d/2} & \text{for } |t|\le s^{-1/2},\\ |t|^{d/2} & \text{for } |t|\ge s^{-1/2}\end{cases}$$

is equivalent to the function $\mathcal{M}$ defined by (1.3), that is,

$$\mathcal{N}(t)\ll\mathcal{M}(t)\ll\mathcal{N}(t) \quad\text{and}\quad \mathcal{M}(t)\ll(|t|s)^{-d/2}+|t|^{d/2}\ll\mathcal{M}(t).$$

Estimation of I_1 *and* I_3. Let us prove that $I_1\ll q^d/s$. Using (3.2) of Lemma 3.1, we have

$$I_1\ll\int_{|t|\le s^{-1+2/d}}\psi(t)\,dt.$$

By (5.4) of Theorem 5.1, $\psi(t)\ll q^d\mathcal{M}(t)\ll q^d\mathcal{N}(t)$. Therefore, using $\psi(t)\le 1$, as well as $s^{-1+2/d}\le s^{-1/2}$ for $d\ge 4$, we obtain

$$\begin{aligned}I_1&\ll q^d\int_0^{s^{-1+2/d}}\min\{1;\mathcal{N}(t)\}\,dt\ll q^d\int_0^{1/s}dt+q^d\int_{1/s}^{s^{-1/2}}\frac{dt}{(ts)^{d/2}}\\ &=\frac{q^d}{s}+\frac{q^d}{s}\int_1^\infty\frac{dt}{t^{d/2}}\ll\frac{q^d}{s}.\end{aligned}$$

Let us prove that $I_3\ll q^d/s$. Using (3.2) of Lemma 3.1 and the inequality $\psi(t)\ll q^d\mathcal{M}(t)\ll q^d\mathcal{N}(t)$, we get

$$\begin{aligned}I_3&\ll\int_{s^{-1+2/d}\le|t|\le s^{-2/d}}\psi(t)\frac{dt}{|t|}\ll q^d\int_{s^{-1+2/d}}^{s^{-2/d}}\mathcal{N}(t)\,\frac{dt}{t}\\ &=q^d\int_{s^{-1+2/d}}^{s^{-1/2}}\frac{1}{(ts)^{d/2}}\,\frac{dt}{t}+q^d\int_{s^{-1/2}}^{s^{-2/d}}t^{d/2}\frac{dt}{t}\end{aligned}$$

$$\ll q^d \int_{s^{2/d}}^{\infty} \frac{1}{t^{d/2}} \frac{dt}{t} + q^d \int_0^{s^{-2/d}} t^{d/2} \frac{dt}{t} \ll \frac{q^d}{s}$$

for $s \geq 4$.

Estimation of I_4. Using (3.3) of Lemma 3.1, we have

$$I_4 \ll \sup_{b,L\in\mathbb{R}^d} \int_{s^{-2/d}}^{1} \varphi(t) \frac{dt}{t}.$$

We show in Section 5 that $\varphi(t)\varphi(t+\varepsilon) \ll q^d \mathcal{M}(\varepsilon)$ for all $t, \varepsilon \in \mathbb{R}$, with $\mathcal{M}$ defined in (1.3). In Section 6 we show that this inequality implies that

$$\int_{s^{-2/d}}^{1} \varphi(t)\, \frac{dt}{t} \ll \frac{q^d}{s}$$

for $d \geq 9$, and the desired estimate of I_4 follows.

4. An estimate of the integral I_2 of Lemma 2.1. In this section we shall show that

$$I_2 = \int_{|t|\leq s^{-1+2/d}} |f(t) - g(t)|\, \frac{dt}{|t|} \ll \frac{q^{d+4}}{s} \quad \text{for } d \geq 5. \tag{4.1}$$

We shall use Taylor expansions in order to reduce the problem to the estimation of integrals like those in Section 3.

We shall show that

$$|f(t) - g(t)| \ll q^4\,(|t| + st^2)\,\psi(t). \tag{4.2}$$

Using the estimate $\psi(t) \ll q^d \min\{1;\, \mathcal{M}(t)\}$ of Theorem 5.1 and integrating in t, we easily derive (4.1). Thus it remains to prove (4.2).

Recall that

$$f(t) = \int \mathrm{e}\{tQ[x-a]\}\, \mu * \omega(dx), \qquad g(t) = \int \mathrm{e}\{tQ[x-a]\}\, \nu * \omega(dx).$$

Let $\tau(dx) = \mathbf{I}\{|x|_\infty \leq 1/2\}\, dx$ denote the uniform measure concentrated on the cube $|x|_\infty \leq 1/2$. Recall that μ is the uniform lattice measure in the cube $|x|_\infty \leq [6r] + 1$, and that ν is the uniform measure in the same cube. Therefore, for any function u we can write the identity

$$\int u(x)\, \nu(dx) = \int u(x+y)\, \mu(dx)\, \tau(dx).$$

Thus we have

$$g(t) = \iint \mathrm{e}\{tQ[x+y-a]\}\, \mu * \omega(dx)\, \tau(dy).$$

We shall expand the function $\mathrm{e}\{tQ[x+y-a]\}$ into a Taylor series in powers of y. Notice that

$$\mathrm{e}\{tQ[x+y-a]\} = \mathrm{e}\{tQ[x-a]\}\,\mathrm{e}\{2t\langle Q(x-a),y\rangle\}\,\mathrm{e}\{tQ[y]\}$$

and introduce the function

$$h(t) = \iint \mathrm{e}\{tQ[x-a]\}\,\mathrm{e}\{2t\langle Q(x-a),y\rangle\}\,\mu*\omega(dx)\,\tau(dy).$$

Then

$$|f(t)-g(t)| \le |f(t)-h(t)| + |h(t)-g(t)|,$$

and the proof of (4.2) reduces to the verification of

$$|f(t)-h(t)| \ll q^4 s t^2 \psi(t) \tag{4.3}$$

and

$$|h(t)-g(t)| \ll q^2 |t| \psi(t). \tag{4.4}$$

Let us prove (4.4). Expanding $\mathrm{e}\{z\} = 1 + \int_0^1 iz\mathrm{e}\{vz\}\,dv$ with $z = tQ[y]$, we obtain

$$|h(t)-g(t)| \le \int_0^1\!\!\int |tQ[y]|\,J\,\tau(dy)\,dv,$$

where

$$J = \Big|\int \mathrm{e}\{tQ[x-a] + 2t\langle Q(x-a),y\rangle\}\,\mu*\omega(dx)\Big|.$$

Using the definition of ψ, we have

$$J \le \int\Big|\iint \mathrm{e}\{tQ[x+w-a]+2t\langle Q(x+w-a),y\rangle\}\,\chi*\chi(dx)\Big|\,\mu*\chi*\chi(dw) \le \psi(t).$$

Consequently, using $|Q[y]| \le q^2|y|^2 \ll q^2|y|_\infty^2$, we derive

$$|h(t)-g(t)| \le \psi(t)\int_0^1\!\!\int |tQ[y]|\,\tau(dy)\,dv \le q^2|t|\psi(t)\int |y|_\infty^2\,\tau(dy) \ll q^2|t|\psi(t),$$

thus proving (4.4).

Let us prove (4.3). Expanding

$$\mathrm{e}\{z\} = 1 + iz + \int_0^1 (1-v)(iz)^2\mathrm{e}\{vz\}\,dv \quad \text{with } z = 2t\langle Q(x-a),y\rangle, \tag{4.5}$$

we obtain

$$|f(t)-h(t)| \ll t^2 \int_0^1\!\!\int J_0\,\tau(dy)\,dv, \tag{4.6}$$

where

$$J_0 = \Big|\int \langle Q(x-a),y\rangle^2 \mathrm{e}\{tQ[x-a] + 2vt\langle Q(x-a),y\rangle\}\,\mu*\omega(dx)\Big|.$$

Notice that the term corresponding to iz in (4.5) is equal to zero since the measure τ is symmetric and therefore $\int \langle L, y\rangle\, \tau(dy) = 0$ for any $L \in \mathbb{R}^d$. Set $x_0 = -a$ and write

$$x = x_1 + x_2 + x_3 + x_4 + x_5,$$
$$\mu * \omega(dx) = \chi(dx_1)\, \chi(dx_2)\, \chi(dx_3)\, \chi(dx_4)\, \mu(dx_5).$$

Then $Q(x-a) = \sum_{j=0}^{5} Qx_j$ and we have

$$\begin{aligned} J_0 &= \Big|\int \Big\langle \sum_{j=0}^{5} Qx_j, y\Big\rangle^2 \operatorname{e}\{tQ[x-a] + 2vt\langle Q(x-a), y\rangle\}\, \mu * \omega(dx)\Big| \\ &\ll \sum_{j=0}^{5} \sum_{k=0}^{5} J_{jk} \end{aligned}$$

with

$$J_{jk} = \Big|\int \langle Qx_j, y\rangle\, \langle Qx_k, y\rangle \operatorname{e}\{tQ[x-a] + 2vt\langle Q(x-a), y\rangle\}\, \mu * \omega(dx)\Big|.$$

Given the variables x_j and x_k, we may choose out of x_1, x_2, x_3, x_4 at least two further variables, say x_l and x_m with $l \neq m$, such that both l and m are different from j and k. Using Fubini's Theorem, we have

$$J_{jk} \le \iiint |\langle Qx_j, y\rangle\, \langle Qx_k, y\rangle|\, J_1\, \mu(dx_5) \prod_{1\le p\le 4, p\neq l, p\neq m} \chi(dx_p),$$

where

$$J_1 = \Big|\iint \operatorname{e}\{tQ[x-a] + 2vt\langle Q(x-a), y\rangle\}\, \chi(dx_l)\, \chi(dx_m)\Big|.$$

Splitting $x - a = x_l + x_m + w$ with some w independent of x_l and x_m we see that $J_1 \le \psi(t)$. Therefore

$$\begin{aligned} J_{jk} &\ll \psi(t) \iiint |\langle Qx_j, y\rangle\, \langle Qx_k, y\rangle|\, \mu(dx_5) \prod_{1\le p\le 4, p\neq l, p\neq m} \chi(dx_p) \\ &= \psi(t) \int \dots \int |\langle Qx_j, y\rangle\, \langle Qx_k, y\rangle|\, \mu(dx_5) \prod_{1\le p\le 4} \chi(dx_p). \end{aligned}$$

Using $|\langle Qx_j, y\rangle| \ll q^2\, |x_j|\, |y|$ and summing the bounds for J_{jk}, we obtain

$$\begin{aligned} J_0 &\ll q^4 |y|^2 \psi(t) \int \dots \int \Big(|a| + \sum_{j=1}^{5} |x_j|\Big)^2 \mu(dx_5) \prod_{1\le p\le 4} \chi(dx_p) \\ &\ll q^4 |y|^2 \psi(t) \Big(|a|^2 + \int |x|^2\, \mu(dx) + \int |x|^2\, \chi(dx)\Big). \end{aligned}$$

The measures μ and χ are concentrated in cubes of size $[6r] + 1/2 \ll \sqrt{s}$ and $[r] + 1/2 \ll \sqrt{s}$ respectively. Thus, we have $\int |x|^2\, \mu(dx) \ll s$ and

$\int |x|^2\,\chi(dx) \ll s$. Using $|a| \ll |a|_\infty \le 1/2$, we obtain $J_0 \ll q^4|y|^2\psi(t)s$. This estimate together with (4.6) implies (4.3).

5. An inequality for trigonometric sums. Let $a, L \in \mathbb{R}^d$. Define the trigonometric sum

$$\varphi(t) = \varphi_{a,L}(t) = \Big|\iint \mathrm{e}\{tQ[x+y-a] + t\langle x+y, L\rangle\}\,\chi(dx)\,\chi(dy)\Big|, \qquad t \in \mathbb{R},$$

and

$$\psi(t) = \sup_{a,L} \varphi_{a,L}(t).$$

In order to illustrate the basic argument in the following inequalities, let

$$\Phi(t,p) = \Big|\sum_{x\in X} p_x \mathrm{e}\{tQ[x]\}\Big|$$

denote a trigonometric sum with weights p_x, $x \in X \subset \mathbb{R}^d$. Then, for any $\varepsilon > 0$, we have

$$\Phi(t-\varepsilon)\,\Phi(t+\varepsilon) \le \sum_{u\in X-X} \Phi(\varepsilon, q(u)) \tag{5.1}$$

with

$$\Phi(\varepsilon, q(u)) = \Big|\sum_{x\in X} q_x(u)\mathrm{e}\{-2\varepsilon Q[x] + 2(t+\varepsilon)\langle x, Qu\rangle\}\Big|$$

and

$$q_x(u) = p_x \sum_{y\in X} p_y\,\mathbf{I}\{x-y=u\}.$$

To obtain (5.1) it suffices to reorder the summation over x and y as summation over $x-y$ and $x+y$, and use the identities

$$Q[x]-Q[y] = \langle Q(x+y), x-y\rangle, \qquad 2(Q[x]+Q[y]) = Q[x+y]+Q[x-y] \tag{5.2}$$

together with the triangle inequality or, e.g., Hölder's inequality as below. The further bounds of $\Phi(\varepsilon, q(u))$ using the double large sieve will depend only on the coefficients of the quadratic part of the exponent in $\Phi(\varepsilon, q(u))$, which are proportional to ε and *independent* of t. In this section we shall prove the following inequality for $\varphi(t)$ defined above.

THEOREM 5.1. *We have*

$$\varphi(t)\varphi(t+\varepsilon) \ll q^d \mathcal{M}(\varepsilon) \qquad \textit{for } t, \varepsilon \in \mathbb{R}. \tag{5.3}$$

In particular,

$$\psi(\varepsilon) \ll q^d \mathcal{M}(\varepsilon). \tag{5.4}$$

Notice that the right hand side of (5.3) is independent of t, a, L. Recall that the function $\mathcal{M}$ is defined by (1.3), and that we assume that the eigenvalues of Q satisfy $1 \le q_1^2 \le \ldots \le q_d^2 = q^2$.

The following double large sieve bound is a consequence of Lemma 2.4 in Bombieri and Iwaniec (1986). For a vector $\mathcal{T} = (T_1, \ldots, T_d)$ with positive coordinates, introduce the cube

$$\begin{aligned} B(\mathcal{T}) &= B(T_1, \ldots, T_d) \\ &= \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : \ |x_j| \le T_j \text{ for } 1 \le j \le d\}. \end{aligned}$$

Write $\mathcal{T}^{-1} = (T_1^{-1}, \ldots, T_d^{-1})$.

LEMMA 5.2. *Let μ, ν denote arbitrary measures on $\mathbb{R}^d$ and let $\mathcal{S}, \mathcal{T}$ be vectors with positive coordinates. Write*

$$J = \Big| \int_{B(\mathcal{S})} \Big(\int_{B(\mathcal{T})} g(x)\, h(y) \mathrm{e}\{\langle x, y\rangle\}\, \mu(dx)\Big)\, \nu(dy)\Big|^2,$$

where $g, h : \mathbb{R}^d \to \mathbb{C}$ denote arbitrary (measurable) functions. Then

$$J \ll Q(2\mathcal{S}^{-1}, g, \mu)\, Q(2\mathcal{T}^{-1}, h, \nu) \prod_{j=1}^{d} (1 + S_j T_j),$$

where

$$Q(\mathcal{S}, g, \mu) = \int \Big(\int_{y \in x + B(\mathcal{S})} |g(y)|\, \mu(dy)\Big) |g(x)|\, \mu(dx).$$

In particular, if $|g(x)| \le 1$ and $|h(x)| \le 1$, then

$$J \ll \sup_{x \in \mathbb{R}^d} \mu(x + B(2\mathcal{S}^{-1})) \sup_{x \in \mathbb{R}^d} \nu(x + B(2\mathcal{T}^{-1})) \prod_{j=1}^{d} (1 + S_j T_j). \tag{5.5}$$

Proof. We shall call a measure μ *discrete* if there is a countable set, say $C_\mu \subset \mathbb{R}^d$, such that $\mu(C_\mu) = 1$. Bombieri and Iwaniec formulated this lemma for discrete measures μ and ν such that $\mu(\{x\}) = 1$ and $\nu(\{y\}) = 1$ for $x \in C_\mu$ and $y \in C_\nu$. Obviously, the lemma extends to the case of discrete μ and ν since the functions g and h are arbitrary. To extend the lemma to arbitrary μ and ν, one uses the fact that any measure can be weakly approximated by discrete measures, as well as the well-known properties of integrals. ■

Remark. Lemma 5.2 can be easily extended to the case of σ-finite complex-valued measures replacing $Q(\cdot, \cdot, \mu)$ and $Q(\cdot, \cdot, \nu)$ by $Q(\cdot, \cdot, \operatorname{var} \mu)$ and $Q(\cdot, \cdot, \operatorname{var} \nu)$, where $\operatorname{var} \mu$ is the total variation measure of μ. We shall use Lemma 5.2 only for uniform lattice measures.

COROLLARY 5.3. *Assume that $|g(x)| \le 1$ and $|h(x)| \le 1$. Let*

$$\mu(\{x \in \mathbb{R}^d : \ |x|_\infty \le T\}) = 1 \quad \textit{and} \quad \nu(\{x \in \mathbb{R}^d : \ |x|_\infty \le S\}) = 1, \tag{5.6}$$

for some $T > 0$ and $S > 0$. Write

$$J_t = \Big|\int\Big(\int g(x)\, h(y) \mathrm{e}\{t\langle Qx, y\rangle\}\, \mu(dx)\Big)\, \nu(dy)\Big|^2, \qquad t \in \mathbb{R}.$$

Then there exists a positive constant c_d depending on the dimension d only such that

$$(5.7)\qquad J_t \ll q^{2d}(1 + (|t|ST)^d) \times \sup_{x\in\mathbb{R}^d} \mu(x + c_d(|t|S)^{-1}B) \sup_{x\in\mathbb{R}^d} \nu(x + c_d(|t|T)^{-1}B),$$

where $B = \{x \in \mathbb{R}^d : |x|_\infty \le 1\}$.

Proof. The operator Q is positive and Q has eigenvalues $1 \le q_1^2 \le \ldots \le q_d^2 = q^2$. Therefore we can decompose $Q = \sqrt{Q}\,\sqrt{Q}$. If Q is symmetric and invertible, we can use the decomposition $Q = \sqrt{|Q|}J\sqrt{|Q|}$, where J is a symmetric isometric operator and the symmetric operator $\sqrt{|Q|}$ has eigenvalues $q_1, \ldots, q_d$. Consider the mappings $x \to \operatorname{sign}(t)\sqrt{|t|Q}x$ and $x \to \sqrt{|t|Q}x$, which map $\mathbb{R}^d$ into $\mathbb{R}^d$. Let μ_t and ν_t denote the induced measures of μ and ν under these mappings, that is,

$$\mu_t(C) = \mu(\operatorname{sign}(t)(|t|Q)^{-1/2}C) \quad \text{and} \quad \nu_t(C) = \nu((|t|Q)^{-1/2}C) \qquad \text{for } C \in \mathcal{B}^d.$$

Then we have

$$(5.8)\qquad J_t = \Big|\int\Big(\int g_0(x)\, h_0(y) \mathrm{e}\{\langle x, y\rangle\}\, \mu_t(dx)\Big)\, \nu_t(dy)\Big|^2,$$

for some functions g_0 and h_0 such that $|g_0| \le 1$ and $|h_0| \le 1$. The obvious inequalities $|\sqrt{Q}x|_\infty \ll |\sqrt{Q}x| \ll q|x| \ll q|x|_\infty$ and (5.6) imply that the integrals with respect to μ_t and ν_t in (5.8) have to be taken over the cubes

$$\{x \in \mathbb{R}^d : |x|_\infty \le c_d q\sqrt{|t|}T\} \quad \text{and} \quad \{x \in \mathbb{R}^d : |x|_\infty \le c_d q\sqrt{|t|}S\}$$

respectively. Thus, we can apply the double large sieve bound (5.5) and get

$$J_t \ll (1 + q^2|t|ST)^d \sup_{x\in\mathbb{R}^d} \mu(x + c_d(q|t|S)^{-1}B) \sup_{x\in\mathbb{R}^d} \nu(x + c_d(q|t|T)^{-1}B).$$

Using $1 \le q$ we obtain (5.7). ∎

Proof of Theorem 5.1. The estimate (5.3) implies (5.4) in the case $t = 0$ since $\varphi(0) = 1$.

Let us prove (5.3). We shall assume that $L = 0$. This will not restrict the generality since the operator Q is invertible, and therefore we can replace a by $a - 2^{-1}Q^{-1}L$. Without loss of generality we shall assume as well that $\varepsilon > 0$. Notice that in order to prove (5.3) it suffices to show that

$$(5.9)\qquad \varphi^2(t-\varepsilon)\varphi^2(t+\varepsilon) \ll q^{2d}\mathcal{M}^2(\varepsilon) \quad \text{for } \varepsilon > 0.$$

We have

$$\varphi^2(t-\varepsilon)\varphi^2(t+\varepsilon) = \Big|\iiiint \mathrm{e}\{I\}\,\chi(dx)\,\chi(dy)\,\chi(d\overline{x})\,\chi(d\overline{y})\Big|^2 \tag{5.10}$$

with

$$\begin{aligned} I &:= (t+\varepsilon)Q[x+y+a] - (t-\varepsilon)Q[\overline{x}+\overline{y}+a] \\ &= t(Q[x+y+a] - Q[\overline{x}+\overline{y}+a]) + \varepsilon(Q[x+y+a] + Q[\overline{x}+\overline{y}+a]). \end{aligned}$$

Using (5.2), we can write

$$\begin{aligned} I &= t\langle Q(x-\overline{x}+y-\overline{y}), x+\overline{x}+y+\overline{y}+2a\rangle \\ &\quad + 2^{-1}\varepsilon(Q[x+\overline{x}+y+\overline{y}+2a] + Q[x-\overline{x}+y-\overline{y}]). \end{aligned}$$

The measure χ assigns equal weights to integer points in the cube $B(r)$. Thus

$$\int h(u)\,\chi(du) = (2A+1)^{-d} \sum_{u\in\mathbb{Z}^d,\, |u|_\infty \le A} h(u),$$

for arbitrary functions h, and we can rewrite the 4-fold integral in (5.10) as a 4-fold sum. We are going to reorder summands in this sum. We shall use coordinatewise the following obvious formula:

$$\begin{aligned} &\sum_{|n|\le A} \sum_{|m|\le A} h(n-m, n+m) \\ &\qquad = \sum_{j:|j|\le 2A} \sum_{k:\,|k|\le 2A-|j|,\, j-k\in 2\mathbb{Z}} h(j,k), \qquad n,m,j,k\in\mathbb{Z}. \end{aligned} \tag{5.11}$$

Introduce the measure θ_x on $\mathbb{R}$ which assigns equal weights to even integer points in the interval $[-2A+|x|-1, 2A-|x|+1]$, if x is even, and respectively to odd points, if x is odd, for $x\in\mathbb{Z}$ such that $|x|\le 2A$. For $z\in\mathbb{Z}^d\cap B(2A+1)$, define the product measure

$$\eta_z(dx) = \prod_{j=1}^{d} \theta_{z_j}(dx_j).$$

Introduce as well the uniform lattice measure μ_{2A} in the cube $B(2A+1/2)$. Using (5.11) and changing variables

$$x-\overline{x} = \overline{u}, \quad y-\overline{y} = \overline{v}, \quad x+\overline{x} = u, \quad y+\overline{y} = v,$$

we can rewrite (5.10) as

$$\varphi^2(t-\varepsilon)\varphi^2(t+\varepsilon) = \Big|\iint p\Big(\iint \mathrm{e}\{I\}\,\eta_{\overline{u}}(du)\,\eta_{\overline{v}}(dv)\Big)\mu_{2A}(d\overline{u})\,\mu_{2A}(d\overline{v})\Big|^2$$

with

$$I = t\langle Q(\overline{u}+\overline{v}), u+v+2a\rangle + 2^{-1}\varepsilon(Q[u+v+2a] + Q[\overline{u}+\overline{v}])$$

and

$$p = (2A+1)^{-4d}\,(4A+1)^{2d} P(\overline{u})P(\overline{v}) \ll A^{-2d}\,P(\overline{u})\,P(\overline{v}),$$

where $P(z) = \prod_{j=1}^{d}(2A - |z_j| + 1)$. By Hölder's inequality we have

$$\varphi^2(t-\varepsilon)\varphi^2(t+\varepsilon) \le \iint p^2 J\,\mu_{2A}(d\overline{u})\,\mu_{2A}(d\overline{v}) \tag{5.12}$$

with

$$J = \Big|\iint \mathrm{e}\{I_0\}\,\eta_{\bar u}(du)\,\eta_{\bar v}(dv)\Big|^2$$

and

$$I_0 = t\langle Q(\overline{u}+\overline{v}), u+v\rangle + 2^{-1}\varepsilon Q[u+v+2a].$$

We can write

$$\mathrm{e}\{I_0\} = g(u)\,h(v)\,\mathrm{e}\{\varepsilon\langle Qu, v\rangle\}$$

with some functions g and h depending on $\overline{u}, \overline{v}, t, \varepsilon, a, Q$ such that $|g| \le 1$ and $|h| \le 1$. Hence

$$J = \Big|\iint g(u)\,h(v)\,\mathrm{e}\{\varepsilon\langle Qu, v\rangle\}\eta_{\bar u}(du)\,\eta_{\bar v}(dv)\Big|^2.$$

In order to estimate J we shall apply the double large sieve bound of Corollary 5.3. Choose

$$T = \sum_{j=1}^{d}(2A - |\overline{u}_j| + 1), \qquad S = \sum_{j=1}^{d}(2A - |\overline{v}_j| + 1).$$

Then

$$J \ll q^{2d}(1 + (\varepsilon ST)^d) \sup_{x\in\mathbb{R}^d} \eta_{\bar u}(x + c_d(\varepsilon S)^{-1}B) \sup_{x\in\mathbb{R}^d} \eta_{\bar v}(x + c_d(\varepsilon T)^{-1}B).$$

The measures $\eta_{\bar u}$ and $\eta_{\bar v}$ are both concentrated on a sublattice of $\mathbb{Z}^d$. Moreover, $\eta_{\bar u}(\{x\}) \le 1/P(\overline{u})$ and $\eta_{\bar v}(\{x\}) \le 1/P(\overline{v})$, for $x \in \mathbb{Z}^d$. Therefore

$$\sup_{x\in\mathbb{R}^d} \eta_{\bar u}(x + c_d(\varepsilon S)^{-1}B) \ll P^{-1}(\overline{u}) \max\{1;\ (\varepsilon S)^{-d}\},$$

and a similar inequality holds with $\overline{u}$ resp. S replaced by $\overline{v}$ resp. T. Collecting these estimates and using $P(\overline{u}) \le T^d$ and $P(\overline{v}) \le S^d$, we get

$$p^2 J \ll q^{2d}\frac{S^dT^d}{A^{4d}}(1 + \varepsilon^d S^d T^d) \max\Big\{1; \frac{1}{\varepsilon^d S^d}\Big\} \max\Big\{1; \frac{1}{\varepsilon^d T^d}\Big\}.$$

Since $1 + \varepsilon^d S^d T^d \ll (1 + \varepsilon^{d/2}S^d)\,(1 + \varepsilon^{d/2}T^d)$, we get

$$p^2 J \ll q^{2d} F(S)F(T),$$

$$F(S) := A^{-2d}S^d(1 + \varepsilon^{d/2}S^d)\,\max\{1;\ \varepsilon^{-d}S^{-d}\}.$$

Elementary estimates for the cases $\varepsilon S \le 1$ resp. $\varepsilon S > 1$ show that $F(S) \ll \mathcal{M}(\varepsilon)$ and similarly $F(T) \ll \mathcal{M}(\varepsilon)$, which together with (5.12) implies (5.3). ■

We close this section with a refinement of inequality (5.3). Let χ^{*m} denote the m-fold convolution of χ. Recall that μ_ϱ is the uniform lattice measure in the cube $B(\varrho)$ and $\chi = \mu_r$. Fix a natural number m. Notice that we can bound $\Delta(s,Q)$ by Fourier integrals with respect to the measure $\mu_{3mr+2r} * \chi^{*3m+2}$ instead of $\mu * \chi^{*4}$. This will lead to bounds of Fourier transforms replacing throughout the function φ by

$$\varphi_m(t) = \Big|\iint \mathrm{e}\{tQ[x-a] + t\langle x, L\rangle\}\,\chi^{*3m}(dx)\Big|.$$

The function φ_m satisfies the following inequality of type (5.3):

$$\varphi_m(t)\varphi_m(t+\varepsilon) \ll \mathcal{M}_0(\varepsilon), \tag{5.13}$$

where

$$\mathcal{M}_0^2(\varepsilon) = \iint \mathrm{e}\{\varepsilon\langle Qx, y\rangle\}\,\mu_{2A}^{*2m}(dx)\,\mu_{2A}^{*2m}(dy).$$

The constant in (5.13) can depend on m as well. Using a modification of the double large sieve bound, we shall show that

$$\mathcal{M}_0(\varepsilon) \ll q^d \mathcal{M}(\varepsilon). \tag{5.14}$$

Thus, (5.13) implies an inequality of type (5.3) for φ_m. The inequality is interesting since the bound $\mathcal{M}_0^2(\varepsilon)$ is again a trigonometric sum, which satisfies as well an inequality of type (5.3). The proof of this fact is more involved than the proof of (5.3), and can be found in Bentkus and Götze (1994b).

Proof of (5.13). The proof is similar to the proof of Theorem 5.1, but instead of the double large sieve bound we shall apply a symmetrization inequality.

In order to simplify the notation we shall assume that $m = 1$. Without loss of generality, we shall assume that $L = 0$ and $\varepsilon > 0$.

Similarly to the proof of (5.10)–(5.12) we obtain

$$\varphi_1^2(t-\varepsilon)\varphi_1^2(t+\varepsilon) \le \iiint p^2 J\,\mu_{2A}(d\overline{u})\,\mu_{2A}(d\overline{v})\,\mu_{2A}(d\overline{w}) \tag{5.15}$$

with $p \ll A^{-3d}P(\overline{u})P(\overline{v})P(\overline{w})$,

$$J = \Big|\iiint \mathrm{e}\{I_0\}\,\eta_{\bar u}(du)\,\eta_{\bar v}(dv)\,\eta_{\bar w}(dw)\Big|^2$$

and

$$I_0 = t\langle Q(\overline{u}+\overline{v}+\overline{w}), u+v+w\rangle + 2^{-1}\varepsilon Q[u+v+w+2a].$$

In order to estimate J let us apply the symmetrization Lemma 7.1 with measures $\mu_1 = \eta_{\bar u}$, $\mu_2 = \eta_{\bar v}$, $\mu_3 = \eta_{\bar w}$. Notice that $\widetilde{\mu} = \mu * \mu$ for symmetric μ. Thus, we get

$$J \le \iint \mathrm{e}\{\varepsilon\langle Qx, y\rangle\}\,\eta_{\bar u}^{*2}(dx)\,\eta_{\bar v}^{*2}(dy) + \iint \mathrm{e}\{\varepsilon\langle Qx, y\rangle\}\,\eta_{\bar u}^{*2}(dx)\,\eta_{\bar w}^{*2}(dy). \tag{5.16}$$

The bounds $P(\overline{w}) \ll A^d$ for the first integral in (5.16), and $P(\overline{v}) \ll A^d$ for the second integral, together with (5.15) imply

$$\varphi_1^2(t-\varepsilon)\varphi_1^2(t+\varepsilon) \le \iint p_0^2 J_0 \, \mu_{2A}(d\overline{u}) \, \mu_{2A}(d\overline{v}) \tag{5.17}$$

with

$$J_0 = \iint \mathrm{e}\{\varepsilon\langle x, y\rangle\} \, \eta_{\bar u}^{*2}(dx) \, \eta_{\bar v}^{*2}(dy),$$

where $p_0 = A^{-2d} P(\overline{u}) P(\overline{v})$.

The estimate (5.17) implies (5.13) provided that we show that

$$J_0 \ll p_0^{-2} \iint \mathrm{e}\{\varepsilon\langle Qx, y\rangle\} \, \mu_{2A}^{*2}(dx) \, \mu_{2A}^{*2}(dy). \tag{5.18}$$

In order to prove (5.18) write

$$J_0 = \iint f(x_1 + x_2) \, \eta_{\bar u}(dx_1) \, \eta_{\bar u}(dx_2),$$

where

$$f(x) = \int \mathrm{e}\{\varepsilon\langle x, y\rangle\} \, \eta_{\bar v}^{*2}(dy) = \Big| \int \mathrm{e}\{\varepsilon\langle x, y\rangle\} \, \eta_{\bar v}(dy) \Big|^2 \ge 0$$

due to the symmetry of the measure $\eta_{\bar v}$. The measure $\eta_{\bar u}$ is defined on a sublattice of $\mathbb{Z}^d$ in the cube

$$\prod_{j=1}^{d} [-2A + |\overline{u}_j|, \, 2A - |\overline{u}_j|] \subset [-2A, \, 2A]^d,$$

and for $x \in \mathbb{Z}^d$ we have $\eta_{\bar u}(\{x\}) \le 1/P(\overline{u})$. Therefore, using the *positivity* of f, we obtain

$$\begin{aligned} J_0 &\le P^{-2}(\overline{u}) \sum_{x_1, x_2 \in \mathbb{Z}^d, \, |x_1|_\infty \le 2A, \, |x_2|_\infty \le 2A} f(x_1 + x_2) \\ &\ll P^{-2}(\overline{u}) \, A^{2d} \iint f(x_1 + x_2) \, \mu_{2A}(dx_1) \, \mu_{2A}(dx_2), \end{aligned}$$

thus replacing $\eta_{\bar u}$ by μ_{2A}. Similar arguments show that we can replace $\eta_{\bar v}$ by μ_{2A}, thus proving (5.18). ∎

Proof of (5.14). Let $\widehat{\mu}(x) = \int \mathrm{e}\{\langle x, y\rangle\} \, \mu(dy)$ denote the Fourier transform of a measure μ on $\mathbb{R}^d$. Then, for any cube $B(S) = \{|x|_\infty \le S\}$, we have

$$\int_{B(S)} |\widehat{\mu}(y)|^2 \, dy \ll S^d \sup_{x \in \mathbb{R}^d} \mu(x + B(S^{-1})). \tag{5.19}$$

The estimate (5.19) follows from Lemma 2.3 in Bombieri and Iwaniec (1986).

Let $\widetilde{\mu} = \mu * \overline{\mu}$, where $\overline{\mu}(C) = \mu(-C)$, denote the symmetrization of a measure μ. Let us prove that

$$J := \iint \mathrm{e}\{\langle x,y\rangle\}\,\widetilde{\mu}(dy)\,\nu(dx) \ll (1+T^{2d})\sup_z \mu(z+B(T^{-1}))\sup_z \nu(z+B(T^{-1})), \tag{5.20}$$

for arbitrary measures μ, ν and $T>0$ such that

$$\mu(B(T))=1,\quad \nu(B(T))=1,\quad \widehat{\nu}(x)\ge 0\quad \text{for all } x\in\mathbb{R}^d.$$

Let

$$\tau(dx)=p(x)\,dx,\qquad \text{where } p(x)=2^{-d}\,\mathbf{I}\{|x|_\infty\le 1\},$$

denote the uniform measure on the cube $B(1)$. Then $\widehat{\tau}(y)=\prod_{j=1}^d(\sin x_j)/x_j$. Using $1\ll\widehat{\tau}((2T)^{-1}x)$, for $x\in B(2T)$, and $\widehat{\widetilde{\mu}}(x)=|\widehat{\mu}(x)|^2$, we have

$$\begin{aligned} J &= \int \widehat{\nu}(x)\,\widetilde{\mu}(dx) \ll \int \widehat{\nu}(x)\,\widehat{\tau}((2T)^{-1}x)\,\widetilde{\mu}(dx) \\ &= \iiint \mathrm{e}\{\langle x, y+(2T)^{-1}z\rangle\}\,\widetilde{\mu}(dx)\,p(z)\,dz\,\nu(dy) \\ &= T^d\iint|\widehat{\mu}(u)|^2p(2Tu-2Ty)\,du\,\nu(dy). \end{aligned} \tag{5.21}$$

Note that the integral in (5.21) with respect to ν is taken over the cube $B(T)$. Therefore

$$p(2Tu-2Ty)\ll\mathbf{I}\{|u-y|_\infty\le T^{-1}\}=\mathbf{I}\{|u-y|_\infty\le T^{-1}\}\,\mathbf{I}\{|u|_\infty\le T+T^{-1}\},$$

for $|y|_\infty\le T$. Hence, (5.21) and (5.19) together imply

$$\begin{aligned} J &\ll T^d\int|\widehat{\mu}(u)|^2\mathbf{I}\{|u|\le T+T^{-1}\}\Big(\int\mathbf{I}\{|u-y|\le T^{-1}\}\,\nu(dy)\Big)\,du \\ &\ll T^d\sup_z\nu(z+B(T^{-1}))\int|\widehat{\mu}(u)|^2\mathbf{I}\{|u|\le T+T^{-1}\}\,du \\ &\ll T^d(T+T^{-1})^d\sup_z\nu(z+B(T^{-1}))\sup_z\mu(z+B(T/(T^2+1))), \end{aligned}$$

thus proving (5.20) since $T/(T^2+1)\le 1/T$.

Similarly to the proof of Corollary 5.3, the inequality (5.20) yields

$$\iint\mathrm{e}\{\varepsilon\langle Qx,y\rangle\}\,\widetilde{\mu}(dy)\,\nu(dx) \ll q^{2d}(1+\varepsilon^dT^{2d})\sup_z\mu(z+B((\varepsilon T)^{-1}))\sup_z\nu(z+B((\varepsilon T)^{-1})), \tag{5.22}$$

for *arbitrary* measures μ, ν and $T>0$, $\varepsilon>0$ such that

$$\mu(B(T))=1,\quad \nu(B(T))=1,\quad \widehat{\nu}(x)\ge 0\quad \text{for all } x\in\mathbb{R}^d.$$

The measures $\mu=\mu_{2A}^{*m}$ and $\nu=\mu_{2A}^{*2m}$ are symmetric and therefore $\widetilde{\mu}=\mu_{2A}^{*2m}$ and $\widehat{\nu}(x)\ge 0$. Thus, the estimate (5.22) via simple calculations implies (5.14). ∎

6. The integration procedure for $A^{-4/d} \le |t| \le 1$

THEOREM 6.1. *Let $\varphi(t)$, $t \ge 0$, denote a continuous nonnegative function such that $\varphi(0) = 1$. Let $\mathcal{M}(\varepsilon)$ be the function defined in* (1.3) *with some $A \ge 1$. Assume that*

$$\varphi(t)\,\varphi(t+\varepsilon) \le \Lambda\mathcal{M}(\varepsilon) \quad \textit{for all } t \ge 0 \textit{ and } \varepsilon \ge 0, \tag{6.1}$$

with some $\Lambda \ge 1$ independent of t and ε. Then

$$\int_L^1 \varphi(t)\frac{dt}{t} \ll \frac{\Lambda}{A^2} \quad \textit{for } d \ge 9,$$

where $L = A^{-4/d}$.

Proof. The inequality (6.1) implies that (put $t = 0$, use $\varphi(0) = 1$ and note that $\Lambda \ge 1$)

$$\varphi(t) \le \Lambda\mathcal{M}(t) \quad \text{and} \quad \varphi(t)\varphi(t+\varepsilon) \le \Lambda^2\mathcal{M}(\varepsilon). \tag{6.2}$$

In order to derive the result starting with (6.2) we may assume that

$$\varphi(t) \le \mathcal{M}(t) \quad \text{and} \quad \varphi(t)\varphi(t+\varepsilon) \le \mathcal{M}(\varepsilon). \tag{6.3}$$

Indeed, we may replace φ in (6.2) by φ/Λ, and we may integrate over φ/Λ instead of φ.

Thus assuming (6.3) we have to prove that

$$\int_L^1 \varphi(t)\frac{dt}{t} \ll \frac{1}{A^2} \quad \text{for } d \ge 9.$$

For $l = 0, 1, 2, \ldots$ introduce the sets

$$B_l = [L, 1] \cap \{t : 2^{-l-1} \le \varphi^2(t) \le 2^{-l}\},$$
$$D_l = [L, 1] \cap \{t : \varphi^2(t) \le 2^{-l-1}\}.$$

Notice that the sets B_l and D_l are closed and that $\bigcup_{l=0}^m B_l \cup D_m = [L, 1]$ since (6.3) yields $\varphi(t) \le 1$ for $t \in [L, 1]$, provided $d \ge 4$. Furthermore, (6.3) implies that $\varphi^2(t) \le t^d$ for $t \ge A^{-1}$, and $B_l \subset [L_l, 1]$, where $L_l = 2^{-(l+1)/d}$ for $d \ge 4$.

We have

$$\int_L^1 \varphi(t)\frac{dt}{t} \le \int_{D_m} \varphi(t)\frac{dt}{t} + \sum_{l=0}^m \int_{B_l} \varphi(t)\frac{dt}{t}$$
$$\le 2^{-m/2}\ln L^{-1} + \sum_{l=0}^m 2^{-l/2} \int_{B_l} \frac{dt}{t}$$

since $\varphi^2(t) \le 2^{-l}$ for $t \in B_l$, $\varphi^2(t) \le 2^{-m}$ for $t \in D_m$, and $D_m \subset [L, 1]$.

We have $L = A^{-4/d}$ and

$$2^{-m/2} \ln L^{-1} = \frac{2}{d} 2^{-m/2} \ln A^2 \leq A^{-2}$$

provided that we choose

$$m = \frac{2}{\ln 2} \ln(A^2 \ln A^2).$$

Therefore it remains to show that

$$\sum_{l=0}^{m} I_l \ll \frac{1}{A^2}, \quad \text{where } I_l = 2^{-l/2} \int_{B_l} \frac{dt}{t}. \tag{6.4}$$

For an estimation of I_l we need a description of the structure of the sets B_l with $l \leq m$. Let $t, t' \in B_l$ denote points such that $t' > t$. The inequality (6.3) and the definition of B_l imply

$$4^{-l-1} \leq \mathcal{M}^2(t' - t). \tag{6.5}$$

If $t - t' \leq A^{-1}$ then by (6.5) and the definition of $\mathcal{M}(\varepsilon)$ we get

$$t' - t \leq \delta, \quad \text{where } \delta = A^{-2} 4^{(l+1)/d}. \tag{6.6}$$

If $t - t' \geq A^{-1}$ then by (6.5) and the definition of $\mathcal{M}(\varepsilon)$ we have as well

$$t' - t \geq \varrho, \quad \text{where } \varrho = 4^{-(l+1)/d}.$$

For $d > 8$ and sufficiently large $A \geq C$ note that

$$\delta < \varrho \quad \text{provided} \quad l \leq m. \tag{6.7}$$

The verification of (6.7) is elementary and is based on the fact that $l \leq m$.

The estimate (6.7) implies that either $t - t' \leq \delta$ or $t - t' \geq \varrho$. Therefore it follows from (6.6) and (6.7) that

$$t \in B_l \Rightarrow B_l \cap (t + \delta, t + \varrho) = \emptyset. \tag{6.8}$$

Assuming that

$$I_l \ll A^{-2} l 2^{-l/2 + 4l/d} \quad \text{for } l \leq m, \tag{6.9}$$

we obtain (6.4) since the series $\sum_{l=0}^{\infty} l\, 2^{-l/2+4l/d}$ is convergent for $d > 8$.

Thus it remains to prove (6.9). If the set B_l is empty then (6.9) is obviously fulfilled. If B_l is nonempty then define $e_1 := \min\{t : t \in B_l\}$. Choosing $t = e_1$ and using (6.8) we see that the interval $(e_1 + \delta, e_1 + \varrho)$ does not intersect B_l. Similarly, let e_2 denote the smallest $t \geq e_1 + \varrho$ such that $t \in B_l$. Then the interval $(e_2 + \delta, e_2 + \varrho)$ does not intersect B_l. Repeating this procedure we construct a sequence $L_l \leq e_1 < e_2 < \ldots < e_s \leq 1$ such that

$$B_l \subset \bigcup_{j=1}^{s} [e_j, e_j+\delta] \quad \text{and} \quad e_{j+1} \ge e_j + \varrho. \tag{6.10}$$

The sequence $e_1 < \ldots < e_s$ cannot be infinite. Indeed, due to (6.10),

$$1 \ge e_s \ge e_1 + (s-1)\varrho \ge L_l + (s-1)\varrho \ge s\varrho,$$

and therefore $s \le \varrho^{-1}$.

From (6.10) we can finally prove (6.9). Indeed, using $\ln(1+x) \le x$, for $x \ge 0$, we have

$$\begin{aligned} I_l &\le 2^{-l/2} \sum_{j=1}^{s} \int_{e_j}^{e_j+\delta} \frac{dt}{t} = 2^{-l/2} \sum_{j=1}^{s} \ln\Big\{1 + \frac{\delta}{e_j}\Big\} \\ &\le 2^{-l/2} \sum_{j=1}^{s} \frac{\delta}{e_j} \ll l 2^{-l/2+4l/d} A^{-2} \end{aligned}$$

since $e_1 \ge L_l \ge \varrho$, $s \le 1/\varrho$, and

$$\sum_{j=1}^{s} \frac{1}{e_j} \le \sum_{j=1}^{s} \frac{1}{e_1 + (j-1)\varrho} \le \frac{1}{\varrho} \sum_{j=1}^{s} \frac{1}{j} \ll \frac{\ln \varrho}{\varrho} \ll l\, 4^{l/d}. \ \blacksquare$$

7. A symmetrization inequality. The following symmetrization inequality slightly improves an inequality due to Götze (1979). This inequality is a generalization of a classical inequality due to Weyl (1915/16); see, e.g. Graham and Kolesnik (1991).

Define the symmetrization $\widetilde{\mu}$ of a measure μ by $\widetilde{\mu}(C) = \int \mu(C+x)\, \mu(dx)$, for $C \in \mathcal{B}^d$.

LEMMA 7.1. *Let $L \in \mathbb{R}^d$ and $C \in \mathbb{R}$. Let μ_1, μ_2, μ_3, ν denote arbitrary measures on $\mathbb{R}^d$. Define a real-valued polynomial of second order by*

$$P(x) = \langle Qx, x\rangle + \langle L, x\rangle + C \quad \text{for } x \in \mathbb{R}^d.$$

Then the integral

$$J = \Big| \int \mathrm{e}\{tP(x)\}\, \mu_1 * \mu_2 * \mu_3 * \nu(dx) \Big|^2$$

satisfies $2J \le J_1 + J_2$, where

$$J_1 = \iint \mathrm{e}\{2t\, \langle Qx, y\rangle\}\, \widetilde{\mu}_1(dx)\, \widetilde{\mu}_2(dy),$$

$$J_2 = \iint \mathrm{e}\{2t\, \langle Qx, z\rangle\}\, \widetilde{\mu}_1(dx)\, \widetilde{\mu}_3(dz).$$

In particular, if $\mu_2 = \mu_3$ then $J \le J_1$.

Proof. Write $\theta = \mu_2 * \mu_3 * \nu$. Then the definition of the convolution, Fubini's Theorem and Hölder's inequality together imply

$$\begin{aligned} J &= \Big|\iint e\{t\,P(x+v)\}\,\mu_1(dx)\,\theta(dv)\Big|^2 \\ &\le \int\Big|\int e\{t\,P(x+v)\}\,\mu_1(dx)\Big|^2\,\theta(dv) \\ &= \iiint e\{t\,(P(x+v)-P(u+v))\}\,\mu_1(dx)\,\mu_1(du)\,\theta(dv). \end{aligned}$$

We have $P(x+v) - P(u+v) = Q[x] - Q[u] + 2\langle Q(x-u), v\rangle + \langle L, x-u\rangle$. Thus

$$\begin{aligned} J &\le \iint\Big|\int e\{2t\langle Q(x-u), v\rangle\}\,\theta(dv)\Big|\,\mu_1(dx)\,\mu_1(du) \\ &= \int\Big|\iiint e\{2t\langle Qx, y+z+w\rangle\}\,\mu_2(dy)\,\mu_3(dz)\,\nu(dw)\Big|\,\widetilde{\mu}_1(dx) \\ &\le \int\Big|\iint e\{2t\langle Qx, y+z\rangle\}\,\mu_2(dy)\,\mu_3(dz)\Big|\,\widetilde{\mu}_1(dx). \end{aligned}$$

Thus, writing

$$I_j = \Big|\int e\{2t\langle Qx, y\rangle\}\,\mu_j(dy)\Big| \quad \text{for } j = 2, 3,$$

we have $J \le \int |I_2\,I_3|\,\widetilde{\mu}_1(dx)$. To conclude the proof it suffices to use the elementary inequality $2|I_2\,I_3| \le |I_2|^2 + |I_3|^2$ and note that

$$\int |I_2|^2\,\widetilde{\mu}_1(dx) = J_1 \quad \text{and} \quad \int |I_3|^2\,\widetilde{\mu}_1(dx) = J_2. \ \blacksquare$$

8. A smoothing lemma

LEMMA 8.1. *Let F and G be arbitrary distribution functions with the Fourier–Stieltjes transforms $\widehat{F}$ and $\widehat{G}$. Then*

$$\sup_x |F(x) - G(x)| \le \frac{1}{2\pi}\int_{-H}^{H} |\widehat{F}(t) - \widehat{G}(t)|\,\frac{dt}{|t|} + R, \tag{8.1}$$

for any $H > 0$, where

$$|R| \le \frac{1}{H}\int_{-H}^{H} |\widehat{F}(t)|\,dt + \frac{1}{H}\int_{-H}^{H} |\widehat{G}(t)|\,dt.$$

Introduce the function $2K(s) = K_1(s) + iK_2(s)/(\pi s)$, where $K_1(s) = K_2(s) = 0$ for $s \ge 1$, and

$$K_1(s) = 1 - |s|, \quad K_2(s) = \pi s(1 - |s|)\cot \pi s + |s| \quad \text{for } |s| \le 1.$$

It is known (Prawitz (1972)) that for all $x \in \mathbb{R}$ and any $H > 0$,

$$F(x+) \le \frac{1}{2} + \text{V.P.}\int_{\mathbb{R}} e\{-xt\}\,\frac{1}{H}\,K\Big(\frac{t}{H}\Big)\widehat{F}(t)\,dt, \tag{8.2}$$

$$F(x-) \geq \frac{1}{2} - \mathrm{V.P.}\int_{\mathbb{R}} \mathrm{e}\{-xt\}\frac{1}{H}K\Big(-\frac{t}{H}\Big)\widehat{F}(t)\,dt, \tag{8.3}$$

where $F(x+) = \lim_{z\downarrow x} F(z)$, $F(x-) = \lim_{z\uparrow x} F(z)$, and V.P. denotes Cauchy's Principal Value,

$$\mathrm{V.P.}\int_{\mathbb{R}} \ldots dt = \lim_{h\downarrow 0} \int_{|t|\geq h} \ldots dt.$$

Note that all integrals are real and that the integrands vanish unless $|t| \leq H$.

The following lemma is elementary.

LEMMA 8.2. *For* $0 \leq s \leq 1$ *we have*

$$K_2(0) = 1, \quad K_2(1) = 0, \quad K_2(1/2) = 1/2,$$
$$K_2'(s) \leq 0, \quad K_2(s) + K_2(1-s) = 1.$$

Furthermore,

$$1 - 2\,(1-s)\sin^2\frac{\pi s}{2} \leq K_2(s) \leq 1 \qquad \textit{for } 0 \leq s \leq 1/2,$$
$$0 \leq K_2(s) \leq 2\,s\sin^2\frac{\pi(1-s)}{2} \qquad \textit{for } 1/2 \leq s \leq 1.$$

It follows from Lemma 8.2 that $|1-K_2(s)| \leq 2|s|$ for all $s \in \mathbb{R}$. Therefore (8.2), (8.3) and the definition of the function K imply

$$F(x) = \frac{1}{2} + \frac{i}{2\pi}\mathrm{V.P.}\int_{-H}^{H} \mathrm{e}\{-xt\}\,\widehat{F}(t)\frac{dt}{t} + R \tag{8.4}$$

for any $H > 0$, where

$$|R| \leq \frac{1}{H}\int_{-H}^{H} |\widehat{F}(t)|\,dt.$$

As a consequence of (8.4) we derive Lemma 8.1.

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