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On addition of two sets of integers

by

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1. Introduction. Let $A=\{a_1,\ldots,a_k\}$ and $B=\{b_1,\ldots,b_l\}$ be two sets of integers with $k=|A|$, $l=|B|$ and $0=a_1<\ldots<a_k$, $0=b_1<\ldots<b_l$. Denote the set $\{a_i+b_j : 1\le i\le k,\ 1\le j\le l\}$ by $A+B$ and $A+A$ by $2A$. Assume that $a_k\ge b_l$.

G. Freiman [1] showed

Theorem 1. (i) *Let* $a_k\le 2k-3$. *Then* $|2A|\ge a_k+k$.

(ii) *Let* $a_k\ge 2k-2$ *and* $(a_1,\ldots,a_k)=1$. *Then* $|2A|\ge 3k-3$.

G. Freiman [2] generalized this to the case of two distinct summands.

Theorem 2. (i) *Let* $a_k\le k+l-3$. *Then* $|A+B|\ge a_k+l$.

(ii) *Let* $a_k\ge k+l-2$ *and* $(a_1,\ldots,a_k,b_1,\ldots,b_l)=1$. *Then* $|A+B|\ge k+l+\min\{k,l\}-3$.

Let

$$\delta=\begin{cases}1 & \text{if } b_l=a_k,\\ 0 & \text{if } b_l\ne a_k.\end{cases}$$

Vsevolod F. Lev and Pavel Y. Smeliansky [5] sharpened the above Theorem 2 as

Theorem 3. (i) *Let* $a_k\le k+l-2-\delta$. *Then* $|A+B|\ge a_k+l$.

(ii) *Let* $a_k\ge k+l-1-\delta$ *and* $(a_1,\ldots,a_k)=1$. *Then* $|A+B|\ge k+2l-2-\delta$.

In this paper we further sharpen the above Theorem 3. The main results are given in the next section.

2. Statements of main results. In the following we do not use the notations of the first section except in the Corollary of Theorem 4. For any set T of integers, let $|T|$ denote the cardinality of T, and d_T denote the greatest common divisor of elements of $T-\{t\}$, where $t\in T$. It is not difficult

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to see that d_T is independent of the choice of t. Let q be a positive integer. We use $\overline{T}$ to denote the set of residue classes modulo q having nonempty intersection with T. Let A, B be two nonempty finite sets of integers and write

$$A=\bigcup_{i=1}^{t} A_i,\quad |\overline{A}_i|=1,\quad \overline{A}_i\cap\overline{A}_j=\emptyset,\quad i\neq j;$$
$$B=\bigcup_{i=1}^{s} B_i,\quad |\overline{B}_i|=1,\quad \overline{B}_i\cap\overline{B}_j=\emptyset,\quad i\neq j.$$

Then $|\overline{A}|=t$ and $|\overline{B}|=s$. In this paper we prove

THEOREM 4. (i) *Let* $q\le|\overline{A}|+|\overline{B}|-1$. *Then*

$$|A+B|\ge q+|B|+(\max_i|A_i|-2)|\overline{B}|.$$

(ii) *Let* $q\ge|\overline{A}|+|\overline{B}|$ *and* $(d_A,d_B,q)=1$. *Then*

$$|A+B|\ge\min\left\{\frac{2q}{(q,d_A)},|\overline{A}|+|\overline{B}|-1\right\}+|B|+(\max_i|A_i|-2)|\overline{B}|.$$

From Theorem 4 we immediately have

COROLLARY. *Let the notations be as in Section* 1. *Then*

(i) *if* $a_k\le k+l-2-\delta$, *then*

$$|A+B|\ge a_k+l;$$

(ii) *if* $a_k\ge k+l-1-\delta$ *and* $(a_1,\ldots,a_k,b_1,\ldots,b_l)=1$, *then*

$$|A+B|\ge\min\left\{\frac{2a_k}{d_A},k+l-2-\delta\right\}+l.$$

Note. Theorem 4 is sharp. For example, let

$$A=\{n:n\equiv1,2\pmod 5,\ 1\le n\le5k\},$$
$$B=\{m:m\equiv0,3,4\pmod 5,\ 1\le m\le5k\}.$$

Then

$$A+B=\{n:n\equiv0,1,2,4\pmod 5,\ 3\le n\le10k-2\}.$$

So $|A+B|=8k-4$. Now we use Theorem 4. Take $q=5k-4$. Then $|\overline{A}|=2k-1$, $|\overline{B}|=3k-2$, $|B|=3k$ and $\max_i|A_i|=2$. By Theorem 4 we have

$$|A+B|\ge\min\{5k-4,5k-4\}+3k=8k-4.$$

3. Proof of the main results

LEMMA 1. *Let* T *be a set of integers*, *and* T_1 *a subset of* T. *Then*

$$|T|-|\overline{T}|\ge|T_1|-|\overline{T}_1|.$$

P r o o f. Write $T = T_1 \cup T_2$, $T_1 \cap T_2 = \emptyset$. Then

$$|T| = |T_1| + |T_2|, \qquad |\overline{T}| = |\overline{T}_1 \cup \overline{T}_2| = |\overline{T}_1| + |\overline{T}_2| - |\overline{T}_1 \cap \overline{T}_2|.$$

Hence

$$|T| - |\overline{T}| = |T_1| - |\overline{T}_1| + |T_2| - |\overline{T}_2| + |\overline{T}_1 \cap \overline{T}_2| \ge |T_1| - |\overline{T}_1|.$$

This completes the proof of Lemma 1.

Lemma 2. *We have*

$$|A_i + B| \ge |\overline{A}_i + \overline{B}| + |B| + (|A_i| - 2)|\overline{B}|, \qquad i = 1, \ldots, t.$$

P r o o f. Since $\overline{A}_i + \overline{B}_1,\ \overline{A}_i + \overline{B}_2, \ldots, \overline{A}_i + \overline{B}_s$ are pairwise disjoint, we have

$$|\overline{A}_i + \overline{B}| = \sum_{j=1}^{s} |\overline{A}_i + \overline{B}_j| = s$$

and

$$|A_i + B| = \sum_{j=1}^{s} |A_i + B_j| \ge \sum_{j=1}^{s} (|A_i| + |B_j| - 1) = s(|A_i| - 1) + |B|.$$

By $|\overline{B}| = s$ Lemma 2 is true.

Lemma 3 (Kneser's Theorem [3], [4]). *Let H be the subgroup of all those elements $h \in \mathbb{Z}_q$ satisfying $\overline{A} + \overline{B} + h = \overline{A} + \overline{B}$. If*

$$|\overline{A} + \overline{B}| \le |\overline{A}| + |\overline{B}| - 1,$$

then

$$|\overline{A} + \overline{B}| + |H| = |\overline{A} + H| + |\overline{B} + H|.$$

Hence $|H| > 1$ if

$$|\overline{A} + \overline{B}| \le |\overline{A}| + |\overline{B}| - 2.$$

P r o o f o f T h e o r e m 4 (following the proof of Lev and Smeliansky [5]). Without loss of generality, we may assume that $0 \in A$, $0 \in B$ and $\max_i |A_i| = |A_1|$, $\overline{A}_1 = \{0\}$. Suppose that

$$|A + B| \le |\overline{A}| + |B| + (|A_1| - 1)|\overline{B}| - 2 \tag{1}$$

and prove that

$$|A + B| \ge q + |B| + (|A_1| - 2)|\overline{B}| \qquad \text{if } q \le |\overline{A}| + |\overline{B}| - 1;$$

$$|A + B| \ge \min\left\{\frac{2q}{(q, d_A)}, q\right\} + |B| + (|A_1| - 2)|\overline{B}|$$
$$\text{if } q \ge |\overline{A}| + |\overline{B}| \text{ and } (d_A, d_B, q) = 1.$$

By Lemmas 1 and 2 we have

$$(2) \qquad |A+B| - |\overline{A}+\overline{B}| \ge |A_1+B| - |\overline{A}_1+\overline{B}| \ge |B| + (|A_1|-2)|\overline{B}|.$$

So it suffices to prove that

$$|\overline{A}+\overline{B}| = q \quad \text{if } q \le |\overline{A}|+|\overline{B}|-1;$$

$$|\overline{A}+\overline{B}| \ge \min\left\{\frac{2q}{(q,d_A)}, q\right\} \quad \text{if } q \ge |\overline{A}|+|\overline{B}| \text{ and } (d_A,d_B,q)=1.$$

By (1) and (2) we have

$$|\overline{A}+\overline{B}| \le |\overline{A}|+|\overline{B}|-2.$$

By Lemma 3 we have $|H|>1$. Then $H=d\mathbb{Z}_q$, $d\,|\,q$, $d>0$. Let $\phi:\mathbb{Z}\to\mathbb{Z}_q$ be the canonical homomorphism of $\mathbb{Z}$ onto $\mathbb{Z}_q$, and $\sigma:\mathbb{Z}_q\to\mathbb{Z}_q/H$ the canonical homomorphism. Let $\mathcal{A}=\sigma(\phi A)$ and $\mathcal{B}=\sigma(\phi B)$. If $d=1$ then $\overline{A}+\overline{B}=\mathbb{Z}_q$. In the following we assume that $d>1$. By Lemmas 1 and 2 we have

$$\begin{aligned}
&|\{c\in A+B:\phi c\in \overline{B}+H\}| - |\{\overline{c}\in \overline{A}+\overline{B}:\overline{c}\in \overline{B}+H\}| \\
&\quad \ge |\{c\in A_1+B:\phi c\in \overline{B}+H\}| - |\{\overline{c}\in \overline{A}_1+\overline{B}:\overline{c}\in \overline{B}+H\}| \\
&\quad = |\{c\in A_1+B\}| - |\{\overline{c}\in \overline{A}_1+\overline{B}\}| \ge |B|+(|A_1|-2)|\overline{B}|.
\end{aligned}$$

As in [5], we may derive that $\mathcal{A}=\{0\}$, that is, $\overline{A}\subseteq H$. So $|\overline{A}|\le|H|=q/d$, that is, $q\ge d|\overline{A}|$. We also have

$$|\overline{B}| \le |\overline{B}+H| = |\mathcal{B}|\cdot|H| = |\mathcal{B}|q/d.$$

If $q\le|\overline{A}|+|\overline{B}|-1$, then

$$q \le |\overline{A}| + |\mathcal{B}|\frac{q}{d} - 1.$$

That is, $q(d-|\mathcal{B}|)\le d(|\overline{A}|-1)$. So $d|\overline{A}|(d-|\mathcal{B}|)\le d(|\overline{A}|-1)$. Hence $d=|\mathcal{B}|$. Therefore

$$|\overline{A}+\overline{B}| = |\overline{A}+\overline{B}+H| = |\overline{B}+H| = |\mathcal{B}|\frac{q}{d} = q.$$

Since $\mathcal{A}=\{0\}$, we have $d\,|\,d_A$, whence $d\,|\,(d_A,q)$. If $q\ge|\overline{A}|+|\overline{B}|$ and $(d_A,d_B,q)=1$, from $d>1$ and $\mathcal{A}=\{0\}$ we have $\mathcal{B}\ne\{0\}$, whence $|\mathcal{B}|\ge 2$. Hence

$$|\overline{A}+\overline{B}| = |\overline{A}+\overline{B}+H| \ge |\overline{B}+H| = |\mathcal{B}|\frac{q}{d} \ge \frac{2q}{d} \ge \frac{2q}{(d_A,q)}.$$

This completes the proof of Theorem 4.

Remark. Let $(d_A,d_B,q)=1$. By the same method we may prove that

$$\begin{aligned}
|A+B| \ge\ & \min\{q, |\overline{A}|+|\overline{B}|-1\} \\
&+ \min\{|B|+(\max_i|A_i|-2)|\overline{B}|, |A|+(\max_i|B_i|-2)|\overline{A}|\}.
\end{aligned}$$

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