ACTA ARITHMETICA
LXXX.1 (1997)

On Waring's problem with quartic polynomial summands

by

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1. Introduction. Let a quartic integral-valued polynomial be represented by (cf. [8, Section 1])

$$f(x) = a_4F_4(x) + a_3F_3(x) + a_2F_2(x) + a_1F_1(x), \tag{1.1}$$

where a_i $(1 \le i \le 4)$ are integers with $(a_1, a_2, a_3, a_4) = 1$ and $a_4 > 0$, and

$$F_i(x) = \frac{1}{i!}x(x-1)\dots(x-i+1) \qquad (1 \le i \le 4). \tag{1.2}$$

Let $G(f(x))$ be the least s such that the equation

$$f(x_1) + \dots + f(x_s) = n, \qquad x_i \ge 0, \tag{1.3}$$

is solvable for all sufficiently large integers n, and let $\mathfrak{S}^*(f(x))$ be the least number such that if $s \ge \mathfrak{S}^*(f(x))$, then $\mathfrak{S}_s(n)$ the singular series corresponding to the equation (1.3) (see [2]) satisfies $\mathfrak{S}_s(n) \ge c > 0$ for some c, independent of n. In [8] we have proved, among other things, that $\mathfrak{S}^*(f(x)) \le 16$ and $G(f(x)) \le 16$, and both equalities hold whenever $f(x)$ satisfies that

$$2 \nmid f(1) \quad \text{and} \quad f(x) \equiv f(1)x^4 \pmod{2^5} \qquad \text{for all } x. \tag{1.4}$$

In this paper we prove the following more precise result.

Theorem 1. *If $f(x)$ does not satisfy* (1.4), *then* $\max_f \mathfrak{S}^*(f(x)) = 11$.

Moreover, we define $G^*(f(x))$ to be the least number such that if $s \ge G^*(f(x))$ and if $\mathfrak{S}_s(n) \ge c > 0$, then the equation (1.3) has solutions for all sufficiently large integers n.

Theorem 2. *We have $G^*(f(x)) \le 13$.*

Combining this with Theorem 1 and (2.3) below we have

Corollary 1. *If $f(x)$ does not satisfy* (1.4), *then*

$$G(f(x)) \le 13 \quad \textit{and} \quad \max_f G(f(x)) \ge 11.$$

Project supported by the National Natural Science Foundation of China.

The proof of Theorem 1 (see Sections 2 and 3) will present a new difficulty, which does not arise in [4, 8]. Hence our argument has certain features of interest. Theorem 2 is a generalization of Theorem 2 of Vaughan [6]. This may be compared with the upper bound $G^*(f(x)) \le 14$ of Theorem 1A of [8], which follows from Davenport's iteration method. The proof of Theorem 2 can be completed by following the lines of Vaughan's argument in [6] with some modifications; the details will be omitted.

It would be more interesting, as the referee comments, if Theorem 1.2 of Vaughan [7] were generalized to the case of quartic polynomials.

2. Preliminaries to the proof of Theorem 1. Let d be the least common denominator of the coefficients of $f(x)$. Then $d \mid 4!$ (see (1.1) and (1.2)). For each prime p, we define $t = t(p)$ by $p^t \parallel d$, and write $\varphi(x) = p^t f(x)$. Let $\theta^{(i)}$ be the greatest integer such that the ith derivative of $\varphi(x)$ satisfies $\varphi^{(i)}(x) \equiv 0 \pmod{p^{\theta^{(i)}}}$ for all x, and let $f^*(x) = p^{-\theta'}\varphi(x)$. Let $\delta = \max_{1\le i\le 3}(\theta^{(i)} - \theta^{(i+1)})$, and let

$$\gamma = \begin{cases} \theta' - t + \delta + 2 & \text{for } p = 2,\\ \theta' - t + \delta + 1 & \text{for } p > 2. \end{cases}$$

Further, let $M_s(p^l, n)$ denote the number of solutions of

$$f(x_1) + \ldots + f(x_s) \equiv n \pmod{p^l}, \quad 0 \le x_i < p^{l+t}, \tag{2.1}$$

and let $\Gamma(f(x), p^l)$ be the least value of s for which the congruence (2.1) has a solution for every n. From Hua [2, Section 7] we see that in order to establish Theorem 1, it will suffice to prove the following results:

If $f(x)$ does not satisfy (1.4), then

$$M_{11}(p^l, n) \ge p^{10(l-8)} \quad \text{for all } n \text{ and } l \ge 8, \tag{2.2}$$

and

$$\max_f \Gamma(f(x), 2^\gamma) = 11. \tag{2.3}$$

Since a direct treatment of $M_s(p^l, n)$ presents certain technical difficulties, we define $N_s(p^l, n)$ as the number of solutions of the congruence (2.1) with not all $f^*(x_i)$'s divisible by p. Then we have (see [2, 3, 5])

$$N_s(p^l, n) = p^{(l-\gamma)(s-1)} N_s(p^\gamma, n) \quad \text{for } l \ge \gamma, \tag{2.4}$$

which is a version of Hensel's Lemma (cf. Theorem 3 of Borevich and Shafarevich [1, Chapter 1, §5.2]). Moreover, for each given n we define $\Gamma_n^*(f(x), p^\gamma)$ to be the least s such that $N_s(p^\gamma, n) \ge 1$. Let $\Gamma^*(f(x), p^\gamma) = \max_n \Gamma_n^*(f(x), p^\gamma)$. It is easily seen from the definition that

$$\Gamma(f(x), p^\gamma) \le \Gamma^*(f(x), p^\gamma) \le \Gamma(f(x), p^\gamma) + 1. \tag{2.5}$$

Lemma 2.1. *The inequality* (2.2) *holds in the following cases*:

(i) $p \geq 3$;

(ii) $p = 2$, *when* $t > 0$ *or* $t = 0$, $0 \leq \theta' \leq 2$ *and* $f(x)$ *does not satisfy* (1.4).

Proof. When $p \geq 5$ we have $\gamma \leq 1$ and $\Gamma^*(f(x), p^\gamma) \leq 8$ (see Hua [3, Lemma 2.3]). Moreover, by the arguments similar to that used in [8, Sections 4 and 5], we have $\Gamma^*(f(x), 3^\gamma) \leq 11$ with $\gamma \leq 3$; and, under the hypothesis of (ii), $\gamma \leq 5$ and $\Gamma^*(f(x), 2^\gamma) \leq 9$. The lemma now follows at once from (2.4) and the obvious inequality $M_s(p^l, n) \geq N_s(p^l, n)$.

We note that if $p = 2$ and $t = 0$ then $0 \leq \theta' \leq 3$ by Lemma 2.4 of [8]. Therefore, in view of Lemma 2.1, to complete the proof of Theorem 1 it suffices to prove (2.2) and (2.3) in the case

$$p = 2, \quad t = 0 \quad \text{and} \quad \theta' = 3. \tag{2.6}$$

Here, however, one is faced with a difficulty that there exist some classes of $f(x)$ such that $\Gamma_n^*(f(x), 2^\gamma) = 12$ for some n (see the proof of Lemma 3.2 below), and thus in these situations the above argument (using (2.4)) fails to provide a proof of (2.2) for $p = 2$. In order to overcome this difficulty the crucial step is to establish Lemma 3.1 below, which is in fact a new version of (2.4).

3. The proof of Theorem 1. In this section we will assume (2.6) and use the notation introduced in Sections 1 and 2 without further reference.

As in [8], we write

$$\frac{a_i}{i!} \equiv b_i \pmod{2^\gamma}, \qquad i = 2, 3, 4. \tag{3.1}$$

Now a_1 must be odd; and we may assume that $a_1 = 1$ (see the beginning of [8, Section 3]). From (2.6), (3.1) and [8, (2.5)] we deduce that

$$2 \nmid b_4, \quad b_2 \equiv -1 \pmod{2^2} \quad \text{and} \quad b_3 \equiv 2 \pmod{2^3}, \tag{3.2}$$

which, together with Lemma 2.4 of [8], gives

$$\theta'' = 2, \quad \theta''' = 3 \quad \text{and} \quad \gamma = 6. \tag{3.3}$$

Furthermore, by (3.2), (3.3), Taylor's expansion and Lemma 2.4 of [8], we have, for any integers x, y and $m \geq 1$,

$$f'(x + 2^m y) - f'(x) \equiv 2^{m+1}(2x - b_4 + 1)y \pmod{2^{m+3}}. \tag{3.4}$$

Lemma 3.1. *Suppose* $l \geq 8$ $(= \gamma + 2)$. *Let* $M_s'(2^l, n)$ *denote the number of solutions of the congruence*

$$f(x_1) + \ldots + f(x_s) \equiv n \pmod{2^l} \tag{3.5}$$

with $0 \leq x_i < 2^{l-\theta'-1}$, $2 \mid f^*(x_i)$ $(1 \leq i \leq s)$ *and* $2 \parallel f^*(x_1)$. *Then*

$$M_s'(2^l, n) \geq 2^{(l-8)(s-1)} M_s'(2^8, n). \tag{3.6}$$

P r o o f. The truth of (3.6) is obvious when $l = 8$. We proceed by induction on l and, accordingly, assume that $l > 8$ and that (3.6) is true with l replaced by $l-1$. We first observe that each x_i with $0 \le x_i < 2^{l-\theta'-1}$ ($1 \le i \le s$) can be uniquely written in the form

$$x_i = y_i + 2^{l-\theta'-2} z_i \quad \text{with } 0 \le y_i < 2^{l-\theta'-2} \text{ and } 0 \le z_i < 2. \tag{3.7}$$

Then, by using Taylor's expansion, (3.3) and $l \ge 9$, (3.5) becomes

$$\sum_{i=1}^{s} f(y_i) + \sum_{i=1}^{s} f^*(y_i) 2^{l-2} z_i \equiv n \pmod{2^l}. \tag{3.8}$$

Moreover, there are $M_s'(2^{l-1}, n)$ s-tuples $(y_1, \ldots, y_s)$ satisfying $f^*(y_i) = 2t_i$ with integral t_i ($1 \le i \le s$) and $2 \nmid t_1$, such that $\sum_{i=1}^{s} f(y_i) - n = 2^{l-1} A$ for some integral A. Hence (3.8) reduces to

$$\sum_{i=1}^{s} t_i z_i + A \equiv 0 \pmod{2}. \tag{3.9}$$

Then, since $2 \nmid t_1, z_i = 0$ or 1 ($i = 2, \ldots, s$) may be chosen arbitrarily in (3.9) and $z_1 = 0$ or 1 is uniquely determined. Also, by (3.4), (3.7) and $l \ge 9$, $f'(x_i) \equiv f'(y_i) \pmod{2^5}$. Therefore, by the induction hypothesis, we have $2 \mid f^*(x_i)$ ($1 \le i \le s$) and $2 \parallel f^*(x_1)$, and so

$$M_s'(2^l, n) \ge 2^{(s-1)} M_s'(2^{l-1}, n) \ge 2^{(l-8)(s-1)} M_s'(2^8, n).$$

This completes the proof of the lemma.

We are now in a position to prove the following result, and thus complete the proof of Theorem 1 (cf. the remark at the end of Section 2).

LEMMA 3.2. *Subject to* (2.6), (2.2) *and* (2.3) *hold.*

P r o o f. We proceed by considering separately the cases $b_4 \equiv -1 \pmod 4$ and $b_4 \equiv 1 \pmod 4$.

(I) $b_4 \equiv -1 \pmod 4$. Then, by (3.1), (3.2) and [8, (2.5)], $b_2 \equiv 3 \pmod{2^3}$. Thus

$$f(2) \equiv 2^3, \quad f(3) \equiv 1 \pmod{2^4} \tag{3.10}$$

and (by using [8, (2.6)])

$$f''(0) \equiv 2^2, \quad f''(2) \equiv 2^2 \pmod{2^3}. \tag{3.11}$$

(i) Suppose first $2 \mid f^*(0)$. Then $2 \nmid f^*(2)$ by (3.4), and $f(4) \equiv 2^5 \pmod{2^6}$ by $(3.11)_1$ and Taylor's expansion. Recall that $\gamma = 6$ and $f(1) = a_1 = 1$. It can be verified that if either $n \not\equiv 2^3 - 1 \pmod{2^6}$ or $2 \nmid f^*(1)$ or $f(3) \not\equiv 1 \pmod{2^6}$ then $\Gamma_n^*(f(x), 2^\gamma) \le 11$, and so (2.2) holds in all these cases (cf. the proof of Lemma 2.1); otherwise $\Gamma_n^*(f(x), 2^\gamma) = 12$ and $\Gamma_n(f(x), 2^\gamma) = 2^3 - 1$.

Therefore, in particular, if $f(x)$ satisfies further

$$2 \mid f^*(1) \quad \text{and} \quad f(3) \equiv 1 \pmod{2^6}, \tag{3.12}$$

then $\Gamma^*(f(x), 2^\gamma) = 12$ and $\Gamma(f(x), 2^\gamma) \le 11$. This, together with (2.5), gives (2.3). Now, in view of Lemma 3.1, to prove the lemma in case (i) it will suffice to verify that $M'_{11}(2^8, n) > 0$ for $n \equiv 2^3 - 1 \pmod{2^6}$, subject to the additional condition (3.12).

For this purpose, we first note that, by (3.3), (3.12) and Taylor's expansion, $f''(1) \equiv 2^3 \pmod{2^4}$. Thus

$$\begin{cases} f(5) \equiv 1 + 2^6 \pmod{2^7} & \text{if } 2^2 \mid f^*(1), \\ f(9) \equiv 1 + 2^7 \pmod{2^8} & \text{if } 2 \parallel f^*(1). \end{cases} \tag{3.13}$$

Moreover, from $2 \mid f^*(1)$ and $2 \mid f^*(0)$ we deduce

$$2 \nmid f^*(x) \text{ if } x \equiv 2 \pmod 4 \quad \text{and} \quad 2 \mid f^*(x) \text{ if } x \not\equiv 2 \pmod 4 \tag{3.14}$$

by (3.4) and

$$b_2 \equiv -5 \pmod{2^4}, \quad \text{i.e.} \quad f(2) \equiv 2^3 + 2^4 \pmod{2^5} \tag{3.15}$$

by using [8, (2.5)]. Similar to the above, we conclude from (3.15) that $f''(0) \equiv 2^2 \pmod{2^4}$. This, together with (3.4), gives

$$\begin{cases} 2 \parallel f^*(4) \text{ and } f(4) \equiv 2^5 \pmod{2^7} & \text{if } 2^2 \mid f^*(0), \\ f(4) \equiv 2^5 + 2^6 \pmod{2^7} & \text{if } 2 \parallel f^*(0). \end{cases} \tag{3.16}$$

By (3.13), (3.14) and (3.16), it can now be verified that $M'_{11}(2^8, n) > 0$ for $n \equiv 2^3 - 1 \pmod{2^6}$.

(ii) Suppose next $2 \nmid f^*(0)$. Similar to case (i), we have $2 \mid f^*(2)$ and so $f(6) \equiv f(2) + 2^5 \pmod{2^6}$ by $(3.11)_2$. Combining this with (3.10) it can be verified that $\Gamma_n^*(f(x), 2^\gamma) \le 11$ unless $n \equiv 2^5 + 2^3 - 1 \pmod{2^6}$ and (3.12) holds, in which case $\Gamma_n^*(f(x), 2^\gamma) = 12$. In the latter case, we will verify that $M'_{11}(2^8, n) > 0$, and the lemma thus follows.

In fact, from $2 \mid f^*(1)$ and $2 \nmid f^*(0)$ we have

$$2 \nmid f^*(x) \text{ if } x \equiv 0 \pmod 4 \quad \text{and} \quad 2 \mid f^*(x) \text{ if } x \not\equiv 0 \pmod 4 \tag{3.17}$$

and

$$f(2) \equiv 2^3 \pmod{2^5}. \tag{3.18}$$

From (3.18) we have $f''(2) \equiv 2^2 \pmod{2^4}$. Hence, in analogy to (3.16),

$$\begin{cases} 2 \parallel f^*(6) \text{ and } f(6) \equiv f(2) + 2^5 \pmod{2^7} & \text{if } 2^2 \mid f^*(2), \\ f(6) \equiv f(2) + 2^5 + 2^6 \pmod{2^7} & \text{if } 2 \parallel f^*(2). \end{cases} \tag{3.19}$$

By (3.13), (3.17)–(3.19), the desired result can be verified directly.

(II) $b_4 \equiv 1 \pmod 4$. In this case we proceed similarly, so that we give a brief sketch only. First we have

$$f(2) \equiv 0, \quad f(3) \equiv 1 + 2^3 \pmod{2^4} \tag{3.20}$$

and

$$f''(1) \equiv 2^2, \quad f''(3) \equiv 2^2 \pmod{2^3}. \tag{3.21}$$

If $2 \mid f^*(1)$, then $2 \nmid f^*(3)$ and $f(5) \equiv 1 + 2^5 \pmod{2^6}$. From this and (3.20) it can be seen that if either $n \not\equiv 2^2 \pmod{2^6}$ or $f(x)$ does not satisfy

$$2 \mid f^*(0) \quad \text{and} \quad f(2) \equiv 0 \pmod{2^6} \tag{3.22}$$

then $\Gamma_n^*(f(x), 2^\gamma) \le 11$; otherwise $\Gamma_n^*(f(x), 2^\gamma) = 12$ and $M'_{11}(2^8, n) > 0$. Thus the lemma follows.

If $2 \nmid f^*(1)$, then $2 \mid f^*(3)$ and $f(7) \equiv f(3) + 2^5 \pmod{2^6}$. Similarly, we have that if either $n \not\equiv 2^5 + 2^2 \pmod{2^6}$ or $f(x)$ does not satisfy (3.22) then $\Gamma_n^*(f(x), 2^\gamma) \le 11$; otherwise $\Gamma_n^*(f(x), 2^\gamma) = 12$ and $M'_{11}(2^8, n) > 0$. The lemma also follows.

The proof of Lemma 3.2, and of Theorem 1 is now complete.

Acknowledgements. The author thanks Professor M. G. Lu for suggesting this problem and Mr. G. Yu for useful discussion. He is also grateful to the referee for the helpful comments.

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Received on 26.7.1996
and in revised form on 21.10.1996 (3029)