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On the Fourier coefficients of triangle functions *

by

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1. Introduction. Consider the class of properly discontinuous groups $\{G_q\}$, q integer, $q \geqslant 3$. Any G_q is generated by the two substitutions:

$$S(\tau) = \tau + \lambda_q, \quad T(\tau) = -1/\tau, \quad \text{where} \quad \lambda_q = 2\cos(\pi/q), \tag{1}$$

and $\operatorname{Im}(\tau) > 0$. To each G_q belongs an invariant, $J_q(\tau)$, such that $J_q(S(\tau)) = J_q(T(\tau)) = J_q(\tau)$. The functions $J_q(\tau)$ are automorphic with respect to G_q. Their general properties were studied by Hecke [3]. It is known that $w = J_q(\tau)$ maps a "triangle" of vertices $-\exp(-\pi i/q)$, i and $i\infty$ on the upper w-plane in such manner that

$$J_q(-\exp(-\pi i/q)) = 0, \quad J_q(i) = 1, \quad J_q(i\infty) = \infty. \tag{2}$$

Moreover, $J_q(\tau)$ has a Fourier expansion, valid at $\tau = i\infty$;

$$J_q(\tau) = \sum_{n=-1}^{\infty} a_n(q) x_q^n, \quad \text{where} \quad x_q = \exp(2\pi i\tau/\lambda_q). \tag{3}$$

Here $J_3(\tau)$ is the well known modular invariant. Its coefficients $a_n(3)$ are known up to $n = 100$ in closed form (cf. [10]) and also for any n as convergent series (cf. [7]). $J_4(\tau)$ and $J_6(\tau)$ were discussed in [8] and they are algebraically related to $J_3(\tau)$. Finally $J_\infty(\tau)$ belongs to a subgroup of the modular group, with the relation:

$$27J_3 \cdot J_\infty = (4J_\infty - 1)^3.$$

Such algebraic relations permit an immediate computation of the first few coefficients of J_4, J_6, J_∞ from those of $J_3(\tau)$:

$$\begin{aligned} 12^3 J_3(\tau) &= x^{-1} + 2^3\cdot 3\cdot 31 + 2^2\cdot 3^3\cdot 1823x + 2^{11}\cdot 5\cdot 2099x^2 + \\ &\qquad + 2\cdot 3^5\cdot 5\cdot 355.679x^3 + \ldots, \\ 2^8 J_4(\tau) &= x^{-1} + 2^3\cdot 13 + 2^2\cdot 1093x + 2^{11}\cdot 47x^2 + 2\cdot 3^3\cdot 22.963x^3 + \ldots, \\ 2^2\cdot 3^3 J_6(\tau) &= x^{-1} + 2\cdot 3\cdot 7 + 3^3\cdot 29x + 2^5\cdot 271x^2 + 3^5\cdot 269x^3 + \ldots, \\ 2^6 J_\infty(\tau) &= x^{-1} + 2^3\cdot 3 + 2^2\cdot 3\cdot 23x + 2^{11}\cdot x^2 + 2\cdot 3\cdot 1867x^3 + \ldots \end{aligned} \tag{4}$$

where, in each case, x stands for x_q, according to (3).

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To date, these are the only four known invariants of the class $\{J_q\}$. As pointed out in [8], we do not have an arithmetic characterization of the remaining groups nor of the singularities of their invariants in a form suitable for calculations.

If we consider the function inverse to $J_q(\tau)$ and write

$$2\pi i\tau/\lambda_q = \Phi_q(J_q)\,, \tag{5}$$

then $\{\Phi_q\}$ is a subclass of inverse triangle functions of Schwarz [9]. An exhaustive presentation of these functions is found in Carathéodory's work, [1].

In [5], J. Lehner shows that, under a particular normalization, the coefficients of the J_q's are rational numbers.

The purpose of this paper is to obtain the residues $a_{-1}(q)$ and the coefficients $a_n(q)$, for $n = 0, 1, 2, 3$, only, for any q, $3 \leqslant q \leqslant \infty$, in closed form. The results are as follows:

$$\text{(I)}\qquad a_{-1}(q) = \exp\{\pi\sec(\pi/q) - 2\psi(1) + \psi(\tfrac{1}{4}+1/2q) + \psi(\tfrac{1}{4}-1/2q)\}$$

where $\psi(u) = \Gamma'(u)/\Gamma(u) =$ logarithmic derivative of $\Gamma(u)$. Formula (I), for q finite, may be written

$$\text{(II)}\quad a_{-1}(q) = 2^{-4+2(-1)^q} q^{-2} \prod_{\nu=1}^{q-1} \exp\left\{2(-1)^\nu \cos\frac{2\nu\pi}{q}\log\left(2-2\cos\frac{\pi\nu}{q}\right)\right\},$$

$$\text{(III)}\begin{cases} a_0(q) = (3q^2+4)2^{-3}q^{-2}, \\ a_1(q) = (69q^4-8q^2-48)2^{-10}q^{-4}(a_{-1})^{-1}, \\ a_2(q) = (27q^6-116q^4+16q^2+64)2^{-7}3^{-3}q^{-6}(a_{-1})^{-2}, \\ a_3(q) = (5601q^8-61.136q^6+166.240q^4-25.856q^2-77.568)2^{-23}q^{-8}(a_{-1})^{-3}. \end{cases}$$

The method is straightforward and perhaps laborious (particularly for $a_3(q)$), but by checking against formulae (4) a control of errors is at hand.

2. The residues $a_{-1}(q)$. For simplicity, instead of $\lambda_q, x_q, J_q, \Phi_q$ we shall write λ, x, J, Φ, respectively. Then the function

$$2\pi i\tau/\lambda = \Phi(J) \tag{5¹}$$

maps the upper J-plane onto a "triangle" in the upper τ-plane with angles π/q, $\pi/2$, 0; the preassigned boundary conditions being given by (2). The formulae for the solution of this mapping problem are given by Carathéodory ([1], p. 166).

We find, except for a bilinear transformation of Carathéodory's formulae:

$$2\pi i\tau/\lambda = -\log J + \frac{F^*(\alpha,\beta,1;J^{-1})}{F(\alpha,\beta,1;J^{-1})} - \pi i\bar{b} \tag{6}$$

where

$$\alpha = \frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right), \qquad \beta = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{q}\right), \tag{7}$$

$$F(\alpha,\beta,1;u) = \sum_{\nu=0}^{\infty} c_\nu u^\nu, \tag{8}$$

$$F^*(\alpha,\beta,1;u) = \sum_{\nu=1}^{\infty} c_\nu e_\nu u^\nu \tag{9}$$

with

$$c_\nu = \frac{\Gamma(\alpha+\nu)\Gamma(\beta+\nu)}{\Gamma(\alpha)\Gamma(\beta)(\nu!)^2}, \qquad e_\nu = \sum_{p=0}^{\nu-1}\left(\frac{1}{\alpha+p}+\frac{1}{\beta+p}-\frac{2}{1+p}\right), \tag{9¹}$$

$$-\pi i\bar{b} = -2\psi(1) + \psi(1-\alpha) + \psi(1-\beta) - \pi\sec(\pi/q)\,. \tag{10}$$

Here $\psi(u) = \Gamma'(u)/\Gamma(u)$. We remark as in [1], pp. 139 & 153, that, since $1-\alpha-\beta = \frac{1}{2}$, series (8) and (9) converge also when $J = 1$.

Let $-\pi i\bar{b} = \log A$. By exponentiation of (6) we have $x = \exp(2\pi i\tau/\lambda)$ $= AJ^{-1}\exp(A_0 J^{-1} + A_1 J^{-2} + \ldots)$ where

$$A_0 = c_1 e_1 = \alpha\beta\left(\frac{1}{\alpha}+\frac{1}{\beta}-2\right) = \frac{3q^2+4}{8q^2}\,. \tag{11}$$

Consequently

$$x/A = J^{-1} + A_0 J^{-2} + \ldots$$

and by inversion

$$J^{-1} = \frac{x}{A}\left(1 - A_0\frac{x}{A} + \ldots\right)$$

whence

$$J = \frac{A}{x} + A_0 + \ldots \tag{12}$$

Comparing (12) with (3) we have:

$$a_{-1}(q) = A = \exp\{-\pi\sec(\pi/q) - 2\psi(1) + \psi(1-\alpha) + \psi(1-\beta)\}\,.$$

Because of (7),

$$\cot\pi\alpha + \cot\pi\beta = 1/(\sin\pi\alpha\sin\pi\beta) = 2\sec(\pi/q)\,.$$

Then, by the formula $\psi(1-x) = \psi(x) + \pi\cot\pi x$, we obtain

$$a_{-1}(q) = \exp\left\{\sec(\pi/q) - 2\psi(1) + \psi\left(\frac{1}{4}+\frac{1}{2q}\right) + \psi\left(\frac{1}{4}-\frac{1}{2q}\right)\right\}$$

as in (I). From (12) and (3) also follows that

$$A_0 = a_0(q) = \frac{3q^2+4}{8q^2},$$

as in (III). Formula (II) is obtained from (I) by using Gauss' representation of $\psi(r)$ where r is rational (see, e.g. [6]).

It is readily verified that formula (I), when $q \to \infty$ gives $a_{-1}(\infty) = 2^{-6}$. From formula (II) we get

$$a_{-1}(3) = 2^{-6}3^{-3}, \quad a_{-1}(4) = 2^{-8}, \quad a_{-1}(6) = 2^{-2}3^{-3}$$

as we expect from (4).

3. The coefficients $a_1(q)$, $a_2(q)$, $a_3(q)$. The role played by the Schwarzian derivative in the mapping of § 2 is well known ([1], Part 7, Chapter II). In our case the function (5¹) satisfies Schwarz's equation:

$$\frac{2\tau'\tau''' - 3(\tau'')^2}{2(\tau')^2} = \frac{1-(1/q^2)}{2J^2} + \frac{1-(1/4)}{2(J-1)^2} + \frac{(1/q^2)+(1/4)-1}{2J(J-1)} \tag{13}$$

where dashes denote differentiation with respect to J. Inverting the order of differentiation in (13) we obtain

$$\frac{3(J'')^2 - 2J'J'''}{(J')^4} = \frac{4q^2J^2 - (5q^2-4)J + 4q^2 - 4}{4q^2J^2(J-1)^2}. \tag{14}$$

Here dashes indicate differentiation with respect to τ (cf. [2], p. 690).

For the purpose of calculation, write (14) as follows:

$$4q^2[3(xJ'')^2 - 2(xJ')(xJ''')](xJ)^2(xJ-x)^2 = (xJ')^4[4q^2(xJ)^2 - (5q^2-4)(xJ)x + (4q^2-4)x^2] \tag{15}$$

where $x = e^{\mu\tau}$, $\mu = 2\pi i/\lambda$, and

$$\begin{aligned} xJ &= a_{-1} + \sum_{n=0}^{\infty} a_n x^{n+1}, & xJ'' &= \mu^2\Big(a_{-1} + \sum_{n=1}^{\infty} n^2 a_n x^{n+1}\Big), \\ xJ' &= \mu\Big(-a_{-1} + \sum_{n=1}^{\infty} n a_n x^{n+1}\Big), & xJ''' &= \mu^3\Big(-a_{-1} + \sum_{n=1}^{\infty} n^3 a_n x^{n+1}\Big). \end{aligned} \tag{16}$$

By substituting formulae (16) in the differential equation (15) and comparing coefficients, formulae (III) are obtained in the usual manner.

4. Remarks. Clearly the above results (I), (II) and (III) are only the first stages of an unsolved problem. Hence remarks, questions and conjectures of varied levels are in order.

(A) As easily verified from (III),

Numerator of $a_2(q) = (27q^4 - 8q^2 - 16)(q^2 - 4)$,

Numerator of $a_3(q) = (5601q^6 - 38.732q^4 + 11.312q^2 + 19.392)(q^2 - 4)$.

For $q = 3$, such factorization confirms the result of D. H. Lehmer [4], that is:

$$C_n \equiv 0 \ (\mathrm{mod}\, 5) \quad \text{if} \quad n \equiv \pm 2 \ (\mathrm{mod}\, 5),$$

where C_n is the Fourier coefficient of $12^3 J_3(\tau)$. It seems reasonable to conjecture that

$$\text{Numerators of } a_n(q) \equiv 0 \ \big(\mathrm{mod}\,(q^2-4)\big) \quad \text{if} \quad n \equiv \pm 2 \ (\mathrm{mod}\, 5).$$

(B) Keeping present the shape of the coefficients (III) let us set $x_q = a_{-1}(q) \cdot q^2 Z_q$. Then from (3)

$$q^2 J_q^*(Z_q) = 1/Z_q + \sum_{n=0}^{\infty} A_n(q) Z_q^n$$

where $A_n(q) = P_{n+1}(q^2)/R_{n+1}$. Here P_{n+1} denotes a polynomial of degree $(n+1)$ in q^2 and R_{n+1} is an integer. Such an expansion would agree with the result of J. Lehner [5].

(C) Using formula (II) we find:

$$a_{-1}(5) = \frac{\sqrt{5}(2+\sqrt{5})^{\sqrt{5}}}{2^6 5^3}, \quad a_{-1}(8) = \frac{(3+2\sqrt{2})^{\sqrt{2}}}{2^{10}},$$

$$a_{-1}(10) = \frac{\sqrt{5}}{2^2 5^3}\left(\frac{1+\sqrt{5}}{2}\right)^{\sqrt{5}}.$$

These numbers are transcendental. The following question arises: Are there values of q, except those of formulae (4), for which $a_{-1}(q)$ is rational or algebraic?

(D) (Pisot's) Besides the four cases of (4) are there other J_q's such that $K_q \cdot J_q(\tau)$ have integral coefficients, with K_q = constant?

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