U such that $T[U] = \begin{pmatrix} T_1^{(l)} & 0 \\ 0 & 0 \end{pmatrix}$. Then the number of (rational) integral representations of T by S is the same as of T_1 by S and so the corresponding formula in [8] (Theorem 5) was easier to prove. For k, we can not always reduce T to this form by a unimodular matrix over k, since the class number of k is greater than 1, in general.

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Contributions to the theory of the distribution of prime numbers in arithmetical progressions III

by

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1. Continuing the research of [1] and [2] I shall prove in this paper some results concerning the distribution of primes $\equiv l_1 \pmod k$ in comparison with those $\equiv l_2 \pmod k$. Once more I shall need the conjecture

(1.1) *In the rectangle* $0<\sigma<1$, $|t| \leqslant \max(c_1, k^7)$, $s=\sigma+it$, *all L-functions* mod k *may vanish only at points of the line* $\sigma = \frac{1}{2}$ (¹).

Writing, as usually,

$$\pi(x,k,l) = \sum_{\substack{p\equiv l (\mathrm{mod}\, k)\\ p\leqslant x}} 1, \quad p \text{ primes},$$

we shall establish the following

Theorem. *Let* $k \geqslant 3$, $0<l_1$, $l_2<k$, $l_1 \neq l_2$, $(l_1,k)=(l_2,k)=1$ *and suppose* (1.1) *to be satisfied. Then*

$$\int_X^T \frac{|\pi(x,k,l_1)-\pi(x,k,l_2)|}{x}\,dx > T^{1/2}\exp\left(-7\frac{\log T}{\log\log T}\right) \tag{1.2}$$

with

$$X = T\exp\left(-(\log T)^{3/4}\right)$$

for

$$T \geqslant \max(c_2, e^{e^k}) \text{ (²)}. \tag{1.3}$$

Remark. In the particular case of $l_1 = 1$ one might prove a similar inequality without assuming (1.1). However, for general l_1, l_2 I have not been able to supply any lower bound (e.g. $T^{1/4}$, as it used to be in the investigation of $\psi(x,k,l_1)-\psi(x,k,l_2)$ performed in [2]) for

$$\int_X^T \frac{|\pi(x,k,l_1)-\pi(x,k,l_2)|}{x}\,dx$$

(¹) c_1 and further $c_2, c_3, \ldots$ stand for positive numerical constants throughout.

(²) Compare the similar, though weaker, Theorem 3 of [2].

or even for

$$\max_{1\leqslant x\leqslant T} |\pi(x, k, l_1)-\pi(x, k, l_2)|,$$

when conjecturing nothing concerning L-zeros.

2. Proof of this Theorem will base on the following two lemmas (for proofs see [4], p. 52, [1], p. 419 and [2], p. 327).

LEMMA 1. *Let m be a non-negative number and $z_1, z_2, \ldots, z_N$ complex numbers such that*

$$1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_h| \geqslant \ldots \geqslant |z_N|, \qquad |z_h| > 2\frac{N}{m+N}.$$

Then there exists an integer ν with $m \leqslant \nu \leqslant m+N$ such that

$$\text{(2.1)} \qquad \frac{|b_1z_1^\nu+b_2z_2^\nu+\ldots+b_Nz_N^\nu|}{(\frac{1}{2}|z_h|)^\nu} \geqslant \min_{h\leqslant j<h_1} |b_1+b_2+\ldots+b_j| \left(\frac{1}{24e}\cdot\frac{N}{2N+m}\right)^N,$$

where $h_1 \leqslant N$ is any integer for which $|z_{h_1}| < |z_h| - \dfrac{N}{m+N}$. In the case when there do not exist numbers h_1 satisfying the latter inequality, we put at the right-hand side of (2.1) $\min\limits_{h\leqslant j\leqslant N} |b_1+b_2+\ldots+b_j|$ *instead.*

LEMMA 2. *Let $k \geqslant 3$, $0 < l_1, l_2 < k$, $l_1 \neq l_2$, $(l_1, k) = (l_2, k) = 1$. Suppose* (1.1) *to be satisfied. Then there exists a number D, $\frac{1}{2}\max(c_3, k^3) \leqslant D \leqslant \max(c_3, k)$, such that*

$$\text{(2.2)} \qquad \left|\frac{1}{\varphi(k)}\sum_{(\chi)}(\overline{\chi}(l_1)-\overline{\chi}(l_2))\sum_{\varrho(\chi)} D^\varrho\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2\right| \geqslant c_4 D\log D,$$

where $\psi = 1/3D$, χ runs through all characters mod k *and $\varrho(\chi)$ through the zeros of $L(s, \chi)$ lying in the strip $0 < \sigma < 1$.*

3. Proof of the Theorem. Similarly to [1] and [2] we shall examine only the case of k sufficiently large. Therefore our conjecture (1.1) can be reduced to

$$\text{(3.1)} \qquad \prod_{\chi \bmod k} L(s, \chi) \neq 0 \quad \text{in} \quad \sigma > \tfrac{1}{2}, \quad |t| \leqslant k^7.$$

We introduce the parameters

$$T_1 = \frac{T}{D}e^{-2\psi} \quad (D, \psi \text{ from Lemma 2}), \quad A = 0.2\log\log T_1,$$

$$B = (\log T_1)^{-0.25}, \quad m = \frac{\log T_1}{A+B} - \log^{3/8} T_1(\log\log T_1)^2,$$

r an integer, to be defined later, with

$$\text{(3.2)} \qquad m \leqslant r \leqslant \frac{\log T_1}{A+B} \left(< 5\frac{\log T_1}{\log\log T_1}\right).$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_\lambda$ and $\alpha_1', \alpha_2', \ldots, \alpha_\mu'$ denote all incongruent solutions mod k of the congruences

$$x^2 \equiv l_1 \pmod k, \qquad x^2 \equiv l_2 \pmod k$$

respectively. Put, further,

$$F_{l_1l_2}(s) = \frac{1}{\varphi(k)}\sum_{(\chi)}(\overline{\chi}(l_2)-\overline{\chi}(l_1))\frac{L'}{L}(s,\chi)$$

and start with the integral

$$\text{(3.3)} \quad J_{l_1l_2} = \frac{1}{2\pi i}\int_{(2)}\Bigg\{D^s\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r F_{l_1l_2}(s) - \frac{D^{s/2}}{2}\left(\frac{e^{\psi s/2}-e^{-\psi s/2}}{\psi s}\right)^2\left(e^{As/2}\frac{e^{Bs/2}-e^{-Bs/2}}{Bs}\right)^r \times$$
$$\times\frac{1}{\varphi(k)}\left(\sum_{j=1}^{\mu}\sum_{(\chi)}\overline{\chi}(\alpha_j')\frac{L'}{L}(s,\chi) - \sum_{j=1}^{\lambda}\sum_{(\chi)}\overline{\chi}(\alpha_j)\frac{L'}{L}(s,\chi)\right)\Bigg\}ds.$$

Using the well-known expansion of $\dfrac{L'}{L}(s,\chi)$ and writing

$$\frac{e^z-e^{-z}}{2z} \overset{\text{def}}{=} K(z),$$

we obtain

$$J_{l_1l_2} = \sum_{n\equiv l_1 (\mathrm{mod}\, k)}\frac{\Lambda(n)}{2\pi i}\int_{(2)}\frac{D^se^{Ars}}{n^s}K^2(\psi s)K^r(Bs)ds -$$
$$-\sum_{n\equiv l_2 (\mathrm{mod}\, k)}\frac{\Lambda(n)}{2\pi i}\int_{(2)}\frac{D^se^{Ars}}{n^s}K^2(\psi s)K^r(Bs)ds -$$
$$-\sum_{j=1}^{\lambda}\sum_{n\equiv \alpha_j (\mathrm{mod}\, k)}\frac{\Lambda(n)}{2\pi i}\int_{(2)}\frac{D^{s/2}e^{Ars/2}}{2n^s}K^2(\psi s/2)K^r(Bs/2)ds +$$
$$+\sum_{j=1}^{\mu}\sum_{n\equiv \alpha_j' (\mathrm{mod}\, k)}\frac{\Lambda(n)}{2\pi i}\int_{(2)}\frac{D^{s/2}e^{Ars/2}}{2n^s}K^2(\psi s/2)K^r(Bs/2)ds.$$

We note that the first two integrals in the above formula disappear if n is outside of the interval

$$(X_1 \overset{\text{def}}{=})\ De^{-2\psi}e^{(A-B)r} < n < De^{2\psi}e^{(A+B)r}\ (\overset{\text{def}}{=} X_2)$$

and similarly do the remaining integrals if n is outside of

$$X_1^{1/2} < n < X_2^{1/2}.$$

The contribution of $n = p^3, p^4, \ldots$ to the sums $\sum\limits_{n\equiv l_1 (\mathrm{mod}\, k)}$, $\sum\limits_{n\equiv l_2 (\mathrm{mod}\, k)}$ and of $n = p^2, p^3, \ldots$ to $\sum\limits_{j=1}^{\lambda} \sum\limits_{n\equiv a_j (\mathrm{mod}\, k)}$, $\sum\limits_{j=1}^{\mu} \sum\limits_{n\equiv a_j' (\mathrm{mod}\, k)}$, as easy to see, does not exceed $c_5 T^{0.4}$.
Hence we have

$$J_{l_1 l_2} = \sum_{\substack{p\equiv l_1 (\mathrm{mod}\, k)\\ X_1\leqslant p\leqslant X_2}} \frac{\log p}{2\pi i} \int\limits_{(2)} \frac{D^s e^{Ars}}{p^s} K^2(\psi s) K^r(Bs)\, ds + $$
$$+ \sum_{\substack{p^2\equiv l_1 (\mathrm{mod}\, k)\\ X_1\leqslant p^2\leqslant X_2}} \frac{\log p}{2\pi i} \int\limits_{(2)} \frac{D^s e^{Ars}}{p^{2s}} K^2(\psi s) K^r(Bs)\, ds - $$
$$- \sum_{\substack{p\equiv l_2 (\mathrm{mod}\, k)\\ X_1\leqslant p\leqslant X_2}} \frac{\log p}{2\pi i} \int\limits_{(2)} \frac{D^s e^{Ars}}{p^s} K^2(\psi s) K^r(Bs)\, ds - $$
$$- \sum_{\substack{p^2\equiv l_2 (\mathrm{mod}\, k)\\ X_1\leqslant p^2\leqslant X_2}} \frac{\log p}{2\pi i} \int\limits_{(2)} \frac{D^s e^{Ars}}{p^{2s}} K^2(\psi s) K^r(Bs)\, ds - $$
$$- \sum_{j=1}^{\lambda} \sum_{\substack{p\equiv a_j (\mathrm{mod}\, k)\\ X_1^{1/2}\leqslant p\leqslant X_2^{1/2}}} \frac{\log p}{2\pi i} \int\limits_{(2)} \frac{D^{s/2} e^{Ars/2}}{2p^s} K^2(\psi s/2) K^r(Bs/2)\, ds + $$
$$+ \sum_{j=1}^{\mu} \sum_{\substack{p\equiv a_j' (\mathrm{mod}\, k)\\ X_1^{1/2}\leqslant p\leqslant X_2^{1/2}}} \frac{\log p}{2\pi i} \int\limits_{(2)} \frac{D^{s/2} e^{Ars/2}}{2p^s} K^2(\psi s/2) K^r(Bs/2)\, ds + O(T^{0.4}) .$$

We can obviously move the line of integration of the above integrals to $\sigma = 0$ and substitute $s = 2w$ in the last two expressions. This makes the integrals concerned equal to

$$\int\limits_{(0)} \frac{D^w e^{Arw}}{p^{2w}} K^2(\psi w) K^r(Bw)\, dw$$

i.e. equal to the ones occurring under sums $\sum\limits_{\substack{p^2\equiv l_1 (\mathrm{mod}\, k)\\ X_1\leqslant p^2\leqslant X_2}}$ and $\sum\limits_{\substack{p^2\equiv l_2 (\mathrm{mod}\, k)\\ X_1\leqslant p^2\leqslant X_2}}$. Since requirements $p^2 \equiv l_1 \,(\mathrm{mod}\, k)$, $X_1 \leqslant p^2 \leqslant X_2$ and $p \equiv a_j \,(\mathrm{mod}\, k)$, $X_1^{1/2} \leqslant p \leqslant X_2^{1/2}$ (and similarly those involved with l_2 and a_j') are clearly equivalent, we obtain finally

$$(3.4)\qquad J_{l_1 l_2} = \sum_{\substack{p\equiv l_1 (\mathrm{mod}\, k)\\ X_1\leqslant p\leqslant X_2}} \frac{\log p}{2\pi i} \int\limits_{(0)} \frac{D^s e^{Ars}}{p^s} K^2(\psi s) K^r(Bs)\, ds - \sum_{\substack{p\equiv l_2 (\mathrm{mod}\, k)\\ X_1\leqslant p\leqslant X_2}} \frac{\log p}{2\pi i} \times$$
$$\times \int\limits_{(0)} \frac{D^s e^{Ars}}{p^s} K^2(\psi s) K^r(Bs)\, ds + O(T^{0.4}) .$$

Using Stieltjes integral we get

$$J_{l_1 l_2} + O(T^{0.4})$$
$$= \int\limits_{X_1}^{X_2} \left\{ \frac{\log x}{2\pi i} \int\limits_{(0)} \frac{D^s e^{Ars}}{x^s} K^2(\psi s) K^r(Bs)\, ds \right\} d\big(\pi(x, k, l_1) - \pi(x, k, l_2)\big)$$
$$= \left\{ \big(\pi(x, k, l_1) - \pi(x, k, l_2)\big) \frac{\log x}{2\pi i} \int\limits_{(0)} \frac{D^s e^{Ars}}{x^s} K^2(\psi s) K^r(Bs)\, ds \right\}_{X_1}^{X_2} -$$
$$- \int\limits_{X_1}^{X_2} \big(\pi(x, k, l_1) - \pi(x, k, l_2)\big)\, d \left\{ \frac{\log x}{2\pi i} \int\limits_{(0)} \frac{D^s e^{Ars}}{x^s} K^2(\psi s) K^r(Bs)\, ds \right\}$$
$$= \int\limits_{X_1}^{X_2} \big(\pi(x, k, l_1) - \pi(x, k, l_2)\big) \times$$
$$\times \left\{ -\frac{1}{\pi x} \int\limits_0^\infty \cos\big(t(\log D + Ar - \log x)\big) \left(\frac{\sin \psi t}{\psi t}\right)^2 \left(\frac{\sin Bt}{Bt}\right)^r dt + \right.$$
$$\left. + \frac{\log x}{\pi} \int\limits_0^\infty \sin\big(t(\log D + Ar - \log x)\big) \left(-\frac{t}{x}\right) \left(\frac{\sin \psi t}{\psi t}\right)^2 \left(\frac{\sin Bt}{Bt}\right)^r dt \right\} ds .$$

Hence

$$|J_{l_1 l_2}| \leqslant \int\limits_{X_1}^{X_2} \frac{|\pi(x, k, l_1) - \pi(x, k, l_2)|}{x} \log x\, dx \times$$
$$\times \int\limits_0^\infty \frac{t+1}{\pi} \left(\frac{\sin \psi t}{\psi t}\right)^2 \left|\frac{\sin Bt}{Bt}\right|^r dt + c_6 T^{0.4} .$$

Noting that

$$\int\limits_0^\infty \frac{t+1}{\pi} \left(\frac{\sin \psi t}{\psi t}\right)^2 \left|\frac{\sin Bt}{Bt}\right|^r dt \leqslant \frac{1}{\pi} \left(\int\limits_0^\infty t \left|\frac{\sin Bt}{Bt}\right|^r dt + \int\limits_0^\infty \left(\frac{\sin \psi t}{\psi t}\right)^2 dt \right)$$
$$\leqslant \frac{1}{\pi} \left(\frac{1}{B^2} \int\limits_0^\infty \left|\frac{\sin u}{u}\right|^r u\, du + \frac{1}{\psi} \int\limits_0^\infty \left(\frac{\sin u}{u}\right)^2 du \right) < (\log T)^{1/2} ,$$

further, by (3.2), that
$X_2 = De^{(A+B)r+2\psi} \leqslant De^{2\psi} T_1 = T$,

$$X_1 = De^{(A-B)r-2\psi} \geqslant D \exp\big(-2\psi - 2Br + \log T_1 - (A+B) \log^{3/8} T_1 (\log\log T_1)^2\big)$$
$$> T \exp\left(-4\psi - 10 \frac{(\log T_1)^{0.75}}{\log\log T_1} - \log^{3/8} T_1 (\log\log T_1)^3\right) > T \exp\big(-(\log T)^{0.75}\big),$$

we get

$$(3.5)\qquad |J_{l_1l_2}| \leqslant (\log T)^{3/2} \int_X^T \frac{|\pi(x,k,l_1)-\pi(x,k,l_2)|}{x}\,dx + c_6 T^{0.4}$$

with

$$X = T\exp\left(-(\log T)^{0.75}\right).$$

4. As in [1] and [2] we consider the infinite broken line U, lying in

$$\tfrac{1}{30} \leqslant \sigma \leqslant \tfrac{1}{20},$$

and such that

$$\left|\frac{L'}{L}(s,\chi)\right| \leqslant c_7 k \log^2\left(k(|t|+1)\right), \qquad \chi \bmod k,$$

on U.

Applying the theorem of residues to the integral (3.3) we get

$$(4.1)\qquad J_{l_1l_2} = \frac{1}{\varphi(k)} \sum_{(\chi)} \left(\overline{\chi}(l_2)-\overline{\chi}(l_1)\right) \sum_{\varrho=\varrho(\chi)>U} D^{\varrho} e^{A r\varrho} K^2(\psi\varrho) K^r(B\varrho) -$$

$$-\frac{1}{2\varphi(k)} \sum_{j=1}^{\mu} \sum_{(\chi)} \overline{\chi}(\alpha_j') \sum_{\varrho=\varrho(\chi)>U} D^{\varrho/2} e^{A r\varrho/2} K^2(\psi\varrho/2) K^r(B\varrho/2) +$$

$$+\frac{1}{2\varphi(k)} \sum_{j=1}^{\lambda} \sum_{(\chi)} \overline{\chi}(\alpha_j) \sum_{\varrho=\varrho(\chi)>U} D^{\varrho/2} e^{A r\varrho/2} K^2(\psi\varrho/2) K^r(B\varrho/2) +$$

$$+\frac{1}{2} D^{1/2} e^{A r/2} \frac{\mu-\lambda}{\varphi(k)} K^2(\psi/2) K^r(B/2) + O(T^{0.48})$$

($\varrho > U$ means that the ϱ's are to be taken to the right of U). The contribution of the ϱ's with $|\Im\varrho| > Y \overset{\text{def}}{=} \log^{3/8} T_1$ will not exceed $c_8 T^{0.48}$, whence all infinite series in the above formula can be reduced to sums $\sum\limits_{\substack{|\Im\varrho|\leqslant Y\\ \varrho>U}}$. Let $\varrho_1 = \frac{1}{2}+i\gamma_1$ be that zero from $0<\sigma<1$, $|t|\leqslant k^{6.5}$ which has the greatest absolute imaginary part. We have then (see [1], (4.8))

$$(4.2)\qquad |K(B\varrho)| \geqslant |K(B\varrho_1)|$$

for all zeros $\varrho = \frac{1}{2}+i\gamma$, $|\gamma| \leqslant |\gamma_1|-1$. Let, further, $\varrho_2 = \frac{1}{2}+i\gamma_2$ be the zero in $|t| \leqslant 2k^{6.5}$ with maximal γ_1'. Lastly, denoting by E the set of $\varrho = \varrho(\chi) \bmod k$, $|\Im\varrho| \leqslant Y$, $\varrho > U$ plus number $\frac{1}{2}$, we introduce the number $\omega \in E$, $\omega = u_0 + iv_0$ such that

$$(4.3)\qquad \max_{z\in E} |e^{Az} K(Bz)| = |e^{A\omega} K(B\omega)|.$$

Now, with aim to apply lemma 1, we define numbers z_j, b_j. These will be of three categories (indices are chosen so as to have $|z_1| \geqslant |z_2| \geqslant \ldots$).

1. $$z_j = e^{A(\varrho-\omega)} \frac{K(B\varrho)}{K(B\omega)}, \quad |\Im\varrho| \leqslant Y, \quad \varrho > U,$$
$$b_j = \frac{1}{\varphi(k)} \left(\overline{\chi}(l_2)-\overline{\chi}(l_1)\right) D^{\varrho} K^2(\psi\varrho);$$

2. $$z_j = e^{A(\varrho/2-\omega)} \frac{K(B\varrho/2)}{K(B\omega)}, \quad |\Im\varrho| \leqslant Y, \quad \varrho > U,$$
$$b_j = \frac{\pm 1}{2\varphi(k)} \overline{\chi}(\delta) D^{\varrho/2} K^2(\psi\varrho/2),$$

where δ is one of α_ν or α_ν' (and ± 1 is to be taken accordingly);

3. $$z_{j_0} = e^{A(1/2-\omega)} \frac{K(B/2)}{K(B\omega)}, \quad b_{j_0} = \frac{1}{2} D^{1/2} \frac{\mu-\lambda}{\varphi(k)} K^2(\psi/2).$$

With this notation we put formula (4.1) simply as follows

$$(4.4)\qquad J_{l_1l_2} = \left(e^{A\omega} K(B\omega)\right)^r \sum_{j=1}^{N} b_j z_j^r + O(T^{0.48})$$

with

$$N = [\log^{3/8} T_1 (\log\log T_1)^4]$$

(if $N > 1+(1+\lambda+\mu)\sum\limits_{(\chi)} \sum\limits_{|\Im\varrho|\leqslant Y,\, \varrho>U} 1$ we can introduce still another category of z_j's : $z_j = b_j = 0$ for the remaining j's). Finally we define

$$(4.5)\qquad z_h = e^{A(\varrho_1-\omega)} \frac{K(B\varrho_1)}{K(B\omega)}$$

and

$$(4.6)\qquad z_{h_1} = e^{A(\varrho_2-\omega)} \frac{K(B\varrho_2)}{K(B\omega)}.$$

5. Now we shall use lemma 1 and estimate $|J_{l_1l_2}|$ from below. First of all we have

$$|z_h| - |z_{h_1}| = e^{A(1/2-u_0)} \frac{|K(B\varrho_1)| - |K(B\varrho_2)|}{|K(B\omega)|}$$
$$\geqslant e^{-A/2} c_9 \{B^2(\gamma_2^2-\gamma_1^2) + O(B^2 k^{6.5})\} > c_{10} k^{13} e^{-A/2} B^2 = c_{10} (\log T_1)^{-0.6} k^{13}.$$

On the other hand

$$\frac{2N}{N+m} < \frac{2N}{m} < \frac{2\log^{3/8} T_1 (\log\log T_1)^4}{\log T_1/\log\log T_1} = \frac{2(\log\log T_1)^5}{(\log T_1)^{5/8}} < \frac{c_{10} k^{13}}{(\log T_1)^{0.6}},$$

whence

$$(5.1)\qquad |z_h|-|z_{h_1}|>\frac{N}{N+m}$$

(and also $|z_h|>2N/(N+m)$). Now I assert that z_j's of the second category are absolutely less than $|z_{h_1}|$. In other words

$$(5.2)\qquad e^{A\beta/2}|K(B\varrho/2)|<e^{A/2}|K(B\varrho_2)|$$

for $\varrho=\beta+i\gamma$, $\varrho>U$, $|\gamma|\leqslant Y$. Using the well-known inequality (see [3], p. 295)

$$(5.3)\qquad \beta<1-\frac{c_{11}}{\max\{\log k,\log^{3/4}(|\gamma|+3)(\log\log(|\gamma|+3))^{3/4}\}},$$

which, owing to (1.3) and $|\gamma|\leqslant Y$, can be put as

$$(5.4)\qquad \beta<1-\frac{1}{(\log\log T_1)^{0.8}},$$

we obtain

$$e^{A\beta/2}|K(B\varrho/2)|\leqslant c_{12}e^{A\beta/2}\leqslant c_{12}e^{A/2}e^{-(\log\log T_1)^{0.1}},$$

while the right-hand side of (5.2) is

$$>c_{13}e^{A/2}.$$

This proves (5.2). Therefore, and also by (4.2), we have

$$\min_{h\leqslant j<h_1}|b_1+b_2+\ldots+b_j|$$
$$\geqslant\left|\frac{1}{\varphi(k)}\sum_{(\chi)}\big(\bar\chi(l_2)-\bar\chi(l_1)\big)\sum_{|\Im\varrho|\leqslant|\nu_1|-1}D^\varrho\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2\right|-c_{14}\sum_{n\geqslant|\nu_1|-2}\frac{D}{\psi^2}\cdot\frac{\log kn}{n^2}-|b_{j_0}|$$
$$\geqslant\left|\frac{1}{\varphi(k)}\sum_{(\chi)}\big(\bar\chi(l_2)-\bar\chi(l_1)\big)\sum_{\varrho(\chi)}D^\varrho\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2\right|-c_{15}D^3\sum_{n\geqslant\frac12k^{6.5}}\frac{\log kn}{n^2}-c_{16}D^{1/2}$$
$$\geqslant\left|\frac{1}{\varphi(k)}\sum_{(\chi)}\big(\bar\chi(l_2)-\bar\chi(l_1)\big)\sum_{\varrho(\chi)}D^\varrho\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2\right|-c_{17}k^{2.5}\log k.$$

Hence, by lemma 2, (2.2), we get

$$(5.5)\qquad \min_{h\leqslant j<h_1}|b_1+b_2+\ldots+b_j|>c_{18}k^3\log k.$$

Using now lemma 1, (4.4), (4.5), (5.1) and (5.5) we have with an appropriate r

$$(5.6)\qquad |J_{l_1l_2}|\geqslant c_{18}\left(\frac{1}{24e}\cdot\frac{N}{2N+m}\right)^N e^{Ar/2}\left|\frac12K(B\varrho_1)\right|^r+O(T^{0.48}).$$

We obtain further the inequalities

$$e^{Ar/2}\geqslant T_1^{1/2}e^{-(\log T_1)^{0.8}}>T^{1/2}e^{-(\log T)^{0.9}}$$

and

$$\left|\frac12K(B\varrho_1)\right|^r=\left|\frac{e^{B\varrho_1}-e^{-B\varrho_1}}{4B\varrho_1}\right|^r>e^{-r}>e^{-5\frac{\log T}{\log\log T}}$$

which together with the (rough) one

$$\left(\frac{1}{24e}\cdot\frac{N}{2N+m}\right)^N>e^{-(\log T)^{0.5}}$$

clearly convert (5.6) to

$$(5.7)\qquad |J_{l_1l_2}|>T^{1/2}e^{-6\frac{\log T}{\log\log T}}.$$

This and (3.5) prove our assertion (1.2).

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