ACTA ARITHMETICA
VIII (1963)

On oscillations of certain means formed from the Möbius series I

by

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1. Let $\mu(n)$ denote the Möbius function; write

$$M(x) = \sum_{n\leqslant x} \mu(n)\,. \tag{1.1}$$

My earlier papers [1], [2], [4], [5] were concerned with the problem of estimating from below the expressions

$$\max_{1\leqslant x\leqslant T} |M(x)| \tag{1.2}$$

and

$$\int_{X(T)}^{T} \frac{|M(x)|}{x}\,dx \tag{1.3}$$

for T sufficiently large. In the present investigation we shall be interested in oscillatory properties of $M(x)$ and of some related functions. It is easy to show—by the well-known theorem of Landau, by the formula

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\,, \qquad s = \sigma+it\,, \qquad \sigma > 1\,, \tag{1.4}$$

(or rather by the one equivalent to (1.4)

$$\frac{1}{\zeta(s)} = s\int_{1}^{\infty} \frac{M(u)}{u^{s+1}}\,du\,, \qquad s = \sigma+it\,, \qquad \sigma > 1)\,,$$

and by the fact that

$$\zeta(\tfrac{1}{2}) \neq 0\,,$$

—that the function (1.1) changes sign infinitely often as $x\to\infty$. In the present paper the latter result will be put in a more definite form, supposing however the Riemann hypothesis. It will be shown, namely, that under this hypothesis

$$\max_{1\leqslant x\leqslant T} M(x) > T^{\frac{1}{2}-o(1)} \tag{1.5}$$

and

$$\min_{1\leqslant x\leqslant T} M(x) < -T^{\frac{1}{2}-o(1)} \tag{1.6}$$

for $T\to\infty$ (Corollary from Theorem 1). Denoting by $\operatorname{osc}_{[a,b]} M(x)$ the oscillation of $M(x)$ on the interval $[a,b]$, i.e. the difference $\max_{a\leqslant x\leqslant b} M(x) - \min_{a\leqslant x\leqslant b} M(x)$, it follows from (1.5), (1.6) that on Riemann's conjecture

$$\operatorname{osc}_{[1,T]} M(x) > T^{\frac{1}{2}-o(1)}, \quad T\to\infty. \tag{1.7}$$

In the second paper of this work I will improve (1.7) by making the concerned interval $[1,T]$ essentially shorter. Finally, in the third paper, I will deal with the oscillatory properties of the Abel-mean

$$\sum_{n=1}^{\infty} \mu(n) x^n, \tag{1.8}$$

as x tends to 1 from the left.

I wish to stress that unlike in the previous papers [1], [2], [4], [5] I can presently dispense altogether with the assumption of the ζ-zeros being simple.

This investigation is based on the method of P. Turán. We shall use the following two lemmas. Before formulating the first of them (which is a modification of P. Turán's one-sided theorem [8] and whose proof can be found in [6]) I will give some introductory explanations. Let m be an arbitrary non-negative integer and

$$1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_n|;$$

suppose, further, that with a $0<\varkappa\leqslant\pi/2$

$$\varkappa \leqslant |\arg z_j| \leqslant \pi, \quad j=1,2,\ldots,n \tag{1.9}$$

holds and let the index h be such that

$$|z_h| > \frac{4n}{m+n(3+\pi/\varkappa)} \tag{1.10}$$

and fixed. Further, let A and the index h_1 be introduced by

$$A = \min_{h\leqslant\xi<h_1} \Re\mathrm{e} \sum_{j\leqslant\xi} b_j \tag{1.11}$$

if there is an h_1 with

$$|z_{h_1}| < |z_h| - \frac{2n}{m+n(3+\pi/\varkappa)} \tag{1.12}$$

and

$$A = \min_{h\leqslant\xi\leqslant n} \Re\mathrm{e} \sum_{j\leqslant\xi} b_j$$

otherwise. Then we have the

LEMMA 1. *If $A>0$ then there are integers ν_1 and ν_2 with*

$$m+1 \leqslant \nu_1, \quad \nu_2 \leqslant m+n(3+\pi/\varkappa) \tag{1.13}$$

such that

$$\Re\mathrm{e} \sum_{j=1}^{n} b_j z_j^{\nu_1} \geqslant \frac{A}{2n+1}\left(\frac{|z_h|}{2}\right)^{m+n(3+\pi/\varkappa)} \cdot \left\{\frac{n}{24(m+n(3+\pi/\varkappa))}\right\}^{2n} \tag{1.14}$$

and

$$\Re\mathrm{e} \sum_{j=1}^{n} b_j z_j^{\nu_2} \leqslant -\frac{A}{2n+1}\left(\frac{|z_h|}{2}\right)^{m+n(3+\pi/\varkappa)} \cdot \left\{\frac{n}{24(m+n(3+\pi/\varkappa))}\right\}^{2n}. \tag{1.15}$$

The other lemma is the following (for its proof see [3])

LEMMA 2. *Let $\varrho=\sigma_\varrho+it_\varrho$ run through the non-trivial ζ-zeros. For $T>c_1$ there exists a y_1 with* [(1)]

$$\tfrac{1}{13}\log_2 T \leqslant y_1 \leqslant \tfrac{1}{10}\log_2 T \tag{1.16}$$

such that

$$\pi \leqslant \left|\arg \frac{e^{it_\varrho y_1}}{\varrho}\right| \geqslant \frac{c_2}{|t_\varrho|^5 \log|t_\varrho|} \tag{1.17}$$

for all ϱ's.

Now we formulate

THEOREM 1. *Suppose all the ζ-zeros in the rectangle $0<\sigma<1$, $|t|\leqslant\omega$, to lie on the line $\sigma=\frac{1}{2}$. Then we have for*

$$c_3 \leqslant T \leqslant e^{\omega^{10}} \tag{1.18}$$

the inequalities

$$\max_{1\leqslant x\leqslant T} M(x) \geqslant T^{1/2} e_1\left(-15\frac{\log T}{\log_2 T}\log_3 T\right) \tag{1.19}$$

and

$$\min_{1\leqslant x\leqslant T} M(x) \leqslant -T^{1/2} e_1\left(-15\frac{\log T}{\log_2 T}\log_3 T\right) \tag{1.20}$$

(c_3 can be numerically evaluated).

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(1) We use the following notation: $e_1(x)=e^x$, $e_{\nu+1}(x)=e_1(e_\nu(x))$, $\log_1 x=\log x$, $\log_{\nu+1}x=\log_1(\log_\nu x)$; $c_1, c_2, \ldots$ denote positive numerical constants.

Corollary. *On Riemann's conjecture* (1.19), (1.20) *hold for all* T *sufficiently large.*

Remark. It may be noted that in the similar theorem concerning the order of growth of the functions (1.2), (1.3) I conjectured additionally [5] the simplicity of ζ-zeros. A new idea in the proof makes the latter conjecture redundant. I will state it explicitly in the second paper of this work.

Let $\tau(T)$ denote the number of sign changes of the function $M(x)$ for $1 \leqslant x \leqslant T$. It is easy to deduce from Theorem 1 the following

Theorem 2. *Suppose all the* ζ*-zeros in the rectangle*

$$0 < \sigma < 1, \quad |t| \leqslant \omega$$

to lie on the line $\sigma = \frac{1}{2}$. *Then, for*

$$c_4 \leqslant T \leqslant e^{\omega^{10}}$$

we have

$$\tau(T) \geqslant c_5 \log_2 T\,. \tag{1.21}$$

Proof. Owing to the trivial inequality

$$|M(x)| \leqslant x$$

we can reduce the interval $[1, T]$ in (1.19) and (1.20) to $[T^{1/3}, T]$, so that $M(x)$ must at least once change sign in $[T^{1/3}, T]$. Hence (1.21).

It is worth while noticing that the methods used in this paper and its continuation apply not only to a study of functions (1.5), (1.6), (1.7), (1.8) but also to a number of other means for the series $\sum\limits_n \mu(n)$.

2. Proof of Theorem 1. Let

$$\varrho_0 = \tfrac{1}{2} + i\cdot 14.13\ldots \tag{2.1}$$

be the ζ-zero with minimal positive imaginary part (as we know, ϱ_0 is a simple zero). I assert that there exists such an integer r, $0 \leqslant r \leqslant 4$ that

$$\Re\mathrm{e}\,\frac{1}{\zeta'(\varrho_0)\,\varrho_0^r} = c_6 > 0\,. \tag{2.2}$$

In fact, writing

$$\frac{1}{\zeta'(\varrho_0)} = \left|\frac{1}{\zeta'(\varrho_0)}\right|\cdot e^{i\psi}, \qquad \varrho_0 = |\varrho_0|\,e^{i\varphi}$$

we have obviously $\pi/4 < \varphi < \pi/2$. Therefore it is evident that out of the numbers $\psi, \psi-\varphi, \psi-2\varphi, \psi-3\varphi, \psi-4\varphi \bmod 2\pi$ one at least must lie between $-\pi/2$ and $\pi/2$, which clearly gives (2.2).

We shall need in the proof the following result (see [7], p. 185, Theorem 9.7)

for every $T \geqslant 2$ *there exists a* $t = t(T)$, $T \leqslant t \leqslant T+1$ *such that*

$$\frac{1}{|\zeta(\sigma+it)|} \leqslant t^{c_7}, \qquad -1 \leqslant \sigma \leqslant 2\,. \tag{2.3}$$

Let us consider the integral

$$I_k = \frac{1}{2\pi i}\int\limits_{4/3-iZ}^{4/3+iZ} \frac{e^{y_1 s k}}{s^{k+r}}\cdot\frac{ds}{\zeta(s)}, \tag{2.4}$$

where y_1 is that given by Lemma 2, r that from (2.2), further

$$Z = t\big((\log T)^{1/10} - 1\big)$$

and integer k subject to the inequalities

$$\frac{\log T}{y_1} - (\log T)^{0.8} \leqslant k \leqslant \frac{\log T}{y_1}\,. \tag{2.5}$$

We obtain from (2.4)

$$I_k = \frac{1}{2\pi i}\int\limits_{(4/3)} \frac{(e^{y_1 k})^s}{s^{k+r}}\cdot\frac{ds}{\zeta(s)} + O(T^{4/3}Z^{-k}),$$

whence by

$$Z^k \geqslant \{(\log T)^{0.1} - 1\}^k = e_1\Big(\frac{k}{10}\log_2 T + k\log(1-\log^{-0.1}T)\Big) > e_1\big(\log T - (\log T)^{0.9}\big)$$

$$I_k = \frac{1}{2\pi i}\int\limits_{(4/3)} \frac{(e^{y_1 k})^s}{s^{k+r}}\cdot\frac{ds}{\zeta(s)} + O(T^{0.4})\,. \tag{2.6}$$

Using the well-known formula

$$\frac{1}{2\pi i}\int\limits_{(4/3)} \frac{\xi^s}{s^{\nu+1}}\,ds = \begin{cases} \dfrac{1}{\nu!}\log^\nu \xi, & \xi \geqslant 1, \\ 0, & 0 < \xi < 1 \end{cases}$$

and (1.4) we get from (2.6)

$$\begin{aligned} I_k &= \sum_{n=1}^{\infty} \frac{\mu(n)}{2\pi i}\int\limits_{(4/3)} \frac{(e^{y_1 k}/n)^s}{s^{k+r}}\,ds + O(T^{0.4}) \\ &= \sum_{n\leqslant e^{y_1 k}} \mu(n)\,\frac{\log^{k+r-1}(e^{y_1 k}/n)}{(k+r-1)!} + O(T^{0.4}) \\ &= \int\limits_{1-0}^{e^{y_1 k}} \frac{\log^{k+r-1}(e^{y_1 k}/x)}{(k+r-1)!}\,dM(x) + O(T^{0.4}) \\ &= \int\limits_{1}^{e^{y_1 k}} M(x)\,d\Big(-\frac{\log^{k+r-1}(e^{y_1 k}/x)}{(k+r-1)!}\Big) + O(T^{0.4})\,. \end{aligned}$$

It is easy to see that

$$\frac{d}{dx}\left\{\log^{k+r-1}\left(\frac{e^{y_1 k}}{x}\right)\right\} \leqslant 0$$

for $1 \leqslant x \leqslant e^{y_1 k}$, whence

(2.7)
$$I_k = \Re I_k \leqslant \max_{1\leqslant x\leqslant T} M(x)\frac{\log^{k+r-1}(e^{y_1 k})}{(k+r-1)!} + c_8 T^{0.4},$$
$$I_k = \Re I_k \geqslant \min_{1\leqslant x\leqslant T} M(x)\frac{\log^{k+r-1}(e^{y_1 k})}{(k+r-1)!} - c_8 T^{0.4}.$$

Finally we note that owing to (2.5) and (1.16)

(2.8)
$$\frac{\log^{k+r-1}(e^{y_1 k})}{(k+r-1)!} \leqslant e_1\left(14\frac{\log T}{\log_2 T}\log_3 T\right).$$

3. Using Cauchy's theorem of residues we have

(3.1)
$$I_k = \sum_{|\Im\varrho|<Z} \operatorname*{Res}_{s=\varrho}\frac{e^{y_1 sk}}{\zeta(s)s^{k+r}} + \operatorname*{Res}_{s=0}\frac{e^{y_1 sk}}{\zeta(s)s^{k+r}} + \frac{1}{2\pi i}\int_{-1-iZ}^{-1+iZ}\frac{e^{y_1 ks}}{s^{r+k}}\cdot\frac{ds}{\zeta(s)} + O(T^{4/3}Z^{-k+c_7}).$$

As easy to see

$$\operatorname*{Res}_{s=0}\frac{e^{y_1 sk}}{\zeta(s)s^{k+r}} = \frac{1}{2\pi i}\int_{|s|=1/y_1}\frac{e^{y_1 sk}}{s^{k+r}}\cdot\frac{ds}{\zeta(s)} = O(e^k y_1^{k+r}) = O\left(e_1\left(14\frac{\log T}{\log_2 T}\log_3 T\right)\right)$$

further

$$\int_{-1-iZ}^{-1+iZ}\frac{e^{y_1 ks}}{s^{k+r}}\cdot\frac{ds}{\zeta(s)} = O\left(\int_{-\infty}^{\infty}\frac{e^{-y_1 k}}{(t^2+1)^{(k+r)/2}}\cdot\frac{dt}{|\zeta(-1+it)|}\right) = O(1)$$

and

$$O(T^{4/3}Z^{-k+c_7}) = O(T^{0.4}),$$

whence, putting

(3.2)
$$S_k \overset{\text{def}}{=} \sum_{|\Im\varrho|<Z}\operatorname*{Res}_{s=\varrho}\frac{e^{y_1 sk}}{\zeta(s)s^{k+r}},$$

we obtain by (3.1)

(3.3)
$$I_k = S_k + O(T^{0.4}).$$

Let $\varrho_j = \frac{1}{2}+i\gamma_j$, $j = 0, 1, \ldots, l$, run through the set of ζ-zeros in $0<\sigma<1$, $0<t<Z$ [2] so that $0<\gamma_0<\gamma_1<\ldots<\gamma_l<Z$, the possible multiple zeros being, however, counted only once. Suppose $\varepsilon>0$ to satisfy the inequalities

(3.4)
$$\varepsilon < \min_{0\leqslant j\leqslant l-1}(\gamma_{j+1}-\gamma_j), \qquad \varepsilon < Z-\gamma_l.$$

[2] $\Re\varrho_j = \frac{1}{2}$ by (1.18).

If the order of multiplicity of ϱ_j is ν, say, we define ν "shifted zeros" corresponding to ϱ_j:

(3.5)
$$\varrho_j^{(1)} = \varrho_j = \tfrac{1}{2}+i\gamma_j, \quad \varrho_j^{(2)} = \tfrac{1}{2}+i\left(\gamma_j+\frac{\varepsilon}{\nu}\right), \quad \ldots, \quad \varrho_j^{(\nu)} = \tfrac{1}{2}+i\left(\gamma_j+\frac{\nu-1}{\nu}\varepsilon\right).$$

We do it for each $j\leqslant l$ and also proceed symmetrically for the ϱ-zeros in the rectangle $0<\sigma<1$, $-Z<t<0$. We get on this way a set E_ε of "shifted zeros", so that to each ϱ with $|\Im\varrho|<Z$ there corresponds an $s_\varepsilon(\varrho)\in E_\varepsilon$. This definition implies in particular that

(3.6)
$$|\varrho - s_\varepsilon(\varrho)| < \varepsilon$$

and

(3.7)
$$s_\varepsilon(\varrho_0) = \varrho_0, \qquad s_\varepsilon(\bar\varrho_0) = \bar\varrho_0.$$

Now we introduce

(3.8)
$$\zeta_\varepsilon(s) \overset{\text{def}}{=} \zeta(s)\prod_{|\Im\varrho|<Z}\frac{s-s_\varepsilon(\varrho)}{s-\varrho};$$

it is easy to see by (3.4) and (3.5) that $\zeta_\varepsilon(s)$ has only simple zeros (namely those at $s_\varepsilon(\varrho)$'s) in the rectangle $0<\sigma<1$, $|t|<Z$. Writing

(3.9)
$$S_k(\varepsilon) \overset{\text{def}}{=} \sum_{|\Im\varrho|<Z}\operatorname*{Res}_{s=s_\varepsilon(\varrho)}\frac{e^{y_1 sk}}{\zeta_\varepsilon(s)s^{k+r}} = \sum_{|\Im\varrho|<Z}\frac{e_1(y_1 s_\varepsilon(\varrho)k)}{\zeta_\varepsilon'(s_\varepsilon(\varrho))\cdot s_\varepsilon(\varrho)^{k+r}},$$

we assert that

(3.10)
$$\lim_{\varepsilon\to 0} S_k(\varepsilon) = S_k.$$

In fact, we have

(3.11)
$$S_k = \frac{1}{2\pi i}\int_C\frac{e^{y_1 sk}}{\zeta(s)s^{k+r}}ds,$$
$$S_k(\varepsilon) = \frac{1}{2\pi i}\int_C\frac{e^{y_1 sk}}{\zeta_\varepsilon(s)s^{k+r}}ds,$$

where the contour of integration C consists of $0\leqslant\sigma\leqslant 2$, $t=\pm Z$; $\sigma=2$, $|t|\leqslant Z$; $\sigma=0$, $\frac{1}{2}\leqslant|t|\leqslant Z$; $\sigma^2+t^2=\frac{1}{4}$, $|t|\leqslant\frac{1}{2}$. Making ε small enough we find a number $\delta>0$ such that

(3.12)
$$|\zeta(s)|\geqslant\delta, \qquad |\zeta_\varepsilon(s)|\geqslant\delta \qquad \text{for} \quad s\in C$$

and

(3.13)
$$\text{distance}(\varrho, C)\geqslant\delta, \qquad \text{distance}(s_\varepsilon(\varrho), C)\geqslant\delta$$

for all ϱ's with $|\Im\varrho|<Z$.

We have then for $s \in C$

$$\left|\frac{\zeta(s)}{\zeta_\varepsilon(s)}-1\right| = \left|\prod_{|\Im\varrho|<Z}\left(1+\frac{s_\varepsilon(\varrho)-\varrho}{s-s_\varepsilon(\varrho)}\right)-1\right|$$

$$= \left|\sum_j \vartheta_j + \sum_{j\neq l}\vartheta_j\vartheta_l + \ldots \vartheta_1\vartheta_2\ldots\vartheta_n\right|,$$

where ϑ's run through the set of numbers $\frac{s_\varepsilon(\varrho)-\varrho}{s-s_\varepsilon(\varrho)}$. Owing to (3.6) and (3.13) ϑ_j's do not exceed absolutely ε/δ, whence

$$\left|\frac{\zeta(s)}{\zeta_\varepsilon(s)}-1\right| \leqslant \left(\frac{\varepsilon}{\delta}+1\right)^n - 1 \leqslant c(n)\cdot\frac{\varepsilon}{\delta}.$$

Hence, using also (3.12), we get for $s \in C$

$$\frac{1}{\zeta_\varepsilon(s)} \rightrightarrows \frac{1}{\zeta(s)};$$

this and (3.11) yield (3.10).

4. We fix our $\varepsilon > 0$, make it however so small that in addition to (3.12), (3.13) we have

$$\pi \geqslant \left|\arg\frac{e_1(y_1 s_\varepsilon(\varrho))}{s_\varepsilon(\varrho)}\right| \geqslant (\log T)^{-2/3} \tag{4.1}$$

(this being a corollary from (1.17)).

Next we define

$$z_j(\varepsilon) = \frac{e_1\left(y_1\left(s_\varepsilon(\varrho)-\frac12\right)\right)}{s_\varepsilon(\varrho)/|\varrho_0|}, \quad j = 1, 2, \ldots, n, \tag{4.2}$$

where ϱ_0 is given by (2.1) and numbers $z_j(\varepsilon)$ are arranged so as to have

$$1 = |z_1(\varepsilon)| \geqslant |z_2(\varepsilon)| \geqslant \ldots \geqslant |z_n(\varepsilon)|.$$

Similarly we define

$$b_j(\varepsilon) = \frac{1}{\zeta_\varepsilon'(s_\varepsilon(\varrho))\, s_\varepsilon^r(\varrho)}, \quad j = 1, 2, \ldots, n. \tag{4.3}$$

Let us put $h = 2$. Thus

$$z_1(\varepsilon) = \frac{e^{y_1(\varrho_0-\frac12)}}{\varrho_0/|\varrho_0|}, \quad z_2(\varepsilon) = z_h(\varepsilon) = \frac{e^{y_1(\bar\varrho_0-\frac12)}}{\bar\varrho_0/|\varrho_0|} \tag{4.4}$$

(whence $|z_1(\varepsilon)| = |z_2(\varepsilon)| = 1$).

Putting next $h_1 = 3$ we have

$$|z_{h_1}(\varepsilon)| = \frac{|\varrho_0|}{|\varrho_1|},$$

where

$$\varrho_1 = \tfrac12 + i\cdot 21.02\ldots$$

so that

$$|z_{h_1}(\varepsilon)| < \tfrac{15}{20} = \tfrac34. \tag{4.5}$$

The number n of ζ-zeros in the considered domain is obviously

$$\leqslant \log^{0.1} T\cdot(\log_2 T)^2; \tag{4.6}$$

thus we can apply Lemma 1 to the sum

$$S_k(\varepsilon)\cdot\frac{|\varrho_0|^k}{e^{\frac12 y_1 k}} = \sum_{j=1}^{n} b_j(\varepsilon) z_j^k(\varepsilon), \tag{4.7}$$

putting

$$m \overset{\text{def}}{=} \left[\frac{\log T}{y_1} - (\log T)^{0.8}\right] \tag{4.8}$$

and (see (4.1))

$$\varkappa = (\log T)^{-2/3}. \tag{4.9}$$

Owing to (4.4) and (4.5) the conditions (1.10) and (1.12) are obviously satisfied. Also

$$A = A(\varepsilon) = \Re\left(b_1(\varepsilon)+b_2(\varepsilon)\right) = 2\Re\frac{1}{\zeta_\varepsilon'(\varrho_0)\,\varrho_0^r}.$$

Since

$$\zeta_\varepsilon'(\varrho_0) \to \zeta'(\varrho_0), \quad \varepsilon \to 0,$$

we get (see (2.2))

$$\lim_{\varepsilon\to 0} A(\varepsilon) = 2\Re\frac{1}{\zeta'(\varrho_0)\varrho_0^r} = 2c_6,$$

so that for $\varepsilon > 0$ sufficiently small

$$A(\varepsilon) > c_6 \quad (>0). \tag{4.10}$$

Hence by (1.14) we have an integer k_ε with

$$m+1 \leqslant k_\varepsilon \leqslant m + n(3+\pi/\varkappa) \tag{4.11}$$

such that

$$\Re\sum_{j=1}^{n} b_j(\varepsilon) z_j^k > \frac{A(\varepsilon)}{2n+1}\left(\frac{|z_h(\varepsilon)|}{2}\right)^{m+n(3+\pi/\varkappa)}\left\{\frac{n}{24\left(m+n(3+\pi/\varkappa)\right)}\right\}^{2n}. \tag{4.12}$$

The right-hand side of (4.12) is owing to (4.4), (4.6), (4.8), (4.9) and (4.10)

$$> e_1\left(-13\frac{\log T}{\log_2 T}\right)$$

so that by (4.7)

$$S_{k_\varepsilon}(\varepsilon) = \Re S_{k_\varepsilon}(\varepsilon) > \frac{e_1(\frac12 y_1 k_\varepsilon)}{|\varrho_0|^{k_\varepsilon}}\, e_1\left(-13\frac{\log T}{\log_2 T}\right). \tag{4.13}$$

It follows that there exists an integer k with

$$m+1 \leqslant k \leqslant m+n(3+\pi/\varkappa)$$

(consequently satisfying (2.5)) and a sequence $\varepsilon \to 0$ for which $k_\varepsilon = k$. For this sequence we get by (4.13) and by the estimation

$$\frac{e_1(\frac{1}{2}y_1 k)}{|\varrho_0|^k} > \frac{T^{1/2}e_1(-(\log T)^{0.9})}{15^{13\log T/\log_2 T}} > T^{1/2}e_1\left(-40\frac{\log T}{\log_2 T}\right),$$

the inequality

$$S_k(\varepsilon) > T^{1/2}e_1\left(-53\frac{\log T}{\log_2 T}\right), \tag{4.14}$$

which yields on letting ε tend to zero

$$S_k \geqslant T^{1/2}e_1\left(-53\frac{\log T}{\log_2 T}\right). \tag{4.15}$$

(4.15) and (3.3) give for T sufficiently large

$$I_k \geqslant T^{1/2}e_1\left(-53\frac{\log T}{\log_2 T}\right). \tag{4.16}$$

This and the first inequality (2.7) together with (2.8) prove (1.19). (1.20) follows on an analogous way (one has to use (1.15) at (4.12) and then properly change the relations (4.13)–(4.16)).

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Reçu par la Rédaction le 14. 1. 1963

ACTA ARITHMETICA
VIII (1963)

Über die Irreduzibilität gewisser Polynome

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In der Literatur begegnen wir uns oft mit der Irreduzibilität algebraischer Polynome von der Form

$$S(x) = F(R(x)),$$

wobei $F(z)$ ein irreduzibles Polynom mit ganzen rationalen Koeffizienten und $z = R(x)$ ein Polynom mit ebenfalls ganzen rationalen Koeffizienten bedeutet.

Der Verfasser hat sich schon in seinen früheren Arbeiten [8], [9] mit der Irreduzibilität von Polynomen eines gewissen Typs beschäftigt; als irreduzibles Polynom $F(z)$ nahm er ein beliebiges n-tes Kreisteilungspolynom $F_n(z)$, in welches er das Polynom

$$z = R(x) = \prod_{k=1}^{m}(x-a_k)Q(x) = P(x)Q(x) \tag{1}$$

substituierte, wo $a_1 < a_2 < \ldots < a_m$ ganze rationale Zahlen bedeuteten und die Koeffizienten des Polynoms

$$Q(x) = x^\mu + b_1 x^{\mu-1} + \ldots + b_\mu$$

ganze rationale Zahlen waren; der Grad von $Q(x)$ war kleiner als m.

Der Beweis der Irreduzibilität von Polynomen dieses Typs ergab die Lösung einer Verallgemeinerung eines Problems von I. Schur, [6], [3]. Für $n = 2^M$ und $Q(x) \equiv 1$ ergab sich die Irreduzibilität des Polynoms

$$S_n(x) = S_{2^M}(x) = \prod_{k=1}^{m}(x-a_k)^{2^M} + 1.$$

Es waren dabei nur einige Ausnahmen.

Im Falle $M = 0$ ist das Polynom $S_1(x)$ für spezielle gegebene $\{a_1, a_2, a_3, a_4\}$ reduzibel über dem Körper K_0 (W. Flügel [3]).

Irreduzibilität ohne Ausnahme ergab sich in den Fällen $M \geqslant 1$. (W. Flügel [4]). Für $M = 1$, $M = 2$ kann der Beweis in dem Buche von G. Pólya und G. Szegö [5] gefunden werden.