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Negative discriminants of binary quadratic forms with one class in each genus *

by

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1. Introduction. Let $0 < k \neq m^2$, $k \equiv 0$ or $k \equiv 3 \pmod 4$, and denote by $h = h(k)$ the number of positive, primitive classes of quadratic forms of discriminant $-k$ and by $g = g(k)$ the corresponding number of genera. If $k = \lambda^2 k_1$, with $-k_1$ a discriminant, implies $\lambda = \pm 1$, then k is called a fundamental discriminant. As there is at least one class in each genus, $h \geqslant g$ holds trivially. Chowla showed in [2] that $\lim_{k\to\infty}(h/g) = \infty$ and this implies in particular, the finiteness of the set $K = \{k_i\}$, where $-k_i$ $(i = 1, 2, \ldots)$ are those negative discriminants for which $h = g$, i.e., with exactly one class in each genus.

The following conjecture is rather old (see [1] and [5], p. 611):

Conjecture. *The element of K of largest absolute value is* $4\cdot 1848 = 7392$.

Dickson and Townes checked this conjecture up to $k = 23\,000$ for odd and up to $k = 400\,000$ for even discriminants (see [4]). Swift [18] extended this result and showed that $k \notin K$ for all $k < 10^7$, except for the 101 known values listed, e.g., in [4]. Recently, J. L. Selfridge, M. Atkinson, and C. MacDonald computed (see [17]) the extension of Swift's results up to

$$k \leqslant 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 44838 = 1\,346\,485\,140\times$$
$$\times(\log_e k \leqslant 21.02\ldots;\ \log_{10} k \leqslant 9.12919\ldots)\,.$$

If $-k$ is a fundamental discriminant, set

$$\chi(m) = \left(\frac{-k}{m}\right) \text{ (Kronecker symbol)}, \quad \text{and} \quad L(s,\chi) = \sum_{m=1}^{\infty}\chi(m)m^{-s}\,.$$

Then,

$$(1) \qquad h(k) = \pi^{-1}k^{1/2}L(1,\chi)\,,$$

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and it is clear that if one knows a lower bound for $L(1, \chi)$, (1) furnishes a lower bound for $h(k)$; this may be taken as a starting point for the proof of above quoted Conjecture. Although (1) applies only to fundamental discriminants, one can handle also the general case, using the fact that if $d = \lambda^2 k$, $-k$ fundamental, then (see [9] or [11], Satz 209 and 214)

$$(2) \qquad h(d) = \lambda \prod_{p \mid \lambda} \left(1 - \left(\frac{-k}{p}\right)\frac{1}{p}\right) \cdot h(k) .$$

Here and in what follows, p, with or without subscripts, stands for a rational prime number.

Simple formulae for the genus are well known; hence, in order to prove the Conjecture it only remains to establish a lower bound for $L(1, \chi)$.

This can be done if one knows that close to s $(= \sigma + it) = 1$ there is a sufficiently large region free of zeros of $L(s, \chi)$-functions. The larger such a zero-free region is, the less computations will be required for the proof of the Conjecture. Actually, it is sufficient to know that $L(s, \chi) \neq 0$ only over some interval $s_0 < s < 1$, with $\frac{1}{2} < s_0 < 1$. It is, in fact, known that such zero-free regions (or intervals) do exist, but the best known result (Bateman and Grosswald, unpublished; see, however, [15]),

$$s_0 = 1 - 6\pi^{-1}k^{-1/2}\{1 + 6\pi^{-1}k^{-1/2}\log k - 12\pi^{-1}(\log\log k + \pi/2)k^{-1/2}\}$$

is not sufficient for the purpose on hand.

Making the assumption that, for $k > 10^{14}$, $L(53/54, \chi) \geqslant 0$, Chowla and Briggs [3] proved that if k is fundamental and $k \in K$, then $k < 10^{14}$. Actually, as they mention in passing, their method permits the extension of the results, from fundamental to all negative discriminants „without difficulty, but with tedium". In the present paper we prove this extension to non-fundamental discriminants ([1]). Next, we use a somewhat stronger version of a lemma of [3], which permits to weaken the unproven hypothesis used by Chowla and Briggs. In fact we shall assume in most of this paper the validity of

Hypothesis L. *If $-k$ is a fundamental discriminant and $k > k_0 = 10^{10}$, then*

$$L(s, \chi) \neq 0 \quad \textit{for} \quad s_0 < s < 1, \quad \textit{where} \quad s_0 = \mathrm{Max}\{1 - 2\log^{-1}k, 53/54\}.$$

Remark. For $\log k \geqslant 108$, $s_0 = 1 - 2\log^{-1}k$; for $\log k \leqslant 108$, $s_0 = 53/54$. For some results, the following, weaker assumption will prove sufficient:

([1]) See „Note", p. 305.

Hypothesis L_0. *If $-k$ is a fundamental discriminant, $k > k_0$, $k = 4n$ and k is not divisible by any prime $p \equiv 3 \pmod 4$, then $L(s, \chi) \neq 0$ for $s_0 < s < 1$.*

The main results will be stated in section 2. For the convenience of reference, some needed, known theorems are quoted in section 3. Section 4 deals with the reduction of the general case to that of a fundamental discriminant by proving Theorem 1. As an easy consequence of a theorem of Hecke and Landau, a useful lemma is derived in section 5. In section 6, this is used in the proof of Theorem 2. The 7th section deals with some consequences of Theorem 2. In particular, as an incidental result, a conjecture formulated in [7] is shown to follow from L_0 and is stated as Theorem 3. Some further considerations (including a numerical table) and conclusions form the 8th and last section.

2. Main results.

Theorem 1. *The nonfundamental negative discriminants $-d$ with $d \in K$ (i.e., with one class per genus) are exactly those of the form $d = 4k$, where $k \equiv 0 \pmod 8$, $-k$ being a fundamental discriminant with $k \in K$.*

Theorem 2. *Hypothesis L implies that if $-k$ is a discriminant with one class in each genus, then either $k \leqslant 7392$ and $-k$ is one of the classically known 101 discriminants or else $10^{9.12919} < k < 4 \cdot 10^{10}$ ($10^{9.12919} < k < 10^{10}$ if $-k$ is fundamental).*

A slight extension of Selfridge et al.'s program, or the individual checking of a few hundred possible exceptional values of k (see section 8) would dispose completely of these unlikely possibilities. Consequently, we may state instead of Theorem 2, the following:

Theorem 2′. *Subject to mentioned numerical verifications, hypothesis L implies that if $-k$ is a discriminant with one class in each genus, then $k \leqslant 7392$ and k has one of the known 101 values.*

Theorem 3. *Subject to mentioned numerical verifications, hypothesis L_0 implies that every natural integer N has a representation as a sum of three positive integral squares,*

$$(3) \qquad N = a^2 + b^2 + c^2, \quad abc \neq 0,$$

except for $N = 4^a N_1$, with integral $a \geqslant 0$ and where either $N_1 \equiv 7 \pmod 8$, or else N_1 is an element of the set $S = \{1, 2, 5, 10, 13, 25, 37, 58, 85, 130\}$.

Corollary. *Let $N(x)$ be the number of integers not exceeding x and having representations* (3). *Then L_0 implies that*

$$N(x) = \frac{5}{6}x - \frac{1}{\log 4}\left(\frac{73}{8} + \frac{\eta_1}{7}\right)\log x - A - 12\eta_2,$$

with

$$A = \frac{7}{6} + \frac{1}{\log 4}\left\{\sum_{n \in S}\log n - \frac{7\log 7}{8}\right\} = 19.68\ldots \quad \textit{and} \quad 0 < \eta_1 < 1, \ 0 < \eta_2 < 1.$$

3. Some needed theorems. We shall need the following known results:

THEOREM A. *Let $-k$ be a negative discriminant, $k > 125$.*

a) *If $k \in K$ and k is odd, then $k \equiv 3 \pmod 8$.*

b) *If k is odd, set $T_j = \frac{1}{4}\{k+(2j+1)^2\}$, with $1 \leqslant (2j+1)^2 < k/3$; if $k = 4n$, set $S_j = n+j^2$ $(4 \leqslant 4j^2 < k/3)$. Then $k \in K$ is possible only if T_j or S_j are not divisible by any prime p, satisfying $2j+1 < p < T_j^{1/2}$, or $2j < p < S_j^{1/2}$, respectively.*

c) *If there exists any prime $p < (k/3)^{1/2}$ such that $(-k/p) = 1$, then $k \notin K$.*

d) *If $k = 4n > 315$ and $n \equiv 3 \pmod 4$, then $k \notin K$.*

THEOREM B. *If $k \in K$ and $k > 315$, then k is not divisible either by 64, or by any odd square.*

THEOREM C. *Let $-k < 0$ be a discriminant; suppose that k is not a square and that it contains exactly r distinct prime factors. Then $g(k) = 2^{r-1}$, except for $k = 4n$, $n \equiv 3 \pmod 4$, when $g(k) = g(n) = 2^{r-2}$ and for $k = 4n$, $n \equiv 0 \pmod 8$, when $g(k) = 2^r$.*

THEOREM D.

a) $\theta(x) = \sum_{p \leqslant x} \log p > x\big(1-1/(2\log x)\big)$ *for* $x \geqslant 563$.

b) $p_n > n(\log n + l_2 n - 3/2)$ *for* $n \geqslant 2$.

Here and in what follows, the iterated logarithm log $(\log n)$ is denoted by $l_2 n$.

THEOREM E. *For real s,*

$$2(k^{1/2}/2\pi)^s \Gamma(s)\zeta(s)L(s,\chi) > \int_1^\infty z^{s-1}\exp(-2\pi z k^{-1/2})\,dz - h(k)/s(1-s). \tag{4}$$

THEOREM F. *If $k \geqslant k_0$ and $L(53/54, \chi) \neq 0$, then*

$$h(k) > (.007495)\,k^{25/54}. \tag{5}$$

Theorem A was presumably known to Dickson and Townes. Parts a) and b) for k odd are proven in [4]; b) for k even, as well as d) are proven by Hall in [8]; a proof of c) may be found in [18]. It should be observed that the restrictions on j are relevant, although they are occasionally omitted in statements of the theorem (even in [4]). Without said restrictions the statement fails to be true, as one may see, for instance, by considering $k = 147$, $j = 14$; $T_{14} = 247 = 13\cdot 19$ and also $(147/13) = (4/13) = 1$; yet $k \in K$.

Theorem B is due to Hall (see [8]). Theorem C is classical (see Gauss [6], especially Art. 228-287); except for a change in notation it is Theorem 75 in [10], where a proof may be found.

Theorem D is due to Rosser and Schoenfeld (see [16], Theorems 3 and 4). Theorem E is due to Hecke; its proof was published by Landau (see [12], p. 289). Theorem F with a coefficient smaller by a factor 1.27151—is due to Chowla and Briggs [3]. Their proof needs only rather trivial modifications (sharper estimates on $|\zeta(\frac{1}{2}+it)|$ and $|L(\frac{1}{2}+it)|$, obtained using results of [14]), in order to lead, on the basis of the present assumptions, to (5).

4. Reduction to fundamental discriminants. Proof of Theorem 1. Let $d = \lambda^2 k$, $-k$ being a fundamental discriminant containing r distinct prime factors. For $d \leqslant 10^7$, all $d \in K$ are known; hence, we may, and whenever convenient shall, assume that $d > 315$. If $d \in K$, then, by Theorem B, $\lambda = 1, 2$, or 4. If $\lambda = 1$, then $-d$ is fundamental. If $\lambda = 4$, then it follows by Theorem B that k must be odd, hence, $k \equiv 3 \pmod 4$. By Theorem C, $g(d) = 2^r$ and $g(k) = 2^{r-1}$. Also, by (2), $h(d) = 4\left(1-\left(\frac{-k}{2}\right)\frac{1}{2}\right)h(k)$, so that $\frac{h(d)}{g(d)} = 2\left(1-\left(\frac{-k}{2}\right)\frac{1}{2}\right)\frac{h(k)}{g(k)}$. If $k \equiv 3 \pmod 8$, then $\left(\frac{-k}{2}\right) = -1$ and $\frac{h(d)}{g(d)} = 3\frac{h(k)}{g(k)} \geqslant 3$. If $k \equiv 7 \pmod 8$, then $\frac{h(k)}{g(k)} \geqslant 2$ (by Theorem A), $\left(\frac{-k}{2}\right) = +1$, and $\frac{h(d)}{g(d)} \geqslant 2\cdot\frac{1}{2}\cdot 2 = 2$. In either case, $d \notin K$, contrary to the assumption. It only remains to consider the case $\lambda = 2$, $d = 4k$. If k is odd, $k \equiv 3 \pmod 4$; hence, $g(k) = g(d) = 2^{r-1}$, by Theorem C. Also, by (2),

$$h(d) = 2\left(1-\left(\frac{-k}{2}\right)\frac{1}{2}\right)h(k), \quad \text{so that} \quad \frac{h(d)}{g(d)} = 2\left(1-\left(\frac{-k}{2}\right)\frac{1}{2}\right)\frac{h(k)}{g(k)}.$$

As before, either $\frac{h(k)}{g(k)} \geqslant 2$ (if $k \equiv 7 \pmod 8$); or else $\left(\frac{-k}{2}\right) = -1$ (if $k \equiv 3 \pmod 8$) and in either case $\frac{h(d)}{g(d)} \geqslant 2$, so that $d \notin K$. Finally, one may have $k = 4n$, where either $n \equiv 1$ or $n \equiv 2 \pmod 4$. If $n \equiv 1 \pmod 4$, then $g(d) = g(k) = 2^{r-1}$ and $\left(\frac{-k}{2}\right) = 0$, so that $h(d) = 2h(k)$, and $\frac{h(d)}{g(d)} = 2\frac{h(k)}{g(k)} \geqslant 2$, whence $d \notin K$. It follows that the only possibility for $-d$ to be a nonfundamental discriminant with one class in each genus and $d > 315$ is that $d = 4k$, with $-k$ a fundamental discriminant of the form $k = 4n$, $n \equiv 2 \pmod 4$. This possibility is actually realized, as can be seen from the numerous examples in the list in [4]. In order to finish the proof of Theorem 1 it only remains to show that now $d \in K$ if and only if $k \in K$. For that, observe that now $g(d) = 2^r$, $g(k) = 2^{r-1}$ and $h(d) = 2h(k)$, so that, indeed, $\frac{h(d)}{g(d)} = \frac{h(k)}{g(k)}$; this ratio equals one, if and only if $k \in K$ and the proof of Theorem 1 is complete.

5. Lemma [2]. *Let $-k$ be a fundamental negative discriminant, $k \neq 1$. If $L(s,\chi) \neq 0$ for $1-2\log^{-1}k < s < 1$, then*

(5′) $$h(k) > (\pi e)^{-1} k^{1/2} \log^{-1} k .$$

Proof. Let $V = 1-2\log^{-1}k$. Then it follows from (4) that $h > V(1-V)I$, with $I = \int_1^\infty z^{V-1}\exp(-2\pi z k^{-1/2})\,dz$. Because $0<V<1$, $\Gamma(V)>1$; also, for $u>0$, $e^{-u}<1$. Using these inequalities, one obtains

$$\begin{aligned} I &= (k^{1/2}/2\pi)^V \int_{2\pi k^{-1/2}}^\infty u^{V-1} e^{-u}\,du \\ &= k^{1/2} k^{-\log^{-1}k} (2\pi)^{-1} (2\pi)^{2\log^{-1}k} \Big\{\Gamma(V) - \int_0^{2\pi k^{-1/2}} u^{V-1} e^{-u}\,du\Big\} \\ &> (2\pi e)^{-1} k^{1/2} \exp\{(2\log 2\pi)\log^{-1}k\}\Big(1-\int_0^{2\pi k^{-1/2}} u^{V-1}\,du\Big) \\ &= (2\pi e)^{-1} k^{1/2} \exp\{-(2\log 2\pi)\log^{-1}k\}\big(1-V^{-1}(2\pi k^{-1/2})^V\big). \end{aligned}$$

Hence replacing V by its value $1-2\log^{-1}k$,

(6) $$h(k) > \lambda(\pi e)^{-1} k^{1/2}\log^{-1}k ,$$

with

(7) $$\lambda = \lambda(k) = (1-2\log^{-1}k)\exp\{(2\log 2\pi)\log^{-1}k\}\times \\ \times\big(1-2\pi e k^{-1/2}(1-2\log^{-1}k)^{-1}\exp\{-(2\log 2\pi)\log^{-1}k\}\big).$$

Setting $\log k = 2/z$ and considering for a moment z as a continuous variable,

$$\frac{d\lambda}{dz} = (2\pi)^z\{\log(2\pi/e) - \varphi(z)\},$$

where $\varphi(z)$ is the monotonically increasing function

$$\varphi(z) = z\log 2\pi + 2\pi e z^{-2}\exp\{-(z\log 2\pi + 1/z)\}.$$

It follows that, for $z>0$, $d\lambda/dz$ vanishes exactly once, when $z = z_1$ ($\approx .142$) and λ attains its maximal value for $\log k_1 = 2/z_1$ (≈ 14.1). For $k<k_1$ λ increases with k; for $k>k_1$, λ decreases monotonically as $k\to\infty$ and $\lim_{k\to\infty}\lambda(k)=1$. For $\log k = 10$, λ (≈ 1.04) >1, so that (5′) holds for $\log k \geqslant 10$. However, if $\log k < 10$, the second member of (5′) is less than $e^5(10\pi e)^{-1} < 2$, while $h(k)\geqslant 2$ for $\log 163 < \log k \leqslant 5\cdot 10^9$ (see [13]); hence, (5′) holds for $k>163$. If $k\leqslant 163$, the second member of (5′) is less than one, while $h(k)\geqslant 1$ trivially and this finishes the proof of the Lemma.

[2] This formulation, which replaces an earlier, weaker one, is due to a remark of Professor P. T. Bateman.

6. Proof of Theorems 2 and 2′. The statements of the two theorems are equivalent. It is, therefore, sufficient to prove only Theorem 2′. Hence, we may assume that if $7392 < k \leqslant k_0$, then $k \notin K$. On account of Theorem 1, it is sufficient to prove that if $-k$ is a fundamental discriminant and $k > k_0$, the $k\notin K$. This we proceed to show, using L, the Lemma and Theorems A to F. We consider separately the cases of odd and of even discriminants.

a) k odd. $-k$ being a fundamental discriminant, $k = p_1p_2\dots p_r \equiv 3 \pmod 4$. Hence, by Theorem C, $g = g(k) = 2^{r-1}$. In order to prove $k\notin K$, it is sufficient to show that $h = h(k) > 2^{r-1}$. On account of L, (5′) applies for $\log k \geqslant 108$, so that it suffices to show that $\frac{1}{2}\log k - l_2 k > 1 + \log\pi + (r-1)\log 2$. Using $\log 2 < .69315$ and $\log\pi < 1.14474$, this condition holds, provided that

(8) $$F(k) = \log k - 2l_2 k > 1.3863r + 2.9032 = f(r).$$

We observe that

(9) $$\log k \geqslant \sum_{p\leqslant p_{r+1}} \log p - \log 2 = \theta(p_{r+1}) - \log 2 .$$

Hence, using Theorem D(a), for any given $r\geqslant 102$ one has

$$\log k > p_{r+1}\big(1-(2\log p_{r+1})^{-1}\big) - \log 2 > p_{r+1}\big(1-(2\log 563)^{-1}\big) - .69315 \\ > .92105\,p_{r+1} - .69315 > .92\,p_{r+1} - .693 .$$

By Theorem D(b),

$$p_{r+1} > (r+1)\log(r+1)\,\big\{1+\big(l_2(r+1)-3/2\big)\big(\log(r+1)\big)^{-1}\big\}.$$

For $r\geqslant 102$, $l_2(r+1) > 3/2$; hence,

$$p_{r+1} > (r+1)\log(r+1) \quad\text{and}\quad \log k > .92(r+1)\log(r+1) - .693 .$$

The function $F(k)$ increases with k (for $k\geqslant 8$), so that

$$F(k) = \log k - 2l_2 k > .92(r+1)\log(r+1) - .693 - 2\log\{.92(r+1)\log(r+1)\} \\ = .92(r+1)\log(r+1) - \{2\log(r+1) + 2l_2(r+1) + .693 + 2\log(.92)\}.$$

The curly brackets represent a function $\varphi(r)$, which, for $r\geqslant 102$, satisfies the inequality $\varphi(r) < .14(r+1)$. Hence,

$$F(k) > (r+1)\big(.92\log(r+1) - .14\big) > 4(r+1) > 1.3863r + 2.9032 = f(r),$$

and (8) holds for $r\geqslant 102$. If $r\leqslant 101$, $f(r)\leqslant 143$ and one easily verifies, using (9) and the monotonicity of $F(k)$, that

$$F(k) = \log k - 2l_2 k \geqslant \theta(p_{r+1}) - \log 2 - 2\log\{\theta(p_{r+1}) - \log 2\} > 146$$

for all $r\geqslant 39$, so that (8) holds also in this case. If, however, $r\leqslant 38$, then $f(r)\leqslant 55.5826$ and (8) holds for all k such that $\log k \geqslant 108$.

For $\log k < 108$, we have to use (5), rather than (5′). The inequality of $h > g$ will now hold, provided that $(.007495)k^{25/54} > 2^{r-1}$, or, setting $\psi(r) = 1.49720r + 9.07282$, if

$$\log k > \psi(r)\,. \tag{10}$$

As now $r \leqslant 38$, $\psi(r) < 66$ and (10) holds, unless $\log k \leqslant 66$. Using (9), it now follows that $r \leqslant 20$. Consequently, (10) holds for $\log k > \psi(20) = 39.01682$. For $\log k \leqslant 39.01682$, however, (9) shows that $r \leqslant 13$. Now $\psi(13) = 28.53642$; hence, (10) holds unless $\log k \leqslant 28.53642$; this is possible only for $r \leqslant 10$, with $\psi(10) = 24.04482$, so that (10) still holds if $\log k > 24.04482$. If $\log k \leqslant 24.04482$ then $r \leqslant 9$ (observe that for $r \geqslant 10$, $\log k \geqslant \sum_{3 \leqslant p \leqslant 31} \log p > 25$), and for $r \leqslant 9$, $\psi(r) \leqslant 22.54762$. However, for $k > k_0$, $\log k > 23.03 > 22.54762 \geqslant \psi(r)$ and (10) holds.

b) k even. If k is even, then $k = 4n$; if, furthermore, $-k$ is a fundamental discriminant with $k > 315$ and $k \in K$ it follows from Theorem A(d) that n is squarefree and $n \equiv 1$ or $2 \pmod 4$. We consider these subcases separately.

b$_1$) $n \equiv 1 \pmod 4$. If r is the number of odd primes dividing n, it follows from Theorem C that $g = 2^r$ and, using (5′), the condition $h > g$ becomes

$$F(n) = \log n - l_2 n > 1.3863r + 2.9032 + 2\log\{1 + \log 4 \log^{-1} n\}$$
$$= f(r) + 2\log\{1 + (\log 4)\log^{-1} n\}\,.$$

As before, we are interested only in values $k > k_0$, so that

$$\log n > \log n_0 = \log(k_0/4) > 21.6395\,.$$

Hence,

$$2\log\{1 + (\log 4)\log^{-1} n\} \leqslant 2\log\{1 + (\log 4)\log^{-1} n_0\} = .1241$$

and, if we set $f_1(r) = f(r) + .1241 = 1.3863r + 3.0273$, we now obtain as a sufficient condition for $h > g$, that

$$F(n) > f_1(r)\,. \tag{8′}$$

The procedure is exactly as before and we conclude that $h > g$, provided that $\log k \geqslant 108$. For $\log k < 108$, we use (5) instead of (5′) and obtain $h > (.007495)(4n)^{25/54} > 2^r = g$, provided that

$$\log n > \psi_1(r)\,, \tag{10′}$$

with

$$\psi_1(r) = 1.49720r + 9.18373\,.$$

The method of descent is used exactly as before and is successful down to $r = 10$, when $\psi_1(10) = 24.15573$ and, for $r > 10$, $\log n > 25$, as already seen. For $r = 9$, $\psi_1(9) = 22.65853$. There are 21 odd, squarefree integers, containing nine distinct primes, with $\log n_0 \leqslant \log n < \psi_1(9)$. Of these, 9 satisfy $n \equiv 1 \pmod 4$; they are considered in section 8 and the corresponding values of $k = 4n$ are eliminated as possible elements of K by Theorem A. If $r \leqslant 8$, $\psi_1(r) \leqslant 21.16133 < 21.6395 \leqslant \log n_0 \leqslant \log n$; hence, (10′) holds also for $r \leqslant 8$.

b$_2$) $n = 2m = 2p_1 \ldots p_r$. As before, $g = 2^{r-1}$. The procedure is identical to that of previous case and does not have to be repeated in detail. For $\log k = \log 8 + \log m \geqslant 108$, one uses (5′). Then (8) is being replaced by

$$F(m) > f_2(r), \qquad f_2(r) = 1.3863r + 2.4007 \tag{8″}$$

as a sufficient condition for $h > g$. (8″) is verified to hold for $\log k \geqslant 108$. If $\log k < 108$, one uses (5); for $h > g$ one obtains as sufficient condition on m:

$$\log m > \psi_2(r), \qquad \psi_2(r) = 1.49720r + 8.49059\,. \tag{10″}$$

We have to check (10″) for values of m satisfying

$$20.94635 \leqslant \log m_0 = \log(k_0/8) \leqslant \log m\,.$$

As before, (10″) is verified for $r \geqslant 10$. For $r = 9$, $\psi_2(9) = 21.96539$ and there are exactly two integers,

$$m_1 = 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29 \quad \text{and} \quad m_2 = 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 31,$$

products of nine distinct odd primes and with $\log m_0 \leqslant \log m < \psi_2(9)$.

We eliminate the two corresponding discriminants

$$-k_1 \ (= -4n_1 = -8m_1) \quad \text{and} \quad -k_2 \ (= -4n_2 = -8m_2)$$

as possible elements of K, by using Theorem A. Indeed, we observe that $n_1 = 2m_1 \equiv 46 \pmod{61}$ and that $n_2 = 2m_2 \equiv 36 \pmod{41}$; hence,

$$\left(\frac{-k_1}{61}\right) = \left(\frac{k_1}{61}\right) = \left(\frac{n_1}{61}\right) = \left(\frac{46}{61}\right) = +1$$

and

$$\left(\frac{-k_2}{41}\right) = \left(\frac{k_2}{41}\right) = \left(\frac{n_2}{41}\right) = \left(\frac{36}{41}\right) = +1$$

and, $k_1 \notin K$, $k_2 \notin K$, by Theorem A(c).

Finally, for $r \leqslant 8$, $\psi_2(r) \leqslant 20.46819 < \log m_0 \leqslant \log m$ so that (10″) holds for all $m \geqslant m_0$. This finishes the proof of Theorem 2′ (hence, of Theorem 2) for fundamental discriminants and, on account of Theorem 1, for all discriminants.

7. Proof of Theorem 3. In [7] it has been shown that for every integer N,

$$N = a^2 + b^2 + c^2\,,$$

has integral solutions with $abc \neq 0$, unless $N = 4^a N_1$ ($a \geqslant 0$, integral) and either $N_1 \equiv 7 \pmod 8$ (in which case a classical result states that N is the sum of no less than four non-vanishing squares); or $N_1 = 25$; or else N_1 is squarefree, is not divisible by any prime $p \equiv 3 \pmod 4$ and, furthermore, is such that

$$2\pi^{-1} N_1^{1/2} L(1, \chi) = 2^r, \tag{11}$$

where

$$\chi(m) = \left(\frac{-4N_1}{m}\right), \quad L(1, \chi) = \sum_{m=1}^{\infty} \chi(m) m^{-1}$$

and r is the number of odd prime factors of N_1. Clearly, (11) is equivalent to $h(4N_1) = g(4N_1)$ and the conditions on N_1 insure that $-4N_1$ is a fundamental discriminant. If we assume hypothesis L_0 of section 1, instead of L, the proof of Theorem 2′ leads to the following, slightly weaker result:

THEOREM 2″. L_0 *implies that there are no even fundamental discriminants* $-k < 0$, *with* $k \not\equiv 0 \pmod p$ *for all* $p \equiv 3 \pmod 4$, *and* $h(k) = g(k)$, *besides the classical ones, except, possibly finitely many, all in the interval* $10^{9.12919} < k < 10^{10}$.

Suppressing from the list of 34 even, fundamental discriminants those 25 that contain prime factors $p \equiv 3 \pmod 4$, we remain with a set of nine integers N of the form $4N_1$. If we adjoin to this set also $N_1 = 25$ (see first sentence of the present section), we obtain precisely the set S of Theorem 3 and this finishes the proof of that theorem. As for the Corollary, it follows (by a method of Landau) directly from Theorem 3, as shown in [7].

8. Final computations and conclusions. In section 6 we postponed consideration of 21 integers n, products of exactly nine distinct, odd primes, satisfying

$$(\log n_0 \approx)\ 21.6395 \leqslant \log n \leqslant 22.65853\ (= \psi_1(9)).$$

In order to show that for the corresponding discriminants $-k = -4n$ one has $k \notin K$, we use Theorem A. First, by A(d) we eliminate all $n \equiv 3 \pmod 4$; we are then left with nine integers $n \equiv 1 \pmod 4$. By A(c) we know that $k \notin K$, if there exists a prime $p < (k/3)^{1/2} = (4n/3)^{1/2}$, such that $\left(\frac{-k}{p}\right) = \left(\frac{-n}{p}\right) = +1$. Equivalently, by A(b), $k \notin K$ if $S_j = n + j^2 \equiv 0 \pmod p$ for some prime p, $2j < p < S_j^{1/2}$. In the following tabulation, each of the nine integers to be investigated is followed by a symbol $S_j(p)$, indicating the test prime p (the same in A(b) and A(c)) and also the value of j, by which $k = 4n$ is eliminated as a possible element of K.

Tabulation

$3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19\cdot23\cdot31$	$S_{15}(37)$
$3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19\cdot23\cdot43$	$S_{16}(37)$
$3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19\cdot23\cdot47$	$S_{13}(31)$
$3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19\cdot23\cdot59$	$S_8(29)$
$3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19\cdot29\cdot37$	$S_5(47)$
$3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19\cdot29\cdot41$	$S_{15}(31)$
$3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot23\cdot29\cdot37$	$S_{11}(31)$
$3\cdot5\cdot7\cdot11\cdot13\cdot19\cdot23\cdot29\cdot31$	$S_4(37)$
$3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19\cdot31\cdot43$	$S_1(29)$

By this method, and using only Swift's results [18], one may prove that hypothesis L implies that K consists exactly of the known 101 integers. But in that case, the number of integers k that have to be checked individually increases considerably.

Note (added November 16, 1962). The content of Theorem 1, concerning non-fundamental discriminants, was obtained already in 1874 by F. Grube. The author is greatly indebted to Dr. A. Schinzel, for bringing Grube's paper (Zeitschrift für Mathematik und Physik, V. 19 (1874), pp. 492-519) to his attention. This happened after the present paper was finished, but before it was submitted for publication. From Grube's paper one is led to conclude that the result must have been known already to Euler (who may not have had a valid proof). In view of the fact that Grube's proof is rather different, much longer, and that actually the result (as formulated in the present Theorem 1) is only implicitly contained in a string of lemmata, it seems justified to keep the statement and proof of Theorem 1 in the present paper.

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ACTA ARITHMETICA
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Über die Ausnahmenullstelle der Heckeschen L-Funktionen

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Für die Dirichletschen L-Funktionen

$$L_d(s) = \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) n^{-s}$$

(d — Diskriminante) bewies Siegel [8] im Jahre 1935 den berühmten Satz

$$\log L_d(1) = o(\log|d|) \qquad (|d|\to\infty) \tag{1}$$

und vermutete den Analogon

$$\log(hR) \sim \log\sqrt{|d|} \tag{2}$$

für jeden algebraischen Körper k festen Grades n ($n \geqslant 2$; h, R und d bezeichnen die Klassenzahl, den Regulator und die Diskriminante des Körpers). Walfisz [9] folgerte von (1), daß für jede Dirichletsche Funktion $L(s,\chi)$ mit den Charakter $\chi \bmod q$:

$$L(s,\chi) \neq 0 \quad \text{in} \quad s > c(\varepsilon) q^{-\varepsilon} \tag{3}$$

(ε beliebig klein >0). Die Vermutung (2) wurde von R. Brauer [1] im Jahre 1947 bewiesen. Den Analogon von (3) für Heckesche L-Funktionen findet man nirgends publiziert, obwohl es eine bedeutende Rolle in der analytischen Theorie der Ideale spielt. Es ist der Zweck dieses Artikels zu zeigen, daß dieses Analogon eine einfache Folge der (von R. Brauer bewiesenen) Residuenabschätzung

$$\operatorname*{Res}_{s=1} \zeta_k(s) = |d|^{o(1)} \qquad (|d|\to\infty) \tag{4}$$

der Dedekindschen Zetafunktion des Körpers k ist.

Für die Heckesche L-Funktionen [5] mit Charakteren mod$\mathfrak{f}$ des Körpers k wird fortan die Landausche [7] Bezeichnung $\zeta(s,\chi)$ benutzt. Wir setzen $D = |d| N\mathfrak{f}$, wo $N\mathfrak{f}$ die Norm des Idealen bezeichnet. Es ist bekannt (siehe [2]) daß für passende $c_1 = c_1(n) > 0$ im Gebiete

$$\sigma > 1 - c_1/\log D(2+|t|)$$