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On the distribution of the k-free integers in residue classes *

by

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Let k be a fixed integer $\geqslant 2$. Denote by Q_k the set of all positive integers n such that if $n = p_1^{e_1} \dots p_t^{e_t}$, where the p_i are the distinct prime divisors of n, then $e_i < k$, $i = 1, \dots, t$. Let L_k be the set of all n for which $e_i \geqslant k$, $i = 1, \dots, t$. The integers of Q_k are sometimes called k-free integers and those of L_k, k-full. Evidently, the only number common to Q_k and L_k is 1.

If $x \geqslant 1$, and if a and h are integers, $h \geqslant 1$, we define the function $Q_k(x; a, h)$ to be the number of integers in Q_k which are $\leqslant x$ and which lie in the residue class $P_{a,h}$ consisting of all n satisfying the congruence, $n \equiv a \pmod{h}$. Another function, $\varphi_s(n)$, will be defined for integral $s \geqslant 1$ by

$$\varphi_s(n) = n^s \prod_{p|n} (1-p^{-s}), \tag{1.1}$$

where the product is over the primes p dividing n. Notice that $\varphi_1(n)$ is the Euler φ-function (see (3.2)).

The function $Q_2(x; a, h)$ has been investigated by Landau ([8], pp. 633-636). If $d = (a, h)$ and if R is the product of all distinct prime factors of d which do not divide h/d, then Landau showed that, with $\zeta(s)$ representing the zeta-function,

$$\frac{Q_2(x; a, h)}{x} \sim \frac{1}{\zeta(2)} \cdot \frac{h\varphi(R)}{\varphi_2(h) R} \quad \text{as} \quad x \to \infty, \tag{1.2}$$

provided $d \in Q_2$. Actually, Landau obtained an estimate with O-term for $Q_2(x; a, h)$. For the case of arbitrary k, Ostmann ([11], p. 23) gives a result corresponding to (1.2) in which $\zeta(2)$ is replaced by $\zeta(k)$, $\varphi_2(h)$ becomes $\varphi_k(h)$ and h becomes h^{k-1}. It will be shown in § 2 that this latter result is incorrect. Moreover, Ostmann's error is carried over to his O-estimate of $Q_k(x; a, h)$.

Section 2 of this paper is devoted to proving a uniform estimate with remainder for $Q_k(x; a, h)$ (Theorem 1). In particular, it is shown

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that the correct generalization of Landau's formulas involves replacing $\varphi(n)$ by its unitary analogue, $\varphi^*(n)$ (see Remark 2).

In § 3 we use the formula derived for $Q_k(x; a, h)$ to show how the distribution of Q_k in a residue class modulo h is influenced by the arithmetical structure of the class. With an appropriate definition of equidistribution $(\bmod h)$ it is proved in Theorem 4 that Q_k is distributed equally among the residue classes $(\bmod h)$ if and only if $h \in L_k$. In general, therefore, equidistribution $(\bmod h)$ cannot occur. For example, the relative density of the even square-free integers is $\frac{1}{3}$ whereas the relative density of the odd square-free integers is $\frac{2}{3}$ (see the Example in § 3). Moreover, we determine in Theorem 5 those values of a for which the relative density of the k-free integers of $P_{a,h}$ is maximal, h assumed fixed. Finally, we note that although equidistribution $(\bmod h)$ is comparatively rare, the relative density function has a surprisingly regular behavior, as Theorem 3 illustrates.

In § 4 we show how the results of § 3 can be applied to give a quite simple and direct proof of a corollary of a theorem due to Page [12]. In particular, we obtain an asymptotic formula for the number of representations of a positive integer n as a sum of a j-free integer and a k-free integer (Corollary 2, Theorem 6).

One of the principal purposes of this paper is to remedy somewhat the unsatisfactory state of the literature concerning the behavior of $Q_k(x; a, h)$. To begin with, Landau's results concerning $Q_2(x; a, h)$ are never collected into a precise statement. For example, he actually obtains a *uniform* O-estimate, but this is never pointed out. Furthermore, his error term is $O(\sqrt{dx})$, where $d = (a, h)$, as compared with $O(\sqrt{x})$ given by Corollary 1 of Theorem 1. Also, even though generalizations of the problem of this paper appear in the literature (for example, Cohen [3]; Wirsing in an unpublished work ([11], footnote on p. 23); Evelyn and Linfoot [4]; Mirsky [10]; and Katz [7]), these are concerned neither with the question of a uniform estimate for $Q_k(x; a, h)$ nor with that of the equidistribution of the k-free integers $(\bmod h)$. Apparently the intrinsic interest of the problem has been overlooked.

The inaccuracy in Ostmann's generalization of Landau's estimate was originally pointed out by the second author of the present paper in an earlier work [13]. In fact, Theorem 1 was proved in an equivalent form in that paper ([13], Theorem 3.9).

In the way of notation, μ will denote the Möbius function, the letter p will be reserved for primes, and the greatest common divisor of a and b is denoted by (a, b). All summations are assumed to extend over positive integers only, and a vacuous product has the value 1. $[x]$ will denote the greatest integer not exceeding x.

Finally, we mention that our methods are completely elementary.

2. Estimate for $Q_k(x; a, h)$. For all integers $n > 0$ we define the *core* of n, denoted by $\nu(n)$, to be the largest square-free divisor of n. Then for $n \in Q_k$, $k \geqslant 2$, we call $\omega_k(n) = \nu^k(n)/n$ the *k-complement* of n.

A divisor $d > 0$ of the positive integer n will be called a *unitary divisor* if $d\delta = n$, $(d, \delta) = 1$. To denote that d is a unitary divisor of n, we write $d \| n$. Put $(a, n)_* = \delta$ if δ is the largest unitary divisor of n which divides a. If $(a, n)_* = 1$ we call a *unitarily prime* to n. The number of integers $a \pmod n$ such that $(a, n)_* = 1$ defines the function $\varphi^*(n)$. Since $\varphi^*(n)$ is multiplicative ([2], Corollary 2.2.1), we have

$$\varphi^*(n) = n \prod_{p^e \| n} (1 - p^{-e}) . \tag{2.1}$$

Now let h and a be given integers with $h \geqslant 1$ and let $d = (a, h)$. Denote by $\varrho_h(a)$ the product of the distinct prime divisors of d which do not divide h/d. Notice that $\varrho_h(a) = \nu\big((a, h)_*\big)$. Let

$$\theta_k(a, h) = \prod_{p \mid \varrho_h(a)} \left(1 - \frac{(p^k, h)}{p^k}\right) . \tag{2.2}$$

Finally, we place $H = (a, h)_*$ and $H' = \omega_k(H)$ if $H \in Q_h$.

With the previous notation we have

LEMMA 1. *If $(a, h)_* \in Q_k$, then $\theta_k(a, h) = \varphi^*(H')/H'$.*

Proof. From (2.2) and the relation, $\varrho_h(a) = \nu\big((a, h)_*\big)$,

$$\theta_k(a, h) = \prod_{p \mid \nu((a,h)_*)} \left(1 - \frac{(p^k, h)}{p^k}\right) = \prod_{p \mid (a,h)_*} \left(1 - \frac{(p^k, h)}{p^k}\right) . \tag{2.3}$$

Furthermore, since by hypothesis, $(a, h)_*$ is k-free, (2.3) becomes

$$\theta_k(a, h) = \prod_{p^e \| (a,h)_*} \left(1 - \frac{p^e}{p^k}\right) = \prod_{p^e \| H} \left(1 - \frac{1}{p^{k-e}}\right) . \tag{2.4}$$

But $p^e \| H$ if and only if $p^{k-e} \| H'$. So finally we get from (2.4) and (2.1) that

$$\theta_k(a, h) = \prod_{p^f \| H'} \left(1 - \frac{1}{p^f}\right) = \varphi^*(H')/H' .$$

Remark 1. If $d = (a, h) \notin Q_k$, then obviously $Q_k(x; a, h) = 0$ for all x.

One other bit of notation will be convenient. Let $S(x; \alpha, \beta, \gamma)$ denote the number of solutions $\leqslant x$ of the congruence $\alpha n \equiv \beta \pmod{\gamma}$ where

α and γ are $\geqslant 1$. The given congruence has no solution if $(\alpha,\gamma)\nmid\beta$ whereas there is a unique solution $\left(\bmod \gamma/(\alpha,\gamma)\right)$ if $(\alpha,\gamma)|\beta$. Therefore we have

$$(2.5)\qquad S(x;\alpha,\beta,\gamma)=\begin{cases} 0 & \text{if}\quad (\alpha,\gamma)\nmid\beta,\\ \left[\dfrac{x}{\gamma}(\alpha,\gamma)\right]+\varepsilon & \text{if}\quad (\alpha,\gamma)|\beta,\end{cases}$$

where ε is a function whose value is either 0 or 1.

We recall that

$$(2.6)\qquad q_k(n)=\sum_{d^k|n}\mu(d),$$

where $q_k(n)$ is 1 or 0 according as $n\in Q_k$ or $n\notin Q_k$ (cf. [5], Theorem 303).

The following theorem is the major result of this section. We shall use the previous notation.

THEOREM 1. *If $d\in Q_k$, then*

$$Q_k(x;a,h)=\frac{h^{k-1}}{\varphi_k(h)}\left(\frac{\varphi^*(H')}{H'}\right)\frac{x}{\zeta(k)}+O(\sqrt[k]{x});$$

the estimate is uniform in h and a.

Proof. By (2.6)

$$Q_k(x;a,h)=\sum_{\substack{z\leqslant x\\ z\equiv a\,(\mathrm{mod}\,h)}}q_k(z)=\sum_{\substack{z\leqslant x\\ z\equiv a\,(\mathrm{mod}\,h)}}\sum_{m^k|z}\mu(m)$$

$$=\sum_{\substack{m,n\\ m^kn\leqslant x\\ m^kn\equiv a\,(\mathrm{mod}\,h)}}\mu(m)=\sum_{m\leqslant\sqrt[k]{x}}\mu(m)\sum_{\substack{n\leqslant x/m^k\\ m^kn\equiv a\,(\mathrm{mod}\,h)}}1$$

$$=\sum_{\substack{m\leqslant\sqrt[k]{x}\\ (m^k,h)|a}}\mu(m)\,S(x/m^k;m^k,a,h).$$

By (2.5) and the fact that $(m^k,h)|a\rightleftarrows(m^k,h)|d$, we get

$$(2.7)\qquad Q_k(x;a,h)=\sum_{\substack{m\leqslant\sqrt[k]{x}\\ (m^k,h)|d}}\mu(m)\{(x/m^kh)(m^k,h)+O(1)\}$$

$$=\frac{x}{h}\sum_{\substack{m=1\\ (m^k,h)|d}}^{\infty}\frac{\mu(m)}{m^k}(m^k,h)-\frac{x}{h}\sum_{\substack{m>\sqrt[k]{x}\\ (m^k,h)|d}}\frac{\mu(m)}{m^k}(m^k,h)+O(\sqrt[k]{x}).$$

Now let

$$f(m)=\begin{cases}\mu(m)(m^k,h)/m^k & \text{if}\quad (m^k,h)|d,\\ 0 & \text{otherwise}.\end{cases}$$

The function f is a multiplicative function of m, and since $\sum_1^\infty f(m)$ is an absolutely convergent series, we may express this series as an Euler product (see [5], p. 249), obtaining

$$\sum_{m=1}^{\infty}f(m)=\prod_p\left(1+f(p)+f(p^2)+\ldots\right).$$

But $f(p^a)=0$ if $a\geqslant 2$ and furthermore $f(p)=-(p^k,h)/p^k$ if $(p^k,h)|d$ and $f(p)=0$ otherwise.

Hence

$$\sum_{m=1}^{\infty}f(m)=\prod_{\substack{p\\ (p^k\nmid h)}}\left(1-\frac{(p^k,h)}{p^k}\right)\prod_{\substack{(p^k,h)\nmid d\\ (p^k\nmid h)}}\left(1-\frac{(p^k,h)}{p^k}\right)^{-1}.$$

Since $(p^k,h)=1\rightleftarrows p\nmid h$, we can write

$$(2.8)\qquad \sum_{m=1}^{\infty}f(m)=\frac{\prod_p(1-p^{-k})}{\prod_{p|h}(1-p^{-k})}\prod_{\substack{p|h\\ (p^k\nmid h)}}\left(1-\frac{(p^k,h)}{p^k}\right)\prod_{\substack{(p^k,h)\nmid d\\ (p^k\nmid h)}}\left(1-\frac{(p^k,h)}{p^k}\right)^{-1}$$

$$=\frac{h^k}{\zeta(k)\varphi_k(h)}\prod_{\substack{p|h\\ (p^k,h)|d}}\left(1-\frac{(p^k,h)}{p^k}\right).$$

The conditions, $p|h$ and $(p^k,h)|d$, are equivalent to the conditions, $p|d$ and $p\nmid h/d$.

By definition of $\varrho_h(a)$, $p|h$ and $p\nmid h/d\rightleftarrows p|\varrho_h(a)$. Therefore, by virtue of Lemma 1, (2.8) may be written as

$$(2.9)\qquad \sum_{m=1}^{\infty}f(m)=\left(\frac{h^k}{\zeta(k)\varphi_k(h)}\right)\theta_k(a,h)=\left(\frac{h^k}{\zeta(k)\varphi_k(h)}\right)\frac{\varphi^*(H')}{H'}.$$

Finally,

$$(2.10)\qquad \left|\sum_{\substack{m>\sqrt[k]{x}\\ (m^k,h)|d}}\frac{\mu(m)}{m^k}(m^k,h)\right|<d\sum_{m>\sqrt[k]{x}}\frac{1}{m^k}=O\left(\frac{d}{x^{1-1/k}}\right).$$

The theorem results on combining (2.7), (2.9), and (2.10) and observing that the constants involved in the O-estimates are independent of h and a.

The case $k=2$ is an important special case of the above theorem.

COROLLARY 1 ($k=2$). *If* $d \in Q_2$, *then*

$$Q_2(x;\, a,\, h) = \frac{h}{\varphi_2(h)} \left(\frac{\varphi(H)}{H}\right) \frac{6x}{\pi^2} + O\left(\sqrt{x}\right),$$

uniformly in h *and* a, *where* $H = (a, h)_*$.

Proof. If $k=2$, then $H = H'$. Furthermore, since H is square-free, $\varphi^*(H) = \varphi(H)$. The corollary results on replacing $\zeta(2)$ by $\pi^2/6$.

Remark 2. The proof of this corollary brings into evidence the very special nature of the case $k=2$. In general, $H \neq H'$ and $\varphi^*(H') \neq \varphi(H')$.

Corollary 1 has been given in an equivalent form by W. Schwarz ([14], Lemma 8).

COROLLARY 2 ($h=1$, $a=0$). *Let* $Q_k(x;\, 0, 1)$ *be denoted by* $Q_k(x)$. *Then*

$$Q_k(x) = \frac{x}{\zeta(k)} + O\left(\sqrt[k]{x}\right).$$

From this corollary it follows that

$$\lim_{x\to\infty} \frac{Q_k(x)}{x} = \frac{1}{\zeta(k)}. \tag{2.11}$$

Corollary 2 is a classical result due to Gegenbauer.

3. Relative density and equidistribution (mod h). A residue a (mod h) will be called k-*admissible* (mod h), or simply *admissible* (mod h) if $(a, h) \in Q_k$. A residue class (mod h) will be called admissible (mod h) if it is defined by an admissible residue (mod h). The *density* of the k-free integers in the residue class $P_{a,h}$ is denoted by $\delta_k(a, h)$ and is defined by

$$\delta_k(a, h) = \lim_{x\to\infty} \frac{Q_k(x;\, a,\, h)}{x}.$$

Theorem 1 guarantees the existence of this limit. The *relative density* of the k-free integers in $P_{a,h}$, denoted by $\delta_k^*(a, h)$, is defined to be

$$\delta_k^*(a, h) = \lim_{x\to\infty} \frac{Q_k(x;\, a,\, h)}{Q_k(x)},$$

so that by (2.11)

$$\delta_k^*(a, h) = \frac{\delta_k(a, h)}{\delta_k(1, 0)} = \zeta(k)\,\delta_k(a, h). \tag{3.1}$$

The symbol $\Phi_k(h)$ will denote the number of residues a (mod h) such that $(a, h) \in Q_k$. The number of k-admissible residue classes (mod h) is therefore $\Phi_k(h)$. The function $\Phi_k(h)$ has the evaluation ([9], [1])

$$\Phi_k(h) = h \prod_{p^k \mid h} (1 - p^{-k}); \tag{3.2}$$

note that $\Phi_k(h) = \Phi_k(h^k)$.

The relationship in (3.1) together with Theorem 1 yields the following result.

THEOREM 2.

$$\delta_k^*(a, h) = \begin{cases} \dfrac{h^{k-1}}{\varphi_k(h)} \cdot \dfrac{\varphi^*(H')}{H'} & \text{if} \quad (a, h) \in Q_k, \\ 0 & \text{if} \quad (a, h) \notin Q_k. \end{cases} \tag{3.3}$$

COROLLARY 1. *The relative density of the* k-*free integers in an admissible residue class* (mod h) *is positive.*

COROLLARY 2. *If a residue class* (mod h) *is admissible, it contains infinitely many* k-*free integers.*

Two other corollaries follow immediately.

COROLLARY 3. *The number of residue classes* (mod h) *containing infinitely many* k-*free integers is* $\Phi_k(h)$, *determined by* (3.2).

COROLLARY 4. *If a residue class* (mod h) *contains at least one* k-*free integer, it contains infinitely many.*

Remark 3. Corollary 3 was stated by Wilkins in an abstract of an unpublished paper [15] (also cf. McCarthy [9]). In the case $k=2$, Corollary 3 becomes a theorem of Haviland [6].

The relative density $\delta_k^*(a, h)$ is a periodic function of a with period h, by definition. Moreover, δ_k^* is further restricted in its values by the following theorem, which is an easy consequence of Theorem 2.

THEOREM 3. *For fixed* h, *the relative density* $\delta_k^*(a, h)$ *is an even function of* a (mod h), *that is,* δ_k^* *depends only on the value of* (a, h). *Moreover, for all pairs* $\{a, h\}$, $h > 0$, *for which* a *is admissible* (mod h), $\left(\varphi_k(h)/h^{k-1}\right) \times \delta_k^*(a, h)$ *is a function of* $(a, h)_*$. *In particular, for each* k-*free value of* h, $\delta_k^*(a, h)$ *is a unitary function of* a (mod h), *that is,* δ_k^* *depends only on the value of* $(a, h)_*$.

Remark 4. Using the notation and terminology of § 2, one infers from the definition of H that $H=1$, or equivalently $H'=1$, if and only if a is unitarily prime to h.

We shall now apply the concept of relative density to a discussion of equidistribution (mod h) of the k-free integers.

DEFINITION. The k-free integers are said to be *equidistributed* (mod h) if the relative density of the k-free integers is the same for each admissible residue class (mod h).

Remark 5. The set Q_k is equidistributed (mod h) if and only if in every admissible residue class Q_k has relative density $1/\Phi_k(h)$.

A simple necessary and sufficient condition for equidistribution is contained in the following theorem.

THEOREM 4. *The k-free integers are equidistributed* $(\mathrm{mod}\, h)$ *if and only if* $h \in L_k$.

Proof. By Theorem 2 and Remark 5, equidistribution $(\mathrm{mod}\, h)$ occurs if and only if

$$\frac{h^{k-1}}{\varphi_k(h)} \cdot \frac{\varphi^*(H')}{H'} = \frac{1}{\Phi_k(h)}, \qquad a \text{ admissible}.$$

From (1.1) and (3.2), the above equation reduces to

$$\frac{\varphi^*(H')}{H'} = \prod_{\substack{p \mid h \\ p^k \nmid h}} (1-p^{-k}), \qquad a \text{ admissible}. \tag{3.4}$$

If $a = 1$, then $(a, h) = 1 \in Q_k$. Furthermore, in this case, $H' = 1$ so that $\varphi^*(H')/H' = 1$. Hence, equidistribution $(\mathrm{mod}\, h)$ must imply that

$$\prod_{\substack{p \mid h \\ p^k \nmid h}} (1-p^{-k}) = 1,$$

and this relation implies that $h \in L_k$. Conversely, if $h \in L_k$, then $H' = 1$ by Remark 4, and (3.4) is satisfied. Q. E. D.

From Theorem 2 we see that for fixed h, $\delta_k^*(a, h)$ is maximal if and only if $\varphi^*(H')/H'$ is maximal. But by definition of φ^*, $\varphi^*(n) \leqslant n$ with equality only when $n = 1$. Therefore, it follows that $\delta_k^*(a, h)$ is maximal if and only if $H' = 1$, so that by Remark 4, (1.1) and (3.3), the following extension of Theorem 4 results:

THEOREM 5. *For fixed h, a necessary and sufficient condition for the relative density of the k-free integers in an admissible residue class $P_{a,h}$ to be maximal is that a be unitarily prime to h; moreover this maximal value is*

$$\delta_k^*(a, h) = \frac{h^{k-1}}{\varphi_k(h)} = \Big(h \prod_{p \mid h} (1-p^{-k})\Big)^{-1}.$$

EXAMPLE. As a simple example of a case in which equidistribution does not occur, we consider the case in which h is an arbitrary prime, p. In this case we have

$$\delta_k^*(0, p) = \frac{p^{k-1}-1}{p^k-1}, \qquad \delta_k^*(a, p) = \frac{p^{k-1}}{p^k-1}$$

if $p \nmid a$,

4. A special case and its application to an additive problem. An important special case of Theorem 1 is

THEOREM 6. *If* $(a, h)_* = 1$, *and in particular if* $h \in L_k$, *then*

$$Q_k(x; a, h) = \frac{h^{k-1}}{\varphi_k(h)} \left(\frac{x}{\zeta(k)}\right) + O(\sqrt[k]{x}), \qquad (a, h) \in Q_k,$$

uniformly in a and h.

For the purposes of this section it will be convenient to have Theorem 1 in a slightly different form. Let $Q_k^*(x; a, h)$ denote the number of k-free integers *strictly less than* x which are congruent to $a\,(\mathrm{mod}\, h)$; Theorem 1 remains valid with $Q_k(x; a, h)$ replaced by $Q_k^*(x; a, h)$. In particular, we can state the following special case of Theorem 6.

COROLLARY 1. *If* $2 \leqslant k \leqslant j$ *and if* $(m^j, a) \in Q_k$, *then*

$$Q_k^*(x; a, m^j) = \left(\frac{m^{k-j}}{\varphi_k(m)\zeta(k)}\right) x + O(\sqrt[k]{x}),$$

uniformly in m and a.

Proof. This result follows by Theorem 6, in connection with the above remarks and the fact that $\varphi_k(m^j) = m^{(j-1)k}\varphi_k(m)$ (see (1.1)).

Let $T_{j,k}(n)$ denote the number of representations of the integer n as the sum of a j-free and a k-free integer, $2 \leqslant k \leqslant j$. We shall use Corollary 1 above to obtain the following estimate for $T_{j,k}(n)$, proved by Page in a somewhat more precise form in [12].

COROLLARY 2. $T_{j,k}(n) \sim n H_{j,k}(n)$ *as* $n \to \infty$, *where*

$$H_{j,k}(n) = \prod_p (1-p^{-k}-p^{-j}) \prod_{p^k \mid n} \left(1 + \frac{1}{p^j - p^{j-k} - 1}\right).$$

Proof. Let q_k denote the characteristic function of the k-free integers (see § 2); then

$$\begin{aligned} T_{j,k}(n) &= \sum_{a+b=n} q_j(a) q_k(b) = \sum_{a+b=n} \Big(\sum_{m^j \mid a} \mu(m)\Big) q_k(b) \\ &= \sum_{\substack{m,r,b \\ m^j r + b = n}} \mu(m) q_k(b) = \sum_{m < n^{1/j}} \mu(m) \sum_{\substack{b<n \\ b \equiv n\,(\mathrm{mod}\, m^j)}} q_k(b) \\ &= \sum_{m < n^{1/j}} \mu(m) Q_k^*(n; n, m^j). \end{aligned}$$

Now let ε denote an arbitrary number such that $0 < \varepsilon < 1/j$. Then, considering separately those values of $m \leqslant n^{1/j-\varepsilon}$ and those $> n^{1/j-\varepsilon}$, we get

$$T_{j,k}(n) = \sum_{m \leqslant n^{1/j-\varepsilon}} \mu(m) Q_k^*(n; n, m^j) + O \sum_{n^{1/j-\varepsilon} < m} Q_k^*(n; n, m^k).$$

By definition of Q_k^* we have $Q_k^*(n;a,h) = O(n/h)$ uniformly in a. Hence by Corollary 1

$$T_{j,k}(n) = \frac{n}{\zeta(k)} \sum_{\substack{m\leqslant n^{1/j-\varepsilon} \\ (m^j,n)\in Q_k}} \frac{\mu(m)}{m^{j-k}\varphi_k(m)} + O(n^{1/k+1/j-\varepsilon}) + O\Big(n \sum_{m>n^{1/j-\varepsilon}} m^{-j}\Big).$$

Since $1/k+1/j<1$, the first O-term is $o(n)$. Moreover, the second O-term is $O(n^{1/j+\varepsilon(j-1)})$ and is therefore also $o(n)$. Dividing by n we get

$$\lim_{n\to\infty} \frac{T_{j,k}(n)}{n} = \frac{1}{\zeta(k)} \sum_{\substack{m=1 \\ (m^j,n)\in Q_k}}^{\infty} \frac{\mu(m)}{m^{j-k}\varphi_k(m)},$$

the series being absolutely convergent because

$$\sum_{m=1}^{\infty} \frac{m^{k-j}}{\varphi_k(m)} = \sum_{m=1}^{\infty} m^{-j} \prod_{p|m} (1-p^{-k})^{-1} < \sum_{m=1}^{\infty} m^{-j} \prod_{p} (1-p^{-k})^{-1} = \zeta(k)\zeta(j).$$

We therefore have

$$\lim_{n\to\infty} \frac{T_{j,k}(n)}{n} = \frac{1}{\zeta(k)} \sum_{m=1}^{\infty} g(m), \tag{4.1}$$

where

$$g(m) = \begin{cases} \dfrac{1}{m^{j-k}} \cdot \dfrac{\mu(m)}{\varphi_k(m)} & \text{if} \quad (m^j,n)\in Q_k, \\ 0 & \text{otherwise}. \end{cases}$$

The function g is a multiplicative function of m, and by an Euler factorization we get $\sum_1^{\infty} g(m) = \prod_p (1+g(p)+g(p^2)+\ldots)$. But if $a\geqslant 2$, $g(p^a)=0$; moreover if $p^k\nmid n$, $g(p) = -1/p^j(1-p^{-k})$. Therefore

$$\sum_{m=1}^{\infty} g(m) = \prod_p \Big(1-\frac{1}{p^j(1-p^{-k})}\Big) \frac{1}{\prod_{p^k|n}\Big(1-\dfrac{1}{p^j(1-p^{-k})}\Big)}.$$

It follows easily that $\big(1/\zeta(k)\big)\sum_1^{\infty} g(m) = nH_{j,k}(n)$, and the corollary results by (4.1) and the fact that $H_{j,k}(n)>0$.

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