On polynomial transformations II

by

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1. We say that a subset $X$ of a field $K$ has property (P) if and only if every polynomial $P(t)$ with coefficients from $K$ such that $P(X) = X$ is linear. We say that $X$ has property (P) if and only if every infinite subset of $X$ has property (P). It has been proved in [1] that the algebraic number fields have property (P) and, moreover, that this property is preserved by any pure transcendental, finitely generated extension of a field having this property. In this paper we prove:

**Theorem I.** If every finite algebraic extension of a number field has property (P), then every field of algebraic functions in any number of variables over $K$ has property (P).

**Theorem II.** If the field $K$ has property (P), then every pure transcendental extension of $K$ has this property also.

From theorem I and from theorem I of [1] it follows that every field finitely generated over the rationals has property (P).

2. Proof of Theorem I.

**Fundamental Lemma** (see [1], Lemma 1). Suppose $T$ is a transformation of the set $X$ onto itself. Suppose there exist two functions $f(x)$ and $g(x)$ defined on $X$, with values in the set of natural numbers, subject to the conditions:

(a) For every constant $c$ the equation $f(x) + g(x) = c$ has only a finite number of solutions.

(b) There exists a constant $C$ such that for $f(x) \geq C$ it follows $f(Tx) > f(x)$.

(c) For every $M$ there exists a constant $B(M)$ such that for $f(x) \leq M$ and $g(x) \geq B(M)$ follows $g(Tx) > g(x)$.

Then $X$ is finite.

Now let $K$ be a number field such that every finite algebraic extension of $K$ has property (P), and let $R$ be a field of algebraic functions over $K$. Without restriction in generality we can assume that $R$ is a normal
Indeed, suppose that in a neighborhood of \( r_i \) the function \( \xi(x) \) is bounded. If \( Q(r_i) \neq 0 \) then
\[
\lim_{x \to r_i} \xi(x) = \frac{1}{Q(r_i)} \sum_{k=1}^{n} P_k(r_i) \omega_k(r_i) \cdot K(\omega_k(r_i), \ldots, \omega_n(r_i)) .
\]

If \( Q(r_i) = Q'(r_i) = \cdots = Q^{(n-1)}(r_i) = 0 \), but \( Q^{(n)}(r_i) \neq 0 \), then
\[
\lim_{x \to r_i} \xi(x) = \frac{1}{Q^{(n)}(r_i)} \lim_{x \to r_i} \frac{d^n}{dx^n} \left( \sum_{k=1}^{n} P_k(x) \omega_k(x) \right) .
\]

From the definition of \( S \) it follows that the functions \( d^n \omega_k / dx^n \) have no poles in \( S \); consequently
\[
\frac{d^n}{dx^n} \left( \sum_{k=1}^{n} P_k(x) \omega_k(x) \right) \bigg|_{x = r_i} \in K(\omega_k(r_i), \ldots, \omega_n(r_i))
\]
and
\[
\lim_{x \to r_i} \xi(x) \in K(\omega_k(r_i), \ldots, \omega_n(r_i)) .
\]

Let us define
\[
W(t) = \frac{1}{A(t)} \sum_{j=0}^{n} A_j(t) U_j(t) , \quad X_t = \left\{ \lim_{x \to t} \xi(x) : \xi \in X, \lim_{x \to t} \xi(x) \neq a \right\} .
\]

The foregoing argument shows that \( X_t \subseteq K(\omega_k(r_i), \ldots, \omega_n(r_i)) \). Moreover, the polynomials \( W_t(t) \) have the degree \( n \geq 2 \), and their coefficients belong to \( K(\omega_k(r_i), \ldots, \omega_n(r_i)) \). Now we prove that \( W_t(X_t) = X_t \). Hence there exists \( E(t) \) such that \( \lim E(t) = \tilde{E} \); thus, since \( W(X_t) = X_t \), there exists \( U(t) \cdot X_t \) such that \( W(U(t)) = \tilde{E}(t) \), i.e.:
Now if \( \lim_{x \to x_1} |U(x)| = \infty \), then
\[
\frac{E(x)}{|U(x)|} > B_n |w(x)| - B \to \infty \quad \text{as} \quad x \to x_1,
\]
but this is impossible, since from \( n \geq 2 \) we infer that
\[
\frac{E(x)}{|U(x)|} \leq B_n |w(x)|^{-1} \to 0 \quad \text{as} \quad x \to x_1.
\]

Thus there exists a finite limit \( u = \lim_{x \to x_1} U(x) \), \( u \in \mathcal{X} \), and obviously \( W(u) = \mathcal{X} \). Hence \( x_1 \in W(U(x)) \). On the other hand, if \( U(x) \in \mathcal{X} \), \( W(U(x)) = \mathcal{X} \) and \( \lim_{x \to x_1} U(x) = u \), then \( \lim_{x \to x_1} E(x) = W(u) \neq \mathcal{X} \) consequently \( W(u) \notin \mathcal{X} \), and so \( x_1 \notin W(U(x)) \), which together with the inclusion formerly established gives \( W(U(x)) = x_1 \).

Now let us remark that the field \( K \{a_0(x), \ldots, a_n(x)\} \) is for \( n \geq 1 \) a finite algebraic extension of \( K \), and so this field possesses property (A1); consequently the sets \( x_i \) must be finite. Since the sequence \( r_i \) is infinite, one infers without difficulty that there can be only a finite number of elements \( f \) in \( \mathcal{X} \) such that \( f(\xi) \neq f(\xi) = f(\xi) \).

We have thus proved that our set \( \mathcal{X} \), the polynomial \( W(t) \), and the functions \( f(\xi) \), \( g(\xi) \), defined as above satisfy condition (a) of our fundamental lemma. Now we are going to prove that the remaining two conditions also are satisfied.

**Lemma 2.** Let \( L \) be a principal ideal domain and \( L' \) its integral closure in \( \mathcal{X} \), a finite algebraic, normal, separable extension of \( \mathcal{K} \) of the quotient field \( K \) of \( L \). Then, for every fixed \( b \) in \( L' \), and every natural number \( n \), there can be only a finite number of \( a \) in \( L \) such that, with some \( e \) in \( L' \), \( a \equiv e \) in \( L \) but no non-unit divisor of \( a \) in \( L \) divides \( e \).

The proof of this lemma in the case where \( L \) is the ring of rational integers was given in [1] (lemma 2). The proof in the general case is almost literally the same. One need only remark that \( L \) is a Dedekind domain (see [3], p. 281) and that in \( L \) there is only a finite number of prime ideals which ramify in \( L' \) (see [3], p. 303).

**Corollary.** Let \( A(\xi) \) be a fixed integral algebraic function from \( R \), and let \( n \) be a positive integer. Then there exist only a finite number of polynomials \( U(\xi) \) with coefficients from \( K \) such that for a certain integral algebraic function \( V(\xi) \) from \( R \), the quotient \( \frac{A(\xi)V(\xi)}{U(\xi)} \) is integral in \( K \), and simultaneously for every non-constant polynomial \( \Phi(\xi) \) over \( K \), dividing \( U(\xi) \) the quotient \( \frac{V(\xi)}{\Phi(\xi)} \) is integral in \( K \).

For the proof one should observe that the ring of polynomials over a field is a principal ideal domain.

**Lemma 3.** Let \( A(\xi) \) be an integral algebraic function from \( R \), \( A(\xi) \neq 0 \), and let \( n \) be a positive integer. Then
\[
\Phi(A) = \inf_{\Phi(\xi) \neq 0} (g(A^2) - ng(\xi)) = -\infty.
\]

**Proof.** Let \( A(\xi) \) be \( A(\xi) = \sum_{j=1}^{m} a_j(\xi) \), where \( A_j \) are polynomials over \( K \). For every \( n \), functions \( f_i(\xi), \ldots, f_{m}(\xi) \) defined in \( S \) the following identity holds:
\[
\sum_{j=1}^{m} A_j(\xi) \alpha_{j}(\xi) \left( \sum_{k=1}^{m} f_k(\xi) \alpha_{k}(\xi) \right)^n = \sum_{k=1}^{m} T_k f_k(\xi) \alpha_{k}(\xi) \quad (r = 1, \ldots, m),
\]
where \( T_k \) are homogeneous forms of \( n \)-th degree of \( m \) variables, with coefficients which depend only on \( A \) and \( n \) and are polynomials over \( K \).

Suppose that
\[
g(A^2) - ng(\xi) \to -\infty, \quad \xi(\xi) = \sum_{j=1}^{m} P_j^{(0)}(\xi) \alpha_{j}(\xi).
\]
We can assume that \( g(\xi) = \deg P_j^{(0)}(\xi) \), \( j = 1, 2, \ldots \), and that
\[
g(A^2) - ng(\xi) \leq -j, \quad j = 1, 2, \ldots
\]
Let us define, for \( \mathfrak{p} \in \mathcal{S} \), \( s(\mathfrak{p}) = \max \{a^{(0)}(\xi)\} \). Evidently
\[
\deg \left( T_k f_k(\xi) \right) \leq [\mathfrak{p}] - 1 + [\mathfrak{p}]
\]
whence for \( |x| \) sufficiently large (say, for \( |x| \gg a^{(0)}(\xi) \)),
\[
\left| T_k f_k(\xi) \right| \left| P_l^{(0)}(\xi) \right| \left| P_m^{(0)}(\xi) \right| \leq B(j) \frac{1}{|x|^n}
\]
and
\[
\left| T_k f_k(\xi) \right| \left| P_l^{(0)}(\xi) \right| \left| P_m^{(0)}(\xi) \right| \leq B(j) \frac{1}{|x|^n}
\]
By multiplication by \( a^{(0)}(\xi) \) and addition, we obtain
\[
\left| \sum_{k=1}^{m} T_k f_k(\xi) \right| \left| P_l^{(0)}(\xi) \right| \left| P_m^{(0)}(\xi) \right| \leq B(j) \frac{s(\mathfrak{p})}{|x|^n} \quad (r = 1, \ldots, m),
\]
whence
\[
\left| \sum_{k=1}^{m} T_k f_k(\xi) \right| \left| P_l^{(0)}(\xi) \right| \left| P_m^{(0)}(\xi) \right| \leq B(j) \frac{s(\mathfrak{p})}{|x|^n},
\]
which gives
\[
\left| \sum_{k=1}^{m} T_k f_k(\xi) \right| \left| P_l^{(0)}(\xi) \right| \left| P_m^{(0)}(\xi) \right| = 0 \quad \frac{1}{a^{(0)}(\xi)} \left( \frac{1}{a^{(0)}(\xi)} \right) \quad (j = 1, 2, \ldots, r = 1, \ldots, m).
\]
(Here and below the constants in \( \Phi \) depend on \( j \) but not on \( x \).)
From $v(\alpha)|\mu(\alpha)|F_\alpha(\alpha)$ follows $v(\alpha)|\sum_{\alpha \leq B} F_\alpha(\alpha)\omega_\alpha(\alpha)$, and thus we have $v(\alpha)|A_\alpha(\alpha)^2(\alpha)$. The application of the corollary to lemma 2 shows us that the degree of the polynomial $v(\alpha)$ is bounded by a constant $B$ depending only on $A_\alpha(\alpha)$ and $n$. Now since $\mu(\alpha) = d_\alpha(\alpha)v(\alpha) = d_\alpha(\alpha)v(\alpha)$ $(d_\alpha, d_\alpha$ relatively prime), $d_\alpha(\alpha)$ divides $A_\alpha(\alpha)^2(\alpha)$ and consequently the degree of $\mu(\alpha)$ is bounded by $nB + \deg \alpha$.

If $f(W(\xi)) < f(\xi)$ then evidently the degree of $g$ is at least equal to degree of $\alpha + \deg g(\alpha) - \deg \mu(\alpha)$ and so the degree of $g(\alpha)$ is at most equal to $nB(n-1)$, which proves the lemma.

**Lemma 5.** For every constant $M$ there exists a constant $B(M)$ such that from $f(\xi) \leq M$ and $g(\xi) \leq B(M)$ follows $g(W(\xi)) > g(\xi)$, for $\xi \in X$.

**Proof.** The following inequalities result directly from the definitions of the functions $f(\xi)$ and $g(\xi)$:

$$f(a+b) \leq f(a) + f(b), \quad f(ab) \leq f(a) + f(b),$$

$$g(a+b) \leq \max(f(a), f(b)) + \max(g(a), g(b)),$$

$$g(ab) \leq B' + g(a) + g(b),$$

with a suitable $B'$.

If $Q$ is a polynomial, then $g(\alpha) - \deg g Q < g(\alpha)(Q) < g(\alpha)$.

Suppose $f(\xi) \leq M$. Then

1. $\left( \frac{1}{\alpha} \sum_{\alpha \leq B} A_\alpha(\alpha)^2(\alpha) \right) \leq \left( \sum_{\alpha \leq B} A_\alpha(\alpha)^2(\alpha) \right) < (n-1) f(\xi) + M$,

with a suitable $M', M$. Let $\xi = (\xi, Q)$, where $\deg Q = f(\xi)$, and $f(\xi) = 0$. Then

$$g(\alpha) \geq (\alpha)^2(\alpha) - \deg \alpha \geq (\alpha)^2(\alpha) - \deg \alpha$$

$$\geq g(\alpha)^2 - n\alpha(\alpha)f(\xi) - \deg \alpha \geq g(\alpha)^2 - n\alpha(\alpha)f(\xi) - nM - \deg \alpha$$

$$\geq n\alpha(\alpha) + \alpha(\alpha) - n\alpha(\alpha)f(\xi) - nM - \deg \alpha \

If for a sequence $\xi_n$, with $f(\xi_n) \leq M$,

2. $\lim_{n \to \infty} [g(W(\xi_n)) - n\alpha(\xi_n)] = -\infty$,

then (since in view of lemma 1, $g(\xi)$ tends to infinity) we have

$$\alpha(\alpha) \leq g(\alpha)^2 - n\alpha(\alpha)f(\xi) + nM + \deg \alpha$$

$$\leq \deg \alpha + nM + g(W(\xi_n)) - n\alpha(\xi_n)$$

$$\leq \deg \alpha + nM + M + \max [g(W(\xi_n)), g\left(\sum_{\alpha \leq B} A_\alpha(\alpha)^2(\alpha)\right)] - n\alpha(\xi_n) \to -\infty$$

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in view of (1) and (2), which is incompatible with lemma 3. Thus
\[ g(W(z)) \geq ng(z) - M, \]
with some \( M \), whence the inequality \( g(W(z)) \leq g(z) \) cannot be true for sufficiently large \( g(z) \). The lemma is thus proved.

To prove the theorem it suffices now to observe that, in view of lemmas 4 and 5, conditions (b) and (c) of the fundamental lemma are satisfied, and lemma 1 ensures condition (a), whence the application of this lemma shows us that the set \( X \) is finite, which is what was to be proved.

3. Proof of theorem II.

Lemma 6 (see [2], p. 188). Let \( L = K(\theta) \) be a simple transcendental extension of a field \( K \). Let us define for \( \xi = A(\theta)/B(\theta) \times L_1 \), \((A(\theta), B(\theta)) = 1\), \( F(\xi) = \max(\deg A, \deg B) \). If \( P(t) \) is a polynomial of degree \( m \) with coefficients from \( K \), then \( P(F(\xi)) = mF(\xi) \).

Now let \( K = E(\theta_1, \ldots, \theta_n) \) be a pure transcendental extension of a field \( E \) which possesses property \( (P_0) \). Let \( Q \) be a polynomial with coefficients from \( K \), of degree \( n \geq 2 \), and let \( X \) be a subset of \( K \) for which \( Q(X) = X \). Since (see theorem II in [1]) every pure transcendental extension of \( K \) obtained by adjoining a finite set of elements has property \( (P_0) \), we can assume that the coefficients of the polynomial \( Q \) belong to \( K \), for otherwise we could obtain this by adjoining to \( K \) all the indeterminates which occur in the coefficients of \( Q \).

Let us now remark that, if \( L \) is a simple transcendental extension of a field \( K \), then, from \( X \subseteq L, X \setminus K \) being non-void, and \( \bar{Q}(X) = X \), where \( \bar{Q} \) is a polynomial over \( K \), it follows that \( \bar{Q} \) is linear. Indeed, let \( a \in X \setminus K \) be such that \( \min \{ F(\bar{Q}(\xi)) \} = F(a) \neq 0 \), and let \( \bar{Q}(\beta) = a \). Then from lemma 6 follows \( nF(\bar{Q}(\xi)) = F(a) \), whence \( F(\bar{Q}(\xi)) = 0 \), i.e. \( \bar{Q} \in X \setminus K \) and so \( F(\bar{Q}(\xi)) = F(a) = nF(\bar{Q}(\xi)) \), which can occur only if \( n = 1 \).

Now, if \( X \subseteq K \), then the set is finite by the assumption. If there exists \( \xi \in X \setminus K \) which, for example, has the form
\[ \xi = A(\theta_1, \ldots, \theta_n)/B(\theta_1, \ldots, \theta_n), \]
then let \( K = K(\theta_i) \) where \( K = K(\theta_1, \ldots, \theta_n) \). Since \( X \setminus K \) is non-void, then from the preceding remark we infer that \( n = 1 \), contrary to our assumption. The theorem is thus proved.

It is worth remarking that, if a field \( K \) has property \( (P_0) \) and \( X \) is a subset of \( K \) such that with a suitable non-linear polynomial \( P(t) \), \( P(X) \supset X \), then \( X \) must be finite. Indeed, let \( Y \) be the smallest set containing \( X \) and closed under the mapping \( z \rightarrow P(z) \). If \( X \) is infinite, then the coefficients of \( P(t) \) belong to \( K \), and so \( Y \) is contained in \( K \). Moreover \( P(Y) = Y \), and so \( Y \) is finite, which is impossible.

References


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