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ACTA ARITHMETICA
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Another note on Hardy-Littlewood's theorem

by

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1. In this paper we return to the subject of [3], i.e. to the investigation of the behaviour of

$$F(y)=\sum_{n=1}^{\infty}\{\Lambda(n)-1\}e^{-ny}, \qquad y>0, \tag{1.1}$$

as $y\to 0+$. Unlike in [3] we shall be interested here in oscillatory properties of the function (1.1). Hardy and Littlewood showed [1] that on the Riemann hypothesis there is a constant K such that each of the inequalities

$$F(y)<-\frac{K}{y^{1/2}}, \qquad F(y)>\frac{K}{y^{1/2}} \tag{1.2}$$

is satisfied for an infinity of values of y tending to zero. In connection with this result we shall supply here inequalities similar, though somewhat weaker, to (1.2) holding however in an explicit form and without any hypothesis. In the proof we shall use the method of Turán (see [5]), particularly its development to the study of oscillatory questions in prime number theory (see [4]). Our result reads as follows:

Theorem. *For* $0<\delta<c_1$ (¹) *we have*

$$\max_{\delta\leqslant y\leqslant\delta^{1/3}} F(y)>\delta^{-1/2}\exp\left(-14\,\frac{\log(1/\delta)\log\log\log(1/\delta)}{\log\log(1/\delta)}\right) \tag{1.3}$$

and

$$\min_{\delta\leqslant y\leqslant\delta^{1/3}} F(y)<-\delta^{-1/2}\exp\left(-14\,\frac{\log(1/\delta)\log\log\log(1/\delta)}{\log\log(1/\delta)}\right). \tag{1.4}$$

Corollary. *Replacing the exponent* $\frac{1}{2}$ *in* (1.2) *by* $\frac{1}{2}-\varepsilon$, $\varepsilon>0$ *and arbitrary, the inequalities are satisfied (without any hypothesis!) for an infinity of values of* y *tending to zero.*

(¹) c_1 and further $c_2, c_3, \ldots$ denote positive, numerically calculable constants.

2. Here are some lemmas which will be used in the following.

LEMMA 1. *For $0 < \delta < c_2$ there is a y_1 with*

$$\frac{1}{12}\log\log\frac{1}{\delta} \leqslant y_1 \leqslant \frac{1}{10}\log\log\frac{1}{\delta} \tag{2.1}$$

such that for all non-trivial zeros $\varrho = \beta + i\gamma$ of $\zeta(s)$ we have

$$\pi \geqslant \left|\arg\frac{e^{i\gamma y_1}}{\varrho}\right| \geqslant \frac{c_3}{|\gamma|^5\log|\gamma|}\,. \tag{2.2}$$

For the proof see [2].

Before formulating the next lemma we give some preliminary explanations. Let m be a positive integer and

$$1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_n| \tag{2.3}$$

and with a $0 < \varkappa \leqslant \pi/2$

$$\varkappa \leqslant |\arg z_j| \leqslant \pi\,. \tag{2.4}$$

Let the index h be such that

$$|z_h| > \frac{4n}{m+n(3+\pi/\varkappa)} \tag{2.5}$$

and fixed. Further

$$B \overset{\text{def}}{=} \min_{h<\xi\leqslant n} \operatorname{re}\sum_{j=1}^{\xi} b_j\,. \tag{2.6}$$

Then we have

LEMMA 2. *If $B > 0$ then there are integers ν_1 and ν_2 with*

$$m+1 \leqslant \nu_1,\ \nu_2 \leqslant m+n(3+\pi/\varkappa) \tag{2.7}$$

such that

$$\operatorname{re}\sum_{j=1}^{n} b_j z_j^{\nu_1} \geqslant \frac{B}{2n+1}\left(\frac{n}{24\big(m+n(3+\pi/\varkappa)\big)}\right)^{2n}\cdot\left(\frac{|z_n|}{2}\right)^{m+n(3+\pi/\varkappa)}$$

and

$$\operatorname{re}\sum_{j=1}^{n} b_j z_j^{\nu_2} \leqslant -\frac{B}{2n+1}\left(\frac{n}{24\big(m+n(3+\pi/\varkappa)\big)}\right)^{2n}\cdot\left(\frac{|z_n|}{2}\right)^{m+n(3+\pi/\varkappa)}.$$

This lemma is a special case of Theorem 4.1 of [4], part III.

Finally we shall need

LEMMA 3. *For $x > 0$ (taking the real value of $\log x$) and positive integer ν we have*

$$\frac{1}{2\pi i}\int_{(2)} \frac{\Gamma(s)}{s^{\nu}}\, x^{-s}ds = \frac{1}{(\nu-1)!}\int_{x}^{\infty} \frac{\log^{\nu-1}(r/x)}{r}\, e^{-r}dr = \frac{1}{(\nu-1)!}\int_{0}^{\infty} y^{\nu-1}e^{-xe^{y}}dy\,.$$

For the proof see [4], part VIII.

3. We turn to the proof. Using the value y_1 given in lemma 1 we consider the integral

$$I_\delta(\nu) \overset{\text{def}}{=} -\frac{1}{2\pi i}\int_{(2)} \left(\frac{e^{y_1 s}}{s}\right)^{\nu} \Gamma(s)\left(\frac{\zeta'}{\zeta}(s)+\zeta(s)\right)ds\,, \tag{3.1}$$

where ν is restricted at the moment only by the inequality

$$\frac{1}{y_1}\log\frac{1}{\delta} - \log^{0.9}\frac{1}{\delta} \leqslant \nu \leqslant \frac{1}{y_1}\log\frac{1}{\delta}\,. \tag{3.2}$$

Developing $\frac{\zeta'}{\zeta}(s)+\zeta(s)$ in Dirichlet series and using lemma 3 we get with $x = ne^{-\nu y_1}$,

$$I_\delta(\nu) = \frac{1}{(\nu-1)!}\sum_{n=1}^{\infty}\big(\Lambda(n)-1\big)\int_{0}^{\infty} y^{\nu-1}e^{-ne^{y-\nu y_1}}dy\,.$$

Changing the order of summation and integration in the above formula (which can be easily justified) we come to

$$I_\delta(\nu) = \frac{1}{(\nu-1)!}\int_{0}^{\infty} y^{\nu-1}\left(\sum_{n=1}^{\infty}\big(\Lambda(n)-1\big)e^{-ne^{y-\nu y_1}}\right)dy\,. \tag{3.3}$$

Using the notation introduced by (1.1) and writing $r = y - \nu y_1$ we put (3.3) as follows

$$I_\delta(\nu) = \operatorname{re} I_\delta(\nu) = \frac{1}{(\nu-1)!}\int_{-\nu y_1}^{\infty} (r+\nu y_1)^{\nu-1}F(e^{r})\,dr\,. \tag{3.4}$$

We have obviously

$$F(y) = O\left(\frac{1}{e^{y}-1}\right), \qquad y > 0\,,$$

so that

$$\frac{1}{(\nu-1)!}\int_{\nu y_1}^{\infty} (r+\nu y_1)^{\nu-1}F(e^{r})\,dr \leqslant \frac{c_4}{(\nu-1)!}\int_{\nu y_1}^{\infty} e^{-e^{r}}(r+\nu y_1)^{\nu-1}dr$$

$$< \frac{c_5}{(\nu-1)!}\int_{2\nu y_1}^{\infty} e^{-e^{\frac{1}{2}r}}r^{\nu-1}dr < \frac{c_5}{(\nu-1)!}\int_{0}^{\infty} e^{-r/2}r^{\nu-1}dr$$

$$= \frac{2^{\nu}c_5}{(\nu-1)!}\,\Gamma(\nu) = c_5\cdot 2^{\nu} \leqslant \exp\left(12\,\frac{\log(1/\delta)}{\log\log(1/\delta)}\right).$$

Hence and from (3.4)

$$\operatorname{re} I_\delta(\nu) \leqslant \frac{1}{(\nu-1)!}\max_{r\geqslant -\nu y_1} F(e^{r})\int_{-\nu y_1}^{\nu y_1}(r+\nu y_1)^{\nu-1}dr + c_6\exp\left(12\,\frac{\log(1/\delta)}{\log\log(1/\delta)}\right).$$

In view of

$$\frac{1}{(\nu-1)!}\int_{-\nu y_1}^{\nu y_1}(r+\nu y_1)^{\nu-1}dr = \frac{1}{\nu!}(2\nu y_1)^\nu < (2ey_1)^\nu$$

$$< \exp\left(13\frac{\log(1/\delta)}{\log\log(1/\delta)}\log\log\log(1/\delta)\right)$$

we can state the inequality

(3.5) $\operatorname{re} I_\delta(\nu)$

$$\leqslant \max_{r\geqslant -\nu y_1} F(e^r)\cdot\exp\left(13\frac{\log(1/\delta)}{\log\log(1/\delta)}\log\log\log(1/\delta)\right)+c_6\exp\left(12\frac{\log(1/\delta)}{\log\log(1/\delta)}\right)$$

which holds with $\nu=\nu_1$, whenever

$$\operatorname{re} I_\delta(\nu_1)-c_6\exp\left(12\frac{\log(1/\delta)}{\log\log(1/\delta)}\right)>0. \tag{3.6}$$

All in all (3.6) would imply

(3.7) $\max_{r\geqslant -\nu y_1} F(e^r)$

$$\geqslant \exp\left(-13\frac{\log(1/\delta)}{\log\log(1/\delta)}\log\log\log(1/\delta)\right)\left\{\operatorname{re} I_\delta(\nu_1)-c_6\exp\left(12\frac{\log(1/\delta)}{\log\log(1/\delta)}\right)\right\}.$$

Similarly the relation

$$\operatorname{re} I_\delta(\nu_2)-c_6\exp\left(12\frac{\log(1/\delta)}{\log\log(1/\delta)}\right)<0 \tag{3.8}$$

would imply

(3.9) $\min_{r\geqslant -\nu_2 y_1} F(e^r)$

$$\leqslant \exp\left(-13\frac{\log(1/\delta)}{\log\log(1/\delta)}\log\log\log(1/\delta)\right)\left\{\operatorname{re} I_\delta(\nu_2)-c_6\exp\left(12\frac{\log(1/\delta)}{\log\log(1/\delta)}\right)\right\}.$$

4. It is easy to see that there exists an infinite connected broken line V, with segments parallel to the real resp. imaginary axis, all lying in the strip $\frac{1}{10}\leqslant\sigma\leqslant\frac{1}{5}$ and such that

$$\left|\frac{\zeta'}{\zeta}(s)\right|\leqslant c_7\log^3(2+|t|)\quad \text{along } V. \tag{4.1}$$

Using now Cauchy's theorem of residues in the domain limited by V and straight line $\sigma=2$ we obtain the following formula for $I_\delta(\nu)$

$$I_\delta(\nu)=\sum_{\varrho>V}-\Gamma(\varrho)\cdot\left(\frac{e^{y_1\varrho}}{\varrho}\right)^\nu-\frac{1}{2\pi i}\int_V\left(\frac{e^{y_1 s}}{s}\right)^\nu\Gamma(s)\left(\frac{\zeta'}{\zeta}(s)+\zeta(s)\right)ds. \tag{4.2}$$

Using (4.1)

$$\left|\frac{1}{2\pi i}\int_V\left(\frac{e^{y_1 s}}{s}\right)^\nu\Gamma(s)\left(\frac{\zeta'}{\zeta}(s)+\zeta(s)\right)ds\right|\leqslant c_8e^{\nu y_1/5}\cdot 10^\nu\leqslant\delta^{-1/4}. \tag{4.3}$$

Further, owing to the fact that the number of ζ-zeros in $r\leqslant t\leqslant r+1$ does not exceed

$$c_9\log(2+|r|),$$

we have

$$\left|\sum_{\substack{\varrho>V\\ |\Im\varrho|\geqslant\log^{0.1}(1/\delta)}}-\Gamma(\varrho)\left(\frac{e^{y_1\varrho}}{\varrho}\right)^\nu\right|\leqslant c_{10}\exp\left(\log^{0.91}(1/\delta)\right). \tag{4.4}$$

Choosing that zero in $|t|<\log^{0.1}(1/\delta)$, $\varrho_1=\sigma_1+i\gamma_1$ say, at which $|e^{y_1\varrho}/\varrho|$ is maximal, and using (4.3) and (4.4), we get from (4.2)

$$I_\delta(\nu)=\left(\frac{e^{y_1\sigma_1}}{|\varrho_1|}\right)^\nu\sum_{\substack{\varrho>V\\ |\Im\varrho|<\log^{0.1}(1/\delta)}}\left(-\Gamma(\varrho)\right)\left(\frac{e^{y_1(\varrho-\sigma_1)}}{\varrho}|\varrho_1|\right)^\nu+O(\delta^{-1.4}). \tag{4.5}$$

5. In order to estimate $\operatorname{re} I_\delta(\nu)$ we shall make use of lemma 2. For this sake we choose as z_j's the numbers

$$\frac{e^{y_1(\varrho-\sigma_1)}}{\varrho}|\varrho_1| \tag{5.1}$$

and as the corresponding b_j's the numbers

$$-\Gamma(\varrho). \tag{5.2}$$

The number n of z_j's is evidently

$$\leqslant\log^{0.1}\frac{1}{\delta}\left(\log\log\frac{1}{\delta}\right)^2; \tag{5.3}$$

further, let m be defined by

$$m=\left[\frac{\log(1/\delta)}{y_1}-\log^{0.9}\frac{1}{\delta}\right]. \tag{5.4}$$

Lemma 1 gives

$$\pi\geqslant|\arg z_j|\geqslant\varkappa$$

with

$$\varkappa=\log^{-2/3}\frac{1}{\delta}. \tag{5.5}$$

Writing

$$\varrho_2=\tfrac{1}{2}+i\cdot 14.13\ldots$$

(i.e. ϱ_2 is this ζ-zero which has the minimal positive imaginary part) we define integer h by

$$z_{h-1}=\frac{e^{y_1(\varrho_2-\sigma_1)}}{\varrho_2}|\varrho_1|,\qquad z_h=\frac{e^{y_1(\bar\varrho_2-\sigma_1)}}{\bar\varrho_2}|\varrho_1|. \tag{5.6}$$

The condition (2.5) is easily verified as follows: first

$$\frac{4n}{m+n(3+\pi/\varkappa)} < \frac{4}{\pi}\varkappa < 2\log^{-2/3}\frac{1}{\delta},$$

on the other hand

$$|z_h| \geqslant \frac{e^{-\frac{1}{2}y_1}}{|\varrho_2|}|\varrho_1| > \left(\log\frac{1}{\delta}\right)^{-1/20}. \tag{5.7}$$

As to the quantity (2.6) we note that

$$B \geqslant -2\,\mathrm{re}\,\Gamma(\tfrac{1}{2}+i\cdot 14.13\ldots) - \sum_{|t_\varrho|>21} |\Gamma(\varrho)|. \tag{5.8}$$

Using the inequality (see [3], p. 165)

$$247° \leqslant \arg\Gamma(\tfrac{1}{2}+i\cdot 14.13\ldots) \leqslant 262°$$

we have

$$-2\,\mathrm{re}\,\Gamma(\tfrac{1}{2}+i\cdot 14.13\ldots) = 2\,|\mathrm{re}\,\Gamma(\tfrac{1}{2}+i\cdot 14.13\ldots)| \geqslant 0.278\,|\Gamma(\tfrac{1}{2}+i\cdot 14.13\ldots)|$$

and, after the well-known formula

$$|\Gamma(\tfrac{1}{2}+it)| = \sqrt{\frac{2\pi}{e^{\pi t}+e^{-\pi t}}}, \quad -\infty < t < +\infty,$$

come to

$$-2\,\mathrm{re}\,\Gamma(\tfrac{1}{2}+i\cdot 14.13\ldots) \geqslant \frac{\sqrt{2\pi}}{e^{24}}. \tag{5.9}$$

By the inequality (see [3], (4.5))

$$\sum_{|t_\varrho|>21} |\Gamma(\varrho)| \leqslant \frac{\sqrt{2\pi}}{e^{27}}$$

and by (5.8), (5.9) we obtain finally

$$B \geqslant \frac{\sqrt{2\pi}}{e^{24}} - \frac{\sqrt{2\pi}}{e^{27}} \geqslant e^{-24}. \tag{5.10}$$

Now we can return to the formula (4.5). Lemma 2 ensures that there exists a ν_1 satisfying

$$\frac{1}{y_1}\log\frac{1}{\delta} - \log^{0.9}\frac{1}{\delta} \leqslant \nu_1 \leqslant \frac{1}{y_1}\log\frac{1}{\delta} \tag{5.11}$$

and such that

$$\mathrm{re}\,I_\delta(\nu_1) \geqslant \frac{(e^{y_1\sigma_1}/|\varrho_1|)^{\nu_1}B}{2n+1}\left(\frac{n}{24\,(m+n(3+\pi/\varkappa))}\right)^{2n}\left(\frac{|z_h|}{2}\right)^{m+n(3+\pi/\varkappa)} + O(\delta^{-1/4}) \tag{5.12}$$

with $B, n, m, \varkappa, h$ subject to conditions (5.3), (5.4), (5.5), (5.6) and (5.10). These easily imply

$$\mathrm{re}\,I_\delta(\nu_1) \geqslant \frac{e^{\frac{1}{2}y_1\nu_1}}{|2\varrho_2|^{\nu_1}}\left(\frac{|z_h|}{2}\right)^{m+n(3+\pi/\varkappa)-\nu_1} e^{-\log^{0.5}(1/\delta)} + O(\delta^{-1/4}). \tag{5.13}$$

Further

$$\left(\frac{|z_h|}{2}\right)^{m+n(3+\pi/\varkappa)-\nu_1} \geqslant \min\left(1, \left(\frac{|z_h|}{2}\right)^{n(3+\pi/\varkappa)}\right)$$

so that by (5.7)

$$\left(\frac{|z_h|}{2}\right)^{m+n(3+\pi/\varkappa)-\nu_1} > e^{-\log^{0.8}(1/\delta)}. \tag{5.14}$$

Also, by (5.11) and (2.1)

$$\frac{e^{\frac{1}{2}y_1\nu_1}}{|2\varrho_2|^{\nu_1}} \geqslant \left(\frac{1}{\delta}\right)^{1/2} \cdot e^{-42\log(1/\delta)/\log\log(1/\delta)}$$

so that by (5.13) and (5.14) we have

$$\mathrm{re}\,I_\delta(\nu_1) > \delta^{-1/2}e^{-43\log(1/\delta)/\log\log(1/\delta)}. \tag{5.15}$$

Similarly we come to the inequality

$$\mathrm{re}\,I_\delta(\nu_2) < -\delta^{-1/2}e^{-43\log(1/\delta)/\log\log(1/\delta)} \tag{5.16}$$

valid with a certain ν_2 satisfying

$$\frac{1}{y_1}\log(1/\delta) - \log^{0.9}(1/\delta) \leqslant \nu_2 \leqslant \frac{1}{y_1}\log(1/\delta). \tag{5.17}$$

Hence, in view of (3.6)-(3.7), we may write

$$\max_{r \geqslant -\nu_1 y_1} F(e^r) \geqslant \delta^{-1/2}\exp\left(-14\frac{\log(1/\delta)}{\log\log(1/\delta)}\log\log\log(1/\delta)\right)$$

which by (5.11) implies

$$\max_{y \geqslant \delta} F(y) \geqslant \delta^{-1/2}\exp\left(-14\frac{\log(1/\delta)}{\log\log(1/\delta)}\log\log\log(1/\delta)\right). \tag{5.18}$$

Since trivially

$$F(y) = O(y^{-1})$$

we obtain (1.3) immediately from (5.18). The inequality (1.4) follows similarly from (5.16) and (3.8)-(3.9).

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On primitive prime factors of Lehmer numbers I

by

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Lehmer numbers are called terms of the sequences

$$P_n(\alpha,\beta)=\begin{cases}(\alpha^n-\beta^n)/(\alpha-\beta), & n \text{ odd},\\ (\alpha^n-\beta^n)/(\alpha^2-\beta^2), & n \text{ even},\end{cases}$$

where α and β are roots of the trinomial $z^2-L^{1/2}z+M$, and L and M are rational integers (cf. [4]). Without any essential lost of generality (cf. [9]) we can assume that

$$(1)\qquad L>0, \quad M\neq 0, \quad K=L-4M\neq 0.$$

Lehmer numbers constitute a generalization of the numbers a^n-b^n (a, b — rational integers). A prime p is called a *primitive prime factor* of a number a^n-b^n if

$$p\,|\,a^n-b^n \quad \text{but} \quad p\nmid a^k-b^k \quad \text{for} \quad k<n.$$

A proper (not merely automatical) generalization of this notion for Lehmer numbers is the notion of a prime factor p such that

$$p\,|\,P_n \quad \text{but} \quad p\nmid KLP_3\ldots P_{n-1}$$

or, which is easily proved to be equivalent,

$$p\,|\,P_n \quad \text{but} \quad p\nmid nP_3\ldots P_{n-1}.$$

D. H. Lehmer [4] calls such primes p primitive extrinsic prime factors of P_n. In a postscript to my paper [7] I stated erroneously that Lehmer calls them intrinsic divisors, the term which has been used in a different sense by M. Ward [9]. To simplify the terminology, I adopt in the present paper the following definition.

DEFINITION. *A prime p is called a primitive prime factor of the number P_n if $p\,|\,P_n$ but $p\nmid KLP_3\ldots P_{n-1}$.*