Since \( r_0 - m < 0 \) we have by Lemma 7
\[
\sum_{k=0}^{k'-1} D_k \left( \frac{b}{b-k}\right)^{k-k'} = O\left( \frac{1}{b^b} \right).
\]

Theorem 5 is now immediate. The following theorem results at once from Theorem 5 when we take \( m = 2 \).

**Theorem 6.** If \( r_0 < 2 \), using the notations of Theorem 5,
\[
D_r(x) = - \left( r+1 \right) \sum_{\substack{\eta_1, \eta_2, \ldots, \eta_{k-r-1} \leq x \\ \eta_1, \ldots, \eta_{k-r-1} \neq 0}} \frac{1}{(\eta_1 \ldots \eta_{k-r-1})^{r+1}} \sum_{\eta_{k-r-1} = \eta_1 \ldots \eta_{k-r-1}} \left( \eta_k \right)^{1/2}.
\]

Particular cases:
1. If we take \( r = 0 \), \( q = -1 \), \( k = 2 \) in Theorem 6, we get
\[
D_2^{(2-1)}(x) = -2 \sum_{n \leq x} \psi(x/n) + O(1),
\]

a result due to Landau [1], which was the starting point of Van der Corput's investigations of the Dirichlet's divisor problem.

2. Taking \( r = 0 \), \( q = -1 \), \( k = 3 \), in Theorem 6, we get
\[
D_3^{(3-1)}(x) = -3 \sum_{\eta_1, \eta_2, \eta_3} \frac{1}{(\eta_1 \eta_2 \eta_3)} + O(x^{3/2}).
\]

We have, from Theorem 6, trivially
\[
D_2^{(2-1)}(x) = O\left( \frac{1}{(\eta_1 \eta_2 \eta_3)} \right) + O(x^{3/2})
\]
\[
= O\left( x^{3/2} \right) \quad \text{if} \quad r_0 < 1, \quad \text{by Lemma 7}.
\]

I shall return to the general problem of the order of \( D_r^{(2-1)}(x) \) in a subsequent paper.

**References**


**85 par la Réduction le 6. 6. 1962**

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**The lattice point problem of many-dimensional hyperboloids II**

by

B. R. Srinivasan (Madras)

To the loving and respectful memory of Prof. Dr E. Vaidyanathan Swamy

1. In many problems in the analytic theory of numbers, it is necessary to obtain non-trivial inequalities for exponential sums of the form
\[
\sum_{n \leq x} e^{\phi(n)},
\]

where \( \phi(n) \) is a real function. An important method of obtaining such inequalities is due to Van der Corput \((1)\). Titchmarsh \([10], [11]\) has extended Van der Corput's method to two-dimensional sums of the type
\[
\sum_{n \leq x} e^{\phi(n)},
\]

We consider here sums of the type
\[
\sum_{n \leq x} e^{\phi(n, x)}
\]

for arbitrary positive integer \( p \) and extend, step by step, Van der Corput's theory in one dimension to these \( p \)-dimensional sums. In the case \( p = 1 \), the present method reduces completely to Van der Corput's method. In the case \( p = 2 \), the present method includes (and in fact, slightly refines) Titchmarsh's method (cf. \([8]\) also).

The method seems to be of general importance, but in each application there are considerable difficulties of detail. As a straightforward illustration, I consider here the lattice point problem of certain many-dimensional hyperboloids which I have considered elsewhere.

\(1\) For an account of the method and references, cf. \([12]\).
Defining $D_k^q(\alpha)$ by

\[ D_k^q(\alpha) = \binom{k}{r} \sum_{n_1, n_2, \ldots, n_k \leq \alpha^{1/k}} \binom{x}{n_1, \ldots, n_k, \ldots, n_k}^{-\frac{1}{2} \alpha}, \]

where $\alpha$, $\rho$ are real $\geq 1$, $\rho > -2$; $r$, $k$ integers such that $0 \leq r \leq k$; and setting

\[ D_k^q(\alpha) = P_k^q(\alpha) + \delta_k^q(\alpha) \]

where

\[ P_k^q(\alpha) = \frac{1}{\alpha} \sum_{x=0}^{k-1} (-1)^{k-r} \binom{k}{r} \alpha^{-(k-r)\frac{1}{2} \alpha} \times \left(1 + \frac{1}{2} \alpha \right) u \left( \frac{y}{\alpha} \right) y^{\frac{r}{2} - \frac{1}{2} \alpha} dy \]

\[ \times \left(1 + \frac{1}{2} \alpha - u \left( \frac{y}{\alpha} \right) \right) \left(1 - \frac{1}{2} \alpha + u \left( \frac{y}{\alpha} \right) \right) \]

if $\alpha > \rho / (r + \varepsilon)$ and $\rho / (r + \varepsilon) > 0$;

\[ \delta_k^q(\alpha) = \text{coeff of } u^{k-1} \ln \left( \frac{y}{\alpha} \right) \left(1 + \frac{1}{2} \alpha - u \left( \frac{y}{\alpha} \right) \right) \left(1 - \frac{1}{2} \alpha + u \left( \frac{y}{\alpha} \right) \right) \]

if $\alpha = 0$, $\rho / (r + \varepsilon) > 0$;

\[ \text{coeff of } u^{k-1} \ln \left( \frac{y}{\alpha} \right) \left(1 + \frac{1}{2} \alpha - u \left( \frac{y}{\alpha} \right) \right) \left(1 - \frac{1}{2} \alpha + u \left( \frac{y}{\alpha} \right) \right) \]

if $\alpha = 0$,

we have (cf. Theorem 6 of [9]) if $r \rho < 2$

\[ A_k^q(\alpha) = -\binom{k}{r+1} \sum_{n_1, n_2, \ldots, n_k \leq \alpha^{1/k}} \binom{x}{n_1, \ldots, n_k, \ldots, n_k}^{-\frac{1}{2} \alpha} \times \psi_1 \left( \frac{x}{\alpha} \right) \left(1 + \frac{1}{2} \alpha \right) \left(1 - \frac{1}{2} \alpha \right) \alpha^{\frac{r}{2} - \frac{1}{2} \alpha}, \]

where

\[ \psi_1(u) = u - (\text{integral part of } u) - \frac{1}{2}. \]

We concern ourselves with the problem of order of $A_k^q(\alpha)$. We have, quite elementarily from (7), the result (cf. [9])

\[ A_k^q(\alpha) = O(x^{k-q-\frac{1}{2} \alpha}) \quad \text{if } r \rho < 1. \]

Let $\alpha_k^q(\alpha)$ denote the lower bound of $\alpha$ where

\[ A_k^q(\alpha) = O(x^q). \]

Then (9) gives

\[ \alpha_k^q(\alpha) \leq \frac{k + r \rho - 1}{k} \quad \text{if } r \rho < 1. \]

By the application of the present method we prove

\[ \alpha_k^q(\alpha) \leq 1 + \max \left( \frac{r \rho - \beta_k}{k}, \frac{r \rho - \beta_k}{r + 1} \right) \quad \text{if } r \rho < 2, \]

where

\[ \beta_k = \frac{1}{x^{k-r}(k-r+1)(k-r+2)} \]

if $k \geq r + 3$, $k = r + 1$, $r + 2$, respectively.

\[ \beta_k = \frac{1}{x^{k-r}(k-r+1)(k-r+2)} \]

if $k \geq r + 3$, $k = r + 2$, respectively.

(12) is an improvement over (11) when

\[ r \rho < \frac{k \beta_k}{k - (r + 1)} (\leq 1). \]

The case $q = -1$ represents the number of lattice points bounded by the coordinate hyperplanes and a certain number of hyperboloids in a $k-r$ dimensional space. If we further take $r = 0$, the problem of the order of $A_k^q(\alpha)$ is precisely the general (Pills) divisor problem.

We have from (12), (13), (14), when $r = 0$

\[ \alpha_k^q(\alpha) \leq \frac{(k-l)(k+3)}{k+1} - 1 \]

if $k \geq 5$, $k = 4$, $3$, $2$, respectively.

When $k = 2$ and $q = -1$, the above result is due to Van der Corput ([22], [7]). In the case $q = -1$ (divisor problem case) further improvements on (15) are known (cf. Theorem 12.3 of [12]) though (15) is better than the classical estimate of Landau [5], who proved that

\[ \alpha_k^q(\alpha) \leq \frac{k-1}{k+1} \quad \text{for } k \geq 2. \]

We first prove a number of lemmas. We then use them to obtain theorems on finite sums of the type

\[ \sum_{n_1, \ldots, n_k \leq \alpha^{1/k}} \alpha^{\frac{r}{2} - \frac{1}{2} \alpha}, \]

and finally use them to obtain the required estimates for $A_k^q(\alpha)$.
To prove this we count the number of times a $g(n_1, \ldots, n_p)$ occurs on the right-hand side of (18) when we substitute for $G(n_1, \ldots, n_p) \text{ etc. using (16).}$

If $r (0 \leq r \leq p)$ of the $n_i$'s are less than the corresponding $m_i$'s while the other $n_i$'s are equal to the corresponding $m_i$'s, the number of times $g(n_1, \ldots, n_p)$ is counted on the right-hand side of (18) is

$$1 - r \left\{ \begin{array}{ll} 1 & \text{if } r \geq 1, \\
1 & \text{if } r = 0. 
\end{array} \right.$$

This proves (18).

Next,

$$S = \sum_{n_1, \ldots, n_p} g(n_1, \ldots, n_p) h(n_1, \ldots, n_p)$$

$$= \sum_{n_1, \ldots, n_p} h(n_1, \ldots, n_p) \left\{ G(n_1, \ldots, n_p) - \sum_{i=1}^p G(n_1, \ldots, n_i-1, \ldots, n_p) \right\}$$

$$+ \ldots + (-1)^p G(n_1, \ldots, n_p-1).$$

by (18). Hence

$$S = \sum_{n_1, \ldots, n_p} G(n_1, \ldots, n_p) A_{n_1} A_{n_2} \ldots A_{n_p} h(n_1, \ldots, n_p)$$

$$= G(N_1, \ldots, N_p) k(N_1, \ldots, N_p) + \sum_{n_1, \ldots, n_p} S_{n_1, \ldots, n_p}$$

where

$$S_{n_1, \ldots, n_p} = \sum_{n_1, \ldots, n_p} G(n_1, \ldots, n_p) \left\{ \prod_{i=1}^p A_{n_i} \right\} h(n_1, \ldots, n_p).$$

In the above sum $\sum_{n_1, \ldots, n_p}$, $n_i = N_i$ if $a_i = 0$ and the sum is taken over $1 \leq n_i \leq N_i-1$ for $j \in J$, the set of $j$'s such that $a_j = 1$.

Now,

$$\left| S_{n_1, \ldots, n_p} \right| \leq G \left| \sum_{n_1, \ldots, n_p} \left\{ \prod_{i=1}^p A_{n_i} \right\} h(n_1, \ldots, n_p) \right|$$

$$= G \left| \sum_{i \in J} \left( \sum_{n_1, \ldots, n_i=N_i-1} \ldots \right) A_{n_i} h(n_1, \ldots, n_p) \right|_{n_p=N_f \text{ for } i \in J}$$

$$\leq GH^{2^{p-1}} \text{ if } r \text{ is cardinality of } J.$$
So

\[ |S| \leq GH + \sum_{i=0}^{p} p(GH)^{2i-1} = \frac{1}{2}(1+3p)GH. \]

Hence the lemma. When \( p = 2 \), the above lemma is Lemma \( \alpha \) of (10).

Remark 1. In particular, the above lemma holds if the condition (17) is replaced by the condition that the \( 2^{p} - 1 \) derivatives

\[ \left( \prod_{i=1}^{p} \frac{\partial^{2}}{\partial x_{i}^{2}} \right)^{n} h(x_{1}, \ldots, x_{p}) \]

are of constant sign for all values of \( x_{1}, \ldots, x_{p} \) considered.

The above Remark at once follows from the observation

\[ \left( \prod_{i=1}^{p} A_{n}^{i} \right)^{n} h(x_{1}, \ldots, x_{p}) = \int \cdots \int \left( \prod_{i=1}^{p} \frac{\partial^{2}}{\partial x_{i}^{2}} \right)^{n} h(x_{1}, \ldots, x_{p}) \int_{D} (d^{2}x). \]

Remark 2. The above lemma is still valid if the hyper-rectangle \( 1 \leq n_{i} \leq N_{i}, i = 1, \ldots, p \), is replaced by an arbitrary region \( D \) contained in the hyper-rectangle (16) is replaced by

\[ G(n_{1}, \ldots, n_{p}) = \sum_{1 \leq n_{1} < \cdots < n_{p} \leq N} g(n_{1}, \ldots, n_{p}). \]

To see this, we have only to apply the above lemma to the new function \( g^{*}(n_{1}, \ldots, n_{p}) \) defined as follows:

\[ g^{*}(n_{1}, \ldots, n_{p}) = \begin{cases} g(n_{1}, \ldots, n_{p}) & \text{if } (n_{1}, \ldots, n_{p}) \in D, \\ 0 & \text{if } (n_{1}, \ldots, n_{p}) \notin D. \end{cases} \]

Lemma 2. Let \( f(x_{1}, \ldots, x_{p}) \) be real in a region \( D \) contained in the rectangle \( a_{i} \leq x_{i} \leq b_{i}, i = 1, \ldots, p \). Then

\[
\sum_{(n_{1}, \ldots, n_{p}) \in D} g(n_{1}, \ldots, n_{p}) \int_{a_{i}}^{b_{i}} \left( \frac{b_{j} - a_{j}}{q_{i}} \right)^{l} \int_{c_{1}}^{d_{1}} \cdots \int_{c_{p}}^{d_{p}} (d_{1} - c_{1})^{l} \cdots (d_{p} - c_{p})^{l} \sum_{(n_{1}, \ldots, n_{p}) \in D} g(n_{1}, \ldots, n_{p}) \left( \prod_{i=1}^{p} \frac{\partial}{\partial x_{i}} \right)^{n} h(x_{1}, \ldots, x_{p}) \]

where the summation on the right-hand side is taken over the lattice points \((n_{1}, \ldots, n_{p})\) for which both

\[(n_{1} + n_{1}, n_{2} + \cdots, n_{p}) \quad \text{and} \quad (n_{1}, \ldots, n_{p}) \]

lie in \( D \), \( n_{i} \) being an integer and the only restriction on \( q_{i} \) being \( 0 < q_{i} < n_{1} - a_{i} \).

The above lemma is due to van der Corput (cf. Satz 1 of (3)).

Lemma 3. Let \( M \) and \( N \) be positive integers, \( n_{m} > 0 \) and \( n_{n} > 0 \) \((1 \leq m < M, 1 \leq n < N)\) denote constants. Let \( A_{m} > 0 \), \( B_{n} > 0 \). Then there exists a \( q \) with the properties \( (Q_{1} \text{ and } Q_{2} \text{ are given non-negative numbers}) \)

\[ Q_{1} \leq q \leq Q_{2} \]

and

\[ \sum_{m=1}^{M} A_{m} q^{m} + \sum_{n=1}^{N} B_{n} q^{-n} < \sum_{m=1}^{M} A_{m} q^{m} + \sum_{n=1}^{N} B_{n} q^{-n}. \]

Proof. Consider

\[ f(x) = \sum_{m=1}^{M} A_{m} x^{m} + \sum_{n=1}^{N} B_{n} x^{-n}, \quad x \text{ real}. \]

Let

\[ \Phi(x) = \max_{1 \leq m \leq M} A_{m} x^{m}, \quad g(x) = \max_{1 \leq n \leq N} B_{n} x^{-n}. \]

Then \( \Phi(x) \) is monotonic increasing, while \( g(x) \) is steadily decreasing. Hence \( \Phi(x) - g(x) \) is steadily increasing and there is a unique \( q_{0} \) such that \( \Phi(q_{0}) - g(q_{0}) = 0 \). Also \( \Phi(x) \geq g(x) \) according as \( x \leq q_{0} \). We consider three cases.

(i) Suppose \( q_{0} > Q_{1} \). Then \( \Phi(Q) < g(Q) \).

\[ f(Q) \leq M \Phi(Q) + N g(Q) < (M + N) g(Q) < \sum_{n=1}^{N} B_{n} Q^{-n}. \]

(ii) Suppose \( q_{0} < Q_{1} \). Then \( g(Q) < \Phi(Q) \).

\[ f(Q) \leq M \Phi(Q) + N g(Q) < (M + N) \Phi(Q) < \sum_{m=1}^{M} A_{m} Q^{m}. \]

(iii) Suppose \( Q_{1} \leq q_{0} < Q_{2} \). Then \( \Phi(q_{0}) - g(q_{0}) = 0 \) gives

\[ A_{m} q^{m} = B_{n} q^{-n} \]

for some \( a, b \) such that \( 1 \leq a < M, 1 \leq b < N \).
We have
\[ f(q_0) \leq (M + N) \Phi(q_0) \leq A_{\delta}^{\delta_{\mu}} \leq \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{\log(1 + \delta_{\mu})}{\log(1 + \delta_n)} \cdot \frac{\log(1 + \delta_m)}{\log(1 + \delta_{\mu})}. \]

In all the above cases, the Lemma is true. Hence, the Lemma.
The above Lemma is Lemma 4 of [7] Since we are using the Lemma often in the sequel, we have reproduced its short proof here. In the case \( Q_1 = 0, Q_2 = \infty \) (so that the parameter \( q \) is unrestricted except that is positive) the Lemma is due to Van der Corput (Hilfsatz 4 of [1]).

Remark 1. The constant involved in the majorisation \( \leq \) depends only on \( M \) and \( N \) (in fact \( \leq M + N \)) and so is absolute if \( M \) and \( N \) are absolute constants.

Remark 2. The inequality above in Lemma 3 corresponds to the best possible choice of \( q \) in the range \( Q_1 \leq q \leq Q_1 \), i.e. the above inequality is stronger than (i.e. implies) any other inequality obtainable by considering any \( q \) in \( Q_1 \leq q \leq Q_1 \).

Proof of the above Remark 2 is easy and is given in [7]. Remark 1 is obvious.

3. Throughout the following lemmas, we suppose that \( D \) is a finite region in a \( p \)-dimensional Euclidean space and that any line parallel to any of the coordinate axes meets it in \( O(1) \) straight line segments, and the same is true for the intersections of \( D \) with regions of the type \( f_n \leq \text{const} \) and \( f_m \geq \text{const} \), \( s = 1, \ldots , p \), where \( f(x_1, \ldots , x_p) \) is a real function defined over \( D \) such that the transformation \( y_i = f_i, i = 1, \ldots , p \), is one-one over \( D \). We suppose further that any line parallel to any of the coordinate axes meets the surface got by equating to zero any of the second order partial derivatives of \( f \) in \( O(1) \) points.

The conditions regarding the regions and the function \( f \) are in particular satisfied if \( D \) is bounded by \( O(1) \) algebraic surfaces of bounded degrees and the surfaces \( f_m = \text{const}, f_{x_{s_1} x_{s_2}} = 0 \) are also algebraic and of bounded degree.

**Lemma 4.** Let
\[
\frac{\partial(f_{x_{s_1} x_{s_2}} \ldots f_{x_{s_p}})}{\partial(x_{s_1}, \ldots , x_{s_p})} \geq A_{r_1} r_{s_1} \ldots r_{s_p} > 0 \quad \text{in} \quad D
\]
\[(1 \leq s_1 < s_2 < \ldots < s_p \leq p, 1 \leq s \leq p),
\]
where the \( r \)'s are independent of the \( x \)'s. Then
\[
\int_D \left( \frac{1}{V_{x_{s_1 + \ldots + r_p}}} \right) \, dx_{s_1} \ldots dx_{s_p} \leq \frac{1}{V_{r_1 + r_p}},
\]

**Proof.** We shall prove this by induction on \( p \). When \( p = 1 \), the above lemma is well known (Lemma 4.4 of [12]). We have
\[
I = \int_D \left( \frac{1}{V_{r_1}} \right) \, dx_{s_1} \ldots dx_{s_p}
\]
\[
= \int_{D_1} \left( \frac{1}{V_{r_1}} \right) \, dx_{s_1} \ldots dx_{s_p}
\]
\[
= \sum_{j=1}^{N} \int_{D_j} \left( \frac{1}{V_{r_1}} \right) \, dx_{s_1} \ldots dx_{s_p}
\]
where \( D_j \) is given by \( [x_{s_1}] < r_{s_1} \) for \( 1 \leq s < j \). Now,
\[
\text{re} J_j = \int \left( \frac{1}{V_{r_1}} \right) \, dx_{s_1} \ldots dx_{s_p}
\]
\[
= \sum_{j=1}^{N} \int_{D_{j_1}} \left( \frac{1}{V_{r_1}} \right) \, dx_{s_1} \ldots dx_{s_p}
\]
by successive application of the second mean value theorem, since \( 1/f \) has \( O(1) \) maxima and minima on any line parallel to one of the coordinate axes. By the above and by the hypotheses on the regions, the sum here contains \( O(1) \) terms (the \( 2^k \)'s and \( D_j \)'s in each of the terms are not necessarily the same). Also \( D_j \) satisfy the general conditions satisfied by \( D \). Hence, by the induction hypothesis,
\[
\text{re} J_j = O \left( \frac{1}{V_{r_1}} \right) \quad \text{and so} \quad J_j = O \left( \frac{1}{V_{r_1}} \right)
\]
Similarly
\[
\text{im} J_j = O \left( \frac{1}{V_{r_1}} \right)
\]
We make the transformation \( y_j = f_{s_1}, j = 1, \ldots , p \) in \( J \). We then have
\[
J = \int \left( \frac{1}{V_{r_1}} \right) \, dx_{s_1} \ldots dx_{s_p}
\]
\[
= \int \left( \frac{1}{V_{r_1}} \right) \, dx_{s_1} \ldots dx_{s_p}
\]
\[
= \int \left( \frac{1}{V_{r_1 + r_p}} \right) \, dx_{s_1} \ldots dx_{s_p}
\]
\[
= O \left( \frac{1}{V_{r_1 + r_p}} \right) = O \left( \frac{1}{V_{r_1 + r_p}} \right)
\]
since $D'$ is contained in the hyper-rectangle $|y_j| \leq \sqrt{r_j}$, $j = 1, \ldots, p$. Hence the lemma.

**Lemma 5.** Let $f(x_1, \ldots, x_p)$ possess continuous partial derivatives up to the third order in $D$. Again let

$$\left| \frac{\partial(f_{x_1}, \ldots, f_{x_j})}{\partial(x_{i_1}, \ldots, x_{i_s})} \right| > A r_{i_1} \cdots r_{i_s}^2 > 0 \quad (1 \leq i_1 < i_2 < \ldots < i_s \leq p, 1 \leq s \leq p)$$

and

$$|f_{xy}| \leq A r_x^2, \quad |f_{xxy}| \leq A^2 r_x^3 r_y^2 \quad \text{throughout } D.$$  

Further, let $f_{x_j}(x_1, \ldots, x_p) = 0$ for $j = 1, \ldots, p$. Let $a_j$ and $\beta_j$ be the values of $f_{x_j}$ at the end-points of the largest segment of the curve $f_{x_j} = \cdots = f_{x_p} = 0$ which contains $(x_1, \ldots, x_p)$ and lies entirely within $D$ for $j = 1, \ldots, p$. Then if $m$ is the number of changes of sign in the sequence

$$+1, \quad \frac{\partial(f_{x_1}, \ldots, f_{x_j})}{\partial(x_1, \ldots, x_j)}, \quad j = 1, \ldots, p;$$

then

$$\int_D e^{\sum_{i=1}^p \frac{t_i}{r_i} - \frac{1}{2} \sum_{i=1}^p a_i t_i^2} \left( \phi(\Phi_{x_1}, \ldots, \Phi_{x_p}) \right)^{\frac{1}{2}} \phi^2(y_1, \ldots, y_p) \frac{dy_1 \cdots dy_p}{(2\pi)^{p/2}} \leq \frac{1}{\prod_{j=1}^p (a_j + 1/\beta_j)^{1/2}} \left( \sum_{i=1}^p a_i y_i^2 + 2 \sum_{1<i<j} a_i a_j y_i y_j \right).$$

**Proof.** We write

$$I = \int_D e^{\sum_{i=1}^p \frac{t_i}{r_i} - \frac{1}{2} \sum_{i=1}^p a_i t_i^2} \left( \phi(\Phi_{x_1}, \ldots, \Phi_{x_p}) \right)^{\frac{1}{2}} \phi^2(y_1, \ldots, y_p) \frac{dy_1 \cdots dy_p}{(2\pi)^{p/2}}$$

$$= \int \cdots \int e^{\sum_{i=1}^p \frac{t_i}{r_i} - \frac{1}{2} \sum_{i=1}^p a_i t_i^2} \left( \phi(\Phi_{x_1}, \ldots, \Phi_{x_p}) \right)^{\frac{1}{2}} \phi^2(y_1, \ldots, y_p) \frac{dy_1 \cdots dy_p}{(2\pi)^{p/2}}$$

$$= J + \sum_{i=1}^p J_i \quad \text{(say),}$$

where $D_i$ is given by $|a_i| \leq \delta r_i$ for $1 \leq s < j$ and $\delta$ is a positive constant to be chosen later. Here we assume that the region $|a_i| \leq \delta r_i$, $j = 1, \ldots, p$, is entirely contained within $D$, that is

$$\delta \leq \min_j \left( \frac{|a_i|}{r_j}, \frac{|\beta_j|}{r_j} \right).$$

By repeated application of the second mean value theorem (as in the proof of Lemma 4) and by Lemma 4, we find

$$J_i = 0 \cdot \left( \frac{1}{\prod_{j=1}^p (a_j + 1/\beta_j)^{1/2}} \right) \frac{dy_1 \cdots dy_p}{(2\pi)^{p/2}}.$$

We put in the integral $J$, $y_j = f_{x_j}$, $j = 1, \ldots, p$, and

$$\Phi(y_1, \ldots, y_p) = \int_0^1 e^{\frac{t_1}{r_1} \frac{t_2}{r_2} \cdots \frac{t_p}{r_p}} \left( \phi(\Phi_{x_1}, \ldots, \Phi_{x_p}) \right)^{\frac{1}{2}} \phi^2(y_1, \ldots, y_p) dy_1 \cdots dy_p.$$

Then $\Phi_{x_j} = a_j + 1/\beta_j$, $j = 1, \ldots, p$, and $\Phi$ has continuous third partial derivatives in the transform $D_i$ of the region $D_i$ by the transformation $y_j = f_{x_j}$, $1 \leq j \leq p$. Then we have

$$J = \int_0^1 \cdots \int_0^1 e^{\frac{t_1}{r_1} \frac{t_2}{r_2} \cdots \frac{t_p}{r_p}} \left( \phi(\Phi_{x_1}, \ldots, \Phi_{x_p}) \right)^{\frac{1}{2}} \phi^2(y_1, \ldots, y_p) dy_1 \cdots dy_p.$$

Now put $a_0 = \Phi_{x_0}(0, \ldots, 0)$ and

$$2 \chi(y_1, \ldots, y_p) = \sum_{1 \leq i < j} a_i a_j y_i y_j + \sum_{1 \leq i} a_i y_i^2.$$

We have

$$\chi(y_1, \ldots, y_p) = \sum_{i=1}^p a_i y_i^2 \quad \text{(say),}$$

where

$$\chi = \sum_{i=1}^p \frac{\partial(\Phi_{x_1}, \ldots, \Phi_{x_p})}{\partial(y_{x_1}, \ldots, y_{x_p})} \left( \phi(\Phi_{x_1}, \ldots, \Phi_{x_p}) \right)^{\frac{1}{2}} \phi^2(y_1, \ldots, y_p) \frac{dy_1 \cdots dy_p}{(2\pi)^{p/2}}.$$

Now the matrices $\Phi_{x_0}$ and $(f_{x_0})$ are inverse matrices since

$$\sum_{i=1}^p \Phi_{x_0 x_i} f_{x_i} = \delta_{x_0} \quad (\text{the Kronecker delta}).$$

And so we find

$$\chi = \sum_{i=1}^p \frac{\partial(\Phi_{x_1}, \ldots, \Phi_{x_p})}{\partial(y_{x_1}, \ldots, y_{x_p})} \left( \phi(\Phi_{x_1}, \ldots, \Phi_{x_p}) \right)^{\frac{1}{2}} \phi^2(y_1, \ldots, y_p) \frac{dy_1 \cdots dy_p}{(2\pi)^{p/2}} = \frac{\delta_{x_0}}{\delta_{x_0}} \quad \text{(say)}.$$
by repeated application of the second mean value theorem, and by Lemma 4. Similarly, for every $j = 2, \ldots, p$, 

\begin{equation}
\int_{\mathbb{R}^p} e^{(1+\theta)\gamma_1 \cdots \gamma_p} \, dy_j = O(\frac{1}{\gamma_1 \cdots \gamma_p}) .
\end{equation}

Next,

\begin{equation}
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{(1+\theta)\gamma_1 \cdots \gamma_p} \, d\gamma_1 \cdots d\gamma_p
\end{equation}

\begin{align*}
= \int_{\mathbb{R}^p} e^{(1+\theta)\gamma_1 \cdots \gamma_p} \, dy_j = O\left(\frac{1}{\gamma_1 \cdots \gamma_p}\right),
\end{align*}

where $m$ is the number of $g_i < 0$ for $j = 1, \ldots, p$, i.e., $m$ is the number of changes of sign in the sequence

\begin{align*}
+1, \frac{\delta f_{x_1}, \ldots, f_{x_p}}{\delta (x_1, \ldots, x_p)} ; \quad j = 1, \ldots, p .
\end{align*}

From (20) to (25), we have

\begin{equation}
I = (2\pi)^{\frac{np}{2}} \frac{\delta f_{x_1}, \ldots, f_{x_p}}{\delta (x_1, \ldots, x_p)} + I' + I'' + O\left(\frac{1}{\gamma_1 \cdots \gamma_p}\right)
\end{equation}

where

\begin{align*}
I' &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{(1+\theta)\gamma_1 \cdots \gamma_p} \frac{\delta (\Phi_{x_1}, \ldots, \Phi_{x_p})}{\delta (y_1, \ldots, y_p)} \, dy_1 \cdots dy_p,
I'' &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{(1+\theta)\gamma_1 \cdots \gamma_p} \, dy_1 \cdots dy_p.
\end{align*}

Now

\begin{align*}
\Phi_{x_{xj}} = \text{cofactor of } f_{x_{xj}} \text{ in } H(f) \cdot \frac{\partial f_{x_1}, \ldots, f_{x_p}}{\partial (x_1, \ldots, x_p)} \leq \frac{1}{r_{x_j} r_{x_j}},
\end{align*}

and

\begin{align*}
\frac{\partial}{\partial y_j} \frac{\delta f_{x_1}, \ldots, f_{x_p}}{\delta (x_1, \ldots, x_j)} = -\sum_{j=1}^{p} \Phi_{x_{xj}} \frac{\partial f_{x_1}, \ldots, f_{x_p}}{\delta (x_1, \ldots, x_j)} H(f)^2
\leq \sum_{j=1}^{p} \frac{1}{r_{x_j} r_{x_j}} R_{x_j} R_{x_j} \leq \frac{R}{r_{x_j} r_{x_j}} .
\end{align*}

\begin{align*}
\Phi_{x_{xj}} &= \frac{\partial}{\partial y_k} \text{ (cofactor of } f_{x_{xj}} \text{ in } H(f))
- \frac{1}{H(f)} \sum_{j=1}^{p} \Phi_{x_{xj}} \frac{\partial}{\partial y_k} (f_{x_{xj}} H(f)) + (\text{cof. of } f_{x_{xj}}) \frac{\partial}{\partial y_k} H(f)
\leq \frac{1}{r_{x_j} r_{x_j}} \int_{-\infty}^{\infty} \frac{1}{r_{x_j} r_{x_j}} R_{x_j} R_{x_j} \leq \frac{1}{r_{x_j} r_{x_j}}
\leq \frac{R}{r_{x_j} r_{x_j}} .
\end{align*}

Now

\begin{align*}
I' = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\delta (\Phi_{x_1}, \ldots, \Phi_{x_p})}{\delta (y_1, \ldots, y_p)} \, dy_1 \cdots dy_p
\end{align*}

(by the continuity of the third partial derivatives of $\Phi$)

\begin{align*}
\leq \delta_{x_1, \ldots, x_p} \frac{R}{r_{x_1} \cdots r_{x_p} r_{x_j}} .
\end{align*}
and

\[ (28) \quad \mathcal{I}' \leq \frac{1}{r_1 \ldots r_p} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \sum_{\alpha \in \mathbb{Z}^d} \Phi(0, \ldots, 0) \, dy_1 \ldots dy_p \]

\[ \leq \frac{1}{r_1 \ldots r_p} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \sum_{\alpha \in \mathbb{Z}^d} \Phi(0, \ldots, 0) \, dy_1 \ldots dy_p \]

\[ \leq \frac{1}{r_1 \ldots r_p} \sum_{\alpha \in \mathbb{Z}^d} \Phi(0, \ldots, 0) \, dy_1 \ldots dy_p \]

\[ \leq \delta^{p+1} \frac{R}{r_1 \ldots r_p} \]

From (26), (27) and (28) we have

\[ I - \frac{(2z)^2}{\delta} \leq \frac{1}{r_1 \ldots r_p} \left( \frac{1}{\alpha_j} + R D^{p+1} + R D^{p+2} \right) \]

provided \( \delta \) satisfies the conditions (19).

Now by Lemma 3, there exists a \( \delta \) satisfying (19) such that

\[ \frac{1}{\alpha_j} + R D^{p+1} + R D^{p+2} \leq R D^{p+2} + R D^{p+2} + \sum_{j=1}^{p} \frac{1}{\alpha_j} + 1 \]

Hence we have, choosing this \( \delta \),

\[ I - \frac{(2z)^2}{\delta} \leq \frac{1}{r_1 \ldots r_p} \left( \frac{1}{\alpha_j} + R D^{p+2} + R D^{p+2} + \sum_{j=1}^{p} \frac{1}{\alpha_j} + 1 \right) \]

If \( R < 1 \), \( R D^{p+2} < R D^{p+4} \) and so we have Lemma 5. If \( R \geq 1 \), Lemma 5 follows trivially from Lemma 4. This completes the proof of Lemma 5.

4. Proposition. (The general Fourier summation formula.)

Let \( D \) be a finite region in the \( p \)-dimensional Euclidean space such that any line parallel to any of the coordinate axes meets it in a finite number of line segments. Further, let \( D \) have no lattice points on the boundary and

Let \( \alpha_i \Phi(1, \ldots, 1) \) be continuous on each of the bounding surfaces of \( D \). If \( \Phi(x_1, \ldots, x_p) \) is any real function with continuous first order partial derivatives in \( D \), then

\[ (29) \quad \sum_{(n_0, n_1, \ldots, n_p) \in \mathbb{Z}^p} \Phi(n_1, \ldots, n_p) \]

\[ = \sum_{n_0 = -\infty}^{\infty} \cdots \sum_{n_p = -\infty}^{\infty} \int_{D} \int \Phi(x_1, x_2, \ldots, x_p) e^{-2\pi i (n_0 x_1 + \ldots + n_p x_p)} \, dx_1 \ldots dx_p \]

In the case \( p = 1 \), the above formula can be proved, under the conditions stated, from Euler’s summation formula (cf. p. 13 of [12]). Proof of the \( p \)-dimensional formula follows by repeated application of the one-dimensional formula. In the one-dimensional case, modifications in the sum on the left-hand side have to be made if the end-points of the range of summation are integers. Analogous modifications have to be made in the left-hand side of (29) if there are lattice points on the boundary of \( D \). To avoid this, we have stipulated the condition that there are no lattice points on the boundary. In any particular case, it is always easy to choose a domain equivalent to the given domain, for which this condition is also satisfied.

Henceforward, \( D \) always satisfies the conditions stipulated above, in the statement of the Fourier summation formula.

Lemma 6. Let \( f(x_1, \ldots, x_p) \) be real, with second order partial derivatives in \( D \). Let \( f_{ij} \) have \( O(1) \) maxima and minima on any straight line parallel to a coordinate axis in \( D \). Further, let \( \alpha_i = \min f_{ij}, \beta_i = \max f_{ij} \) and \( \eta_i \) be any real constants, \( 0 < \eta_i < 1 \) for \( j = 1, \ldots, p \).

Let

\[ \frac{1}{(a_1, \ldots, a_p)} \geq A r_{a_1} \ldots r_{a_s} > 0 \quad (1 \leq i_1 < \ldots < i_s \leq p, 1 \leq s < p) \]

throughout \( D \), where the \( r_i \)’s are independent of the \( x_i \)’s. Then

\[ \sum_{D} e^{2\pi i (a_1 x_1 + \ldots + a_p x_p)} - \sum_{j=1}^{s} \sum_{a_1, \ldots, a_s} \int_{D} \sum_{x_1}^{a_1} e^{2\pi i (a_1 x_1 + \ldots + a_p x_p)} \, dx_1 \ldots dx_p \]

\[ \leq \sum_{J, J'} \prod_{j=1}^{s} \left( \frac{\beta_j - a_j + 1}{\beta_j} \right) \int_{r_{a_j} P} \log(\beta_j - a_j + 2) \]

where the sum is taken over every partition \( J, J' \) of the set of integers \( 1, 2, \ldots, p \), \( J' \) being non-null.
Proof. We may suppose without loss of generality that
\[ \eta_j - 1 < a_j < \eta_j, \quad j = 1, \ldots, p, \]
so that \( \eta_j \geq 0, j = 1, \ldots, p. \) For if \( k_j \) is an integer such that
\[ \eta_j - 1 \leq a_j - k_j = a' < \eta_j, \quad j = 1, \ldots, p, \]
and
\[ g(x_1, \ldots, x_p) = f(x_1, \ldots, x_p) - k_1 x_1 - k_2 x_2 - \cdots - k_p x_p, \]
we have to prove
\[ \sum_{e} e^{2\pi i (x_{e_{\eta_j}} - a_{\eta_j})} = \sum_{e' \in \mathbb{Z}^p} \prod_{j=1}^{p} \int_{D_j} e^{2\pi i (x_{e_{\eta_j}} - a_{\eta_j})} d\xi_j \cdots d\xi_p, \]
i.e. the same formula for \( g(x_1, \ldots, x_p). \)

We have, by the Fourier summation formula,
\[ S = \sum_{e} e^{2\pi i (x_{e_{\eta_j}} - a_{\eta_j})} = \sum_{e' \in \mathbb{Z}^p} \prod_{j=1}^{p} \int_{D_j} e^{2\pi i (x_{e_{\eta_j}} - a_{\eta_j})} d\xi_j \cdots d\xi_p. \]

Call the range of integers \( a_j - \eta_j < q_j \leq a_j + \eta_j \) or equivalently the range of integers \( 0 < \eta_j < q_j \) by \( R_j \) and let \( R_j \) denote the complement of \( R_j \) in the set of all integers, for \( j = 1, \ldots, p. \) We have
\[ S = \sum_{J} S_{J,J'}, \]
where
\[ S_{J,J'} = \sum_{e \in \mathbb{Z}^p} \prod_{j=1}^{p} \int_{D_j} e^{2\pi i (x_{e_{J_j}} - a_{\eta_j})} d\xi_j \cdots d\xi_p. \]

and the sum is taken over every partition \( J, J' \) of the set of integers \( 1, 2, \ldots, p. \)

We first consider the simple case when \( J' \) consists of unity alone.
Assuming \( n_j \in R_{j}, \) so that, in particular, \( n_j \neq 0, \) we have
\[ \int_{D_j} e^{2\pi i (x_{e_{J_j}} - a_{\eta_j})} d\xi_j \cdots d\xi_p = \sum_{e' \in \mathbb{Z}^p} \prod_{j=1}^{p} e^{2\pi i (x_{e_{J_j}} - a_{\eta_j})} d\xi_j \cdots d\xi_p, \]
and
\[ b_{\eta_j} e^{2\pi i (x_{e_{J_j}} - a_{\eta_j})} d\xi_j = \left( \frac{e^{2\pi i (x_{e_{J_j}} - a_{\eta_j})}}{2\pi i n_j} + b_{\eta_j} \right) \left( \frac{f_{a_{\eta_j}, n_j}^{2\pi i (x_{e_{J_j}} - a_{\eta_j})}}{2\pi i n_j} \right). \]

Now if \( \eta_j < 0 \) or \( n_j \geq \beta_j + \eta_j, \) and \( f_{a_{\eta_j}} \) is monotonic, \( \pm f_{a_{\eta_j}} (x_{e_{\eta_j}} - \eta_j) \) is bounded and monotonic in the same sense as \( f_{a_{\eta_j}}. \) Hence if \( \eta_j \in R_{j}, \) we have
\[
\begin{align*}
(31) \int_{D} e^{2\pi i (x_{e_{J_j}} - a_{\eta_j})} d\xi_j \cdots d\xi_p &= \sum_{e'' \in \mathbb{Z}^p} \left( \int_{D_j} e^{2\pi i (x_{e_{J_j}} - a_{\eta_j})} d\xi_j \cdots d\xi_p \right) \left( \frac{f_{a_{\eta_j}}}{2\pi i n_j} \right) + \\
&+ \sum_{e'' \in \mathbb{Z}^p} \left( \int_{D_{\eta_j}} e^{2\pi i (x_{e_{J_j}} - a_{\eta_j})} d\xi_j \cdots d\xi_p \right) \left( \frac{f_{a_{\eta_j}}}{2\pi i n_j} \right) \left( \frac{f_{a_{\eta_j}}}{2\pi i n_j} \right)
\end{align*}
\]
by repeated application of the second mean value theorem, since the domain of integration can be split up into \( O(1) \) domains in each of which \( f_{a_{\eta_j}} \) is monotonic separately in each of the variables.

Now we have
\[
\begin{align*}
(32) \left| \sum_{e'' \in \mathbb{Z}^p} e^{2\pi i (x_{e_{J_j}} - a_{\eta_j})} \right| &\leq \frac{1}{2} + \sum_{e'' \in \mathbb{Z}^p} \left| \frac{2\pi i n_j}{f_{a_{\eta_j}}} \right| \lesssim \left( \frac{\beta_j + \eta_j}{2\pi i n_j} \right), \\
(33) \sum_{e'' \in \mathbb{Z}^p} \left| \frac{f_{a_{\eta_j}}}{f_{J_j}(x_{e_{J_j}} - \eta_j)} \right| &\lesssim \sum_{e'' \in \mathbb{Z}^p} \left( \frac{\beta_j}{f_{a_{\eta_j}}} \right) + \sum_{e'' \in \mathbb{Z}^p} \left( \frac{\beta_j}{f_{a_{\eta_j}}} \right) + \left( \frac{\beta_j}{f_{a_{\eta_j}}} \right).
\end{align*}
\]
From (31), (32), (33) and Lemma 4, we find
\[
\sum_{e'' \in \mathbb{Z}^p} \int_{D} e^{2\pi i (x_{e_{J_j}} - a_{\eta_j})} d\xi_j \cdots d\xi_p \lesssim \left( \frac{\beta_j - a_{\eta_j} + 1}{2\pi i n_j} \right),
\]
In an exactly similar manner, we can prove the more general result
\[
\begin{align*}
(34) \quad S_{J,J'} &\lesssim \int_{J' \subset J} \left( \frac{\beta_j - a_{\eta_j} + 1}{2\pi i n_j} \right) \log(\beta_j - a_{\eta_j} + 2) \int_{J} \left( \frac{\beta_j - a_{\eta_j} + 1}{2\pi i n_j} \right).
\end{align*}
\]
for every partition \( J, J' \) of the set of integers \( 1, 2, \ldots, p, \) \( J' \) being non-null.

From (30) and (34) follows Lemma 6.

Lemma 7. Under the conditions of Lemma 6, let \( \Delta_n \) be defined as the smallest region containing the lattice points \( (\eta_1, \ldots, \eta_p) \) such that
\[
D \cap [\eta_1 - \eta_1] \cap \ldots \cap [\eta_p - \eta_p] \neq \Phi.
\]
Then we have

\[
\sum_{\mathbf{a}} e^{2 \pi i (\mathbf{a} \cdot \mathbf{u})} = \sum_{\mathbf{a}} \prod_{i=1}^{p} \int_{\mathbf{x} \in [0, 1]^2} e^{2 \pi i (\mathbf{a} \cdot \mathbf{x})} d\mathbf{x} = \prod_{i=1}^{p} \log |\beta_i - \alpha_i| + 2)
\]

where the sum is taken over every partition \( \mathcal{J}, \mathcal{J}' \) of the set of integers \( 1, 2, \ldots, p, \) \( \mathcal{J}' \) being non-null.

Proof. We have, using the notations of the the proof of Lemma 6,

\[
\sum_{\mathbf{a}} e^{2 \pi i (\mathbf{a} \cdot \mathbf{u})} = \sum_{\mathbf{a}} \prod_{i=1}^{p} \int_{\mathbf{x} \in [0, 1]^2} e^{2 \pi i (\mathbf{a} \cdot \mathbf{x})} d\mathbf{x} = \prod_{i=1}^{p} \log |\beta_i - \alpha_i| + 2)
\]

where

\[
I_i = \int_{\mathbf{x} \in [0, 1]^2} e^{2 \pi i (\mathbf{a} \cdot \mathbf{x})} d\mathbf{x}
\]

for \( i = 1, 2, \ldots, p \).}

We can now assume without loss of generality that

\[
\eta_j < \alpha_i < \eta_{j+1}, \quad j = 1, \ldots, p
\]

so that \( \eta_j > 0 \) if \( \eta_j \in B_i, \) \( i = 1, \ldots, p. \) We have then, as in the proof of Lemma 6,

\[
\sum_{\mathbf{a}} I_i = \sum_{\mathbf{a} = 0}^{\infty} \sum_{j=1}^{p} e^{2 \pi i (\mathbf{a} \cdot \mathbf{u})} \int_{\mathbf{x} \in [0, 1]^2} e^{2 \pi i (\mathbf{a} \cdot \mathbf{x})} d\mathbf{x} = \prod_{i=1}^{p} \log |\beta_i - \alpha_i| + 2)
\]

since the domain can be split up into \( O(1) \) domains in each of which \( \pm f_{\mathbf{a}}(\mathbf{x} - \mathbf{u}) \) is monotonic separately in each of the variables and bounded because \( |f_{\mathbf{a}} - \mathbf{u}| > \eta_j \) throughout.

Hence we find, as in Lemma 6,

\[
\sum_{\mathbf{a} \in \mathcal{J}} I_i \leq \log |\beta_i - \alpha_i| + 2 \prod_{i=1}^{p} \log |\beta_i - \alpha_i| + 2)
\]

A similar result holds for every other term on the right-hand side of (35) and Lemma 7 now follows from 35 and Lemma 6.

5. Theorem 1. Let \( f(x_1, \ldots, x_p) \) possess continuous second order partial derivatives in the hyper-rectangle \( D', x_i < x_j < \beta_i, j = 1, \ldots, p, \) containing \( D. \)

Let

\[
|f_{\mathbf{u}}| \leq A \lambda_{\mathbf{u}} \quad (1 \leq i, j \leq p)
\]

and

\[
\frac{\partial^2 (f_{\mathbf{u}})}{\partial x_i \partial x_j} \geq A \lambda_{\mathbf{u}} > 0 \quad \text{for} \quad 1 \leq i_1 < i_2 < \cdots < i_r < p, 1 \leq r \leq p,
\]

throughout \( D', \) where \( \lambda_\mathbf{i} \) are positive numbers independent of \( x, \) satisfying the relations \( (b_i - a_i) \lambda_{\mathbf{i}} \leq (b_i - a_i) \lambda_{\mathbf{i}} \) and \( b_i > a_i + 1. \) Then

\[
S = \sum_{\mathbf{a}} e^{2 \pi i (\mathbf{a} \cdot \mathbf{u})} \leq \prod_{i=1}^{p} ((b_i - a_i) \lambda_{\mathbf{i}} ^{1/2} + \lambda_{\mathbf{i}} ^{1/2}).
\]

Proof. We have from Lemmas 6 and 4

\[
S \leq \prod_{j=1}^{p} \log |\beta_j - \alpha_j| + 2 \prod_{i=1}^{p} \log |\beta_i - \alpha_i| + 2)
\]

Now

\[
\beta_i - \alpha_i \leq \sum_{i=1}^{p} (h_i - a_i) \lambda_{\mathbf{i}} \leq (b_i - a_i) \lambda_{\mathbf{i}}
\]

and

\[
\log (\beta_i - \alpha_i + 2) \leq \log ((b_i - a_i) \lambda_{\mathbf{i}} + 2)
\]

\[
\leq \sum_{i=1}^{p} (h_i - a_i) \lambda_{\mathbf{i}} + 2
\]

\[
= \lambda_{\mathbf{i}} ^{1/2} ((b_i - a_i) \lambda_{\mathbf{i}} + 2) ^{1/2}
\]

\[
\leq \lambda_{\mathbf{i}} ^{1/2} (b_i - a_i + 1) \lambda_{\mathbf{i}}
\]

since \( b_i > a_i + 1. \) Theorem 1 is now immediate.

Theorem 2 (The general Van der Corput transformation.)

Let \( f(x_1, \ldots, x_p) \) possess continuous third order partial derivatives in the hyper-rectangle \( D', x_i < x_j < \beta_i, j = 1, \ldots, p, \) where \( b_i \geq a_i + 1. \) Let

\[
\frac{\partial^3 (f_{\mathbf{u}})}{\partial x_i \partial x_j \partial x_k} \geq A \prod_{j=1}^{p} (b_i - a_i) ^{k/2}
\]

for \( 1 \leq i < \cdots < i_k \leq p, \) \( 1 \leq k \leq p. \)
and

\[ |f_{xy}| \leq A \frac{R^4}{(b_i - a_i)(b_j - a_j)}, \quad |f_{x_2y_2}| \leq A \frac{R^4}{(b_1 - a_1)(b_2 - a_2)(b_3 - a_3)} \]

throughout \( D' \supset D \).

Let \((x_1, ..., x_p) \in D\), where \(A = \text{const. of the region} \) \(D\) by the transformation \(y_i = f_{x_i} = \alpha_{x_i}, i = 1, ..., p\). Then, if \(m\) is the number of changes of sign in the sequence

\[ +1, \frac{\partial (f_{x_1}, ..., f_{x_p})}{\partial (x_1, ..., x_p)}, \ldots, \frac{\partial (f_{x_1}, ..., f_{x_p})}{\partial (x_{j-1}, x_j, x_{j+1}, ..., x_p)} \]

we have

\[
\sum_{(x_1, ..., x_p) \in D} \exp\left(\frac{m-2m-1}{4} \sum_{(x_1, ..., x_p) \in D} \frac{\partial (f_{x_1}, ..., f_{x_p})}{\partial (x_1, ..., x_p)} \right)
\]

\[
\leq R^{m-2m-1} + \sum_{j=1}^{p} \prod_{x_{j-1} \neq x_j, x_{j+1}} \left(1 + \frac{b_j - a_j}{r} + \log \left(1 + \frac{b_j - a_j}{r} \right) \right).
\]

Proof. In the following proof \(a_j\) and \(b_j\) (\(a_j < b_j\)) denote the values of \(f_{x_j}\) at the end-points of the largest segment of the curve \(f_{x_j} = \alpha_j, i = 1, ..., j-1, j+1, ..., p\), which lies entirely within \(D\). Also

\[ a_j = \text{Min}_{D} f_{x_j}, \quad b_j = \text{Max}_{D} f_{x_j}, \quad j = 1, ..., p. \]

The region \(D_1\) is defined as in Lemma 7. The region \(D_2\) is defined as the largest region contained in \(D\) containing lattice points \((n_1, ..., n_p)\) such that the hyper-rectangle \(|y_j - \alpha_j| \leq 1, j = 1, ..., p,\) is entirely contained in \(D_1\).

We have, by Lemma 5,

\[
\sum_{(n_1, ..., n_p) \in D_1} \prod_{j=1}^{p} \exp\left(\frac{m-2m-1}{4} \sum_{(n_1, ..., n_p) \in D_1} \frac{\partial (f_{x_1}, ..., f_{x_p})}{\partial (x_1, ..., x_p)} \right)
\]

\[
\leq \prod_{j=1}^{p} \left(1 + \frac{b_j - a_j}{r} + \log \left(1 + \frac{b_j - a_j}{r} \right) \right).
\]

Now

\[ \beta_j - a_j < \sum_{j=1}^{p} \left( \frac{b_j - a_j}{r} \right)^2 \leq \frac{r^2}{(b_j - a_j)} \]

and

\[ \beta_j - a_j < \sum_{j=1}^{p} \frac{(b_j - a_j)^2}{r} \leq \frac{r^2}{(b_j - a_j)} \]

Also, the error term introduced by replacing the domain of summation \(D_1\) by \(D_2\) and \(D\) respectively in the first and second terms of the left-hand side of (36) is

\[
\sum_{j=1}^{p} \left( \frac{b_j - a_j}{r} \right)^2 \left( \sum_{x_j \neq x_{j-1}} \left(1 + \frac{b_j - a_j}{r} + \log \left(1 + \frac{b_j - a_j}{r} \right) \right) \right).
\]

As in the proof of Theorem 1,

\[ \log(\beta_j - a_j + 2) < \frac{b_j - a_j}{r} \]

and so

\[
\sum_{j=1}^{p} \log(\beta_j - a_j + 2) \leq \frac{b_j - a_j}{r}.
\]

Theorem 2 now follows from (36) to (39) and Lemma 7.

Theorem 3. Let \(f\) possess continuous partial derivatives up to \(k+2\)-th order in the hyper-rectangle \(D\), \(a_2 < \alpha_2 \leq b_2, j = 1, ..., p,\) containing \(D\). Let \(k = \alpha_1 + \ldots + \alpha_p \geq 1\) and \(k_j > 0, j = 1, ..., p.\) Let \(g = f_{x_1, ..., x_p}\) and

\[
\frac{\partial (g_{x_1}, ..., g_{x_p})}{\partial (x_1, ..., x_p)} \geq A x_{2k} \ldots x_{2k} > 0 \quad 1 \leq i < k < \ldots < k_j \leq \alpha_j, 1 \leq i, j \leq p,
\]

\[
[g_{x_2}, ..., g_{x_p}] \leq A x_{k_j}
\]

in \(D\), where \(x\)'s are positive numbers independent of \(x\), satisfying the conditions \((b_j - a_j) x_{2k} < (b_j - a_j) x_{2k} \).

Further, let

\[ b_j \geq a_j + 1, \quad K = 2^{k_j}, \quad K_j = 2^{k_j+1} \quad \text{for} \ j = 1, ..., p, \]

Then

\[
S = \frac{1}{k} \prod_{j=1}^{p} (b_j - a_j) \sum_{(x_1, ..., x_p) \neq (0, ..., 0)} \exp\left(\frac{m-2m-1}{4} \sum_{(x_1, ..., x_p) \neq (0, ..., 0)} \frac{\partial (f_{x_1}, ..., f_{x_p})}{\partial (x_1, ..., x_p)} \right)
\]

where

\[
N = \sum_{j=1}^{p} (b_j - a_j) x_{2k} + \sum_{j=1}^{p} \frac{1}{x_{2k} + 1} - \sum_{j=1}^{p} x_{2k} \]

in \(D, \quad k = 2^{k_j}, \quad K_j = 2^{k_j+1} \quad \text{for} \ j = 1, ..., p. \)
Hence, by the given conditions of Theorem 3, the conditions of Theorem 1 are satisfied for the functions \( A \) with \( U \beta \) in the place of \( \beta \) and so we have by Theorem 1

\[
S' = \sum_{i=1}^{n} g(x; (b_i - a_i) \in \prod_{i=1}^{n} \{b_i - a_i\}^{1/2} (U \beta)^{-1/2} + 1) .
\]

And, by successive applications (\( k \) times w.r.t the variable \( n \)) of Lemma 2 to the sum \( S' \) we find

\[
S = \prod_{i=1}^{n} (b_i - a_i) (q_1^{1/2} + q_2^{1/2} + \ldots + q_k^{1/2}) + \sum_{i=1}^{n} \frac{1}{q_1 \cdots q_k} \sum_{|S'| \leq k} |S'|^{1/2} .
\]

where

\[
0 < q_1 \leq b_k - a_k \quad \text{for} \quad \beta = k_1 + \ldots + k_{n-1} + 1, \ldots, k_1 + \ldots + k_n, \quad a = 1, \ldots, p , \quad \text{and} \quad u_0 \in \{S' \} \quad \text{runs through}
\]

\[
l \leq u_0 = u_0' 
\]

\[
\text{where} \quad u_0 \quad \text{being the} \quad a \text{th} \quad u \quad \text{in the sequence} \quad u_0, \ldots, u_k \quad \text{for} \quad a = 1, \ldots, k ,
\]

\[
\text{From (41), (42) and (43) we deduce, putting} \quad q_0 = q_1 \ldots q_k \quad \text{(} 1 \leq a \leq k),
\]

\[
\frac{S}{\prod_{i=1}^{n} (b_i - a_i)} \leq \psi_1^{1/2} + \psi_2^{1/2} + \ldots + \psi_k^{1/2} + \psi_{k+1}^{1/2} + \psi_{k+2}^{1/2} + \ldots + \psi_{k+2}^{1/2} + \psi_{k+3}^{1/2} + \ldots + \psi_{k+3}^{1/2} + \ldots + \psi_{k+3}^{1/2} + \ldots
\]

\[
\sum_{i=1}^{n} \frac{1}{q_1 \cdots q_k} \sum_{|S'| \leq k} |S'|^{1/2} .
\]

provided the \( \psi \) satisfy the conditions

\[
\frac{q_0}{b_i - a_i} \leq \frac{q_0}{b_i - a_i} \leq \frac{q_0 + 1}{b_i - a_i} \leq q_0 \quad \text{(} 1 < b < k\)
\]

where \( \beta = k_1 + \ldots + k_{n-1} + 1, \ldots, k_1 + \ldots + k_n, \quad a = 1, \ldots, p , \quad 0 < q_0 \leq \alpha .
\]

Now it can easily be shown by induction, by appealing to Lemma 3, that there exist \( \varphi_1, \ldots, \varphi_{n-1} \) satisfying (45) such that

\[
\frac{\psi_1^{1/2} + \psi_2^{1/2} + \ldots + \psi_k^{1/2} + \psi_{k+1}^{1/2} + \psi_{k+2}^{1/2} + \ldots + \psi_{k+2}^{1/2} + \ldots + \psi_{k+3}^{1/2} + \ldots}{b_0 - a_0} < \sum_{i=1}^{n} \frac{1}{b_i - a_i} \frac{1}{\varphi_1^{1/2} + \varphi_2^{1/2} + \ldots + \varphi_{k+3}^{1/2} + \ldots} .
\]
Lattice point problem, many-dimensional hyperboloids. II

Theorem 3'. Suppose \( f(x) \) to be real and has derivatives up to \((b+2)\)-th order \((b \geq 3)\) in \((a,b)\). Let

\[
0 < \lambda_{k+2} < \lambda^{b+2}(a) < A \lambda_{k+2} \quad \text{or} \quad \lambda_{k+2} < -\lambda^{b+2}(a) < A \lambda_{k+2}
\]

throughout \((a,b)\). Let \( b > a + 1 \), \( K = 2^k \). Then

\[
\sum_{\alpha \in \mathbb{Z}^d} e^{2\pi i \alpha \cdot \beta (a-b)^{2(b+2)}} \leq (b-a)^{2(b+2)} A \lambda_{k+2}^{-1/2} (a-b)^{-1/2} K.
\]

Theorem 4'. Under the hypotheses of Theorem 3'

\[
\sum_{\alpha \in \mathbb{Z}^d} e^{2\pi i \alpha \cdot \beta (a-b)^{2(b+2)}} \leq (b-a)^{2(b+2)} A \lambda_{k+2}^{-1/2} (a-b)^{-1/2} K.
\]

Theorem 3' is Van der Corput's inequality (Satz 4 of [3]) for one-dimensional exponential sums, while Theorem 4' is the Titchmarsh inequality (Theorem 5.13 of [12]). In view of the above remark it follows in particular that Van der Corput's inequality is stronger than Titchmarsh's inequality. A direct proof of this statement is given in [7].

6. The lattice point problem. We require the following lemmas.

Lemma 8. For arbitrary \( M > 0 \) and for any function \( g \),

\[
\sum_{(n_1, \ldots, n_p) \in D} \psi_i(g(n_1, \ldots, n_p)) \leq \frac{\|D\|}{M} + \sum_{m=1}^{\infty} \sum_{(n_1, \ldots, n_p) \in D} e^{-2\pi i \text{Im}(\zeta m \langle n_1, \ldots, n_p \rangle)} \min \left( \frac{1}{m}, \frac{M^*}{m^{1+1}} \right),
\]

where \( \psi_i \) is the function defined in (8), \( |D| \) is the volume of the region \( D \) and \( s \) is any positive integer.

Proof. We have

\[
|D| \lesssim M^* \int_0^\infty \left( \int_0^\infty \psi_i(g(n_1, \ldots, n_p)) dy_j \right)^2 \, dy_j,
\]

provided

\[
\sum_{m=1}^{\infty} \sum_{n_1, \ldots, n_p} e^{-2\pi i \text{Im}(\zeta m \langle n_1, \ldots, n_p \rangle)} \min \left( \frac{1}{m}, \frac{M^*}{m^{1+1}} \right) = \sum_{m=1}^{\infty} c_m e^{-2\pi i \text{Im}(\zeta m \langle n_1, \ldots, n_p \rangle)}
\]

where \( c_m \) is \( 1/m \) and \( c_m \ll M^*/m^{1+1} \) and the dash denotes that the term corresponding to \( m = 0 \) is omitted. Hence

\[
\sum_{m=1}^{\infty} \frac{e^{-2\pi i \text{Im}(\zeta m \langle n_1, \ldots, n_p \rangle)}}{3} \left( \int_0^\infty \psi_i(g(n_1, \ldots, n_p)) dy_j \right)^2 \sim \sum_{m=1}^{\infty} |c_m| e^{-2\pi i \text{Im}(\zeta m \langle n_1, \ldots, n_p \rangle)} \min \left( \frac{1}{m}, \frac{M^*}{m^{1+1}} \right).
\]
Again \( \psi(u) - \psi(v) \leq u - v \) if \( u \geq v \). Hence we have

\[
\sum_{D} M^{t} \int_{0}^{1} \ldots \int_{0}^{1} \int_{0}^{y_{1}} \ldots \int_{0}^{y_{1}} dy_{1} \ldots dy_{t} \\
\leq \sum_{D} \psi_{r}(g(n_{1}, \ldots, n_{p})) + \sum_{D} y_{1} \int_{0}^{y_{1}} \ldots \int_{0}^{y_{1}} dy_{1} \\
\leq \sum_{D} \psi_{r}(g(n_{1}, \ldots, n_{p})) + \frac{1}{2M} \sum_{D} |D|.
\]

Similarly

\[
\sum_{D} M^{t} \int_{-M}^{0} \ldots \int_{-M}^{0} \int_{0}^{y_{1}} \ldots \int_{0}^{y_{1}} dy_{1} \ldots dy_{t} \\
\geq \sum_{D} \psi_{r}(g(n_{1}, \ldots, n_{p})) - \frac{1}{2M} |D|.
\]

So

\[
\sum_{(n_{1}, \ldots, n_{p}) \in D} \psi_{r}(g(n_{1}, \ldots, n_{p})) \\
\leq \sum_{D} M^{t} \int_{0}^{1} \ldots \int_{0}^{1} \int_{0}^{y_{1}} \ldots \int_{0}^{y_{1}} dy_{1} \ldots dy_{t} \\
\leq \sum_{D} \psi_{r}(g(n_{1}, \ldots, n_{p})) + \frac{1}{2M} |D|.
\]

(49) now follows from (50) and (51).

**LEMMA 9.** If \( a_{1}, \ldots, a_{p} \) are any real numbers, then

\[
\sum_{a_{1}^{r_{1}} a_{2}^{r_{2}} \ldots a_{p}^{r_{p}}} = \sum_{b_{1}^{r_{1}} b_{2}^{r_{2}} \ldots b_{p}^{r_{p}}} \frac{a_{1}^{r_{1}} a_{2}^{r_{2}} \ldots a_{p}^{r_{p}}}{b_{1}^{r_{1}} b_{2}^{r_{2}} \ldots b_{p}^{r_{p}}} \leq \sum_{b_{1}^{r_{1}} b_{2}^{r_{2}} \ldots b_{p}^{r_{p}}} \frac{a_{1}^{r_{1}} a_{2}^{r_{2}} \ldots a_{p}^{r_{p}}}{b_{1}^{r_{1}} b_{2}^{r_{2}} \ldots b_{p}^{r_{p}}} + \log x.
\]

Proof. When \( p = 1 \) we obviously have

\[
\sum_{b_{1}^{r_{1}}} a_{1}^{r_{1}} \leq a_{1}^{r_{1}} \log x.
\]

We prove the lemma by induction on \( p \). We have, if \( p \geq 1 \),

\[
\sum_{a_{1}^{r_{1}} a_{2}^{r_{2}} \ldots a_{p}^{r_{p}}} = \sum_{b_{1}^{r_{1}} b_{2}^{r_{2}} \ldots b_{p}^{r_{p}}} \frac{a_{1}^{r_{1}} a_{2}^{r_{2}} \ldots a_{p}^{r_{p}}}{b_{1}^{r_{1}} b_{2}^{r_{2}} \ldots b_{p}^{r_{p}}} \\
\leq \sum_{b_{1}^{r_{1}} b_{2}^{r_{2}} \ldots b_{p}^{r_{p}}} \frac{a_{1}^{r_{1}} a_{2}^{r_{2}} \ldots a_{p}^{r_{p}}}{b_{1}^{r_{1}} b_{2}^{r_{2}} \ldots b_{p}^{r_{p}}} (if \ b_{1} \neq 0)
\]

Hence the lemma follows by induction on \( p \) in the case \( b_{1} \neq 0 \).

If

\[
a_{1} = \ldots = a_{s} = 0, \quad a_{s+1} \neq 0, \quad 1 < s < p,
\]

then

\[
\sum_{a_{1}^{r_{1}} a_{2}^{r_{2}} \ldots a_{p}^{r_{p}}} \leq \sum_{a_{1}^{r_{1}} a_{2}^{r_{2}} \ldots a_{p}^{r_{p}}} + \log x,
\]

and the lemma again follows in this case. If \( h \), however, all the \( a_{i} \)'s are zero, the lemma is trivial. Hence the lemma.

We now consider the exponential sum

\[
S = \sum_{D} e^{2\pi i \frac{a_{1} n_{1} + \ldots + a_{p} n_{p}}{n_{1}^{p+1}}}
\]

where \( D \) is the region \( a_{1} \leq n_{1} \leq 2a_{1}, \ldots, a_{p+1} \leq n_{p+1} \leq \leq 2a_{p+1}, \ldots, a_{p} \leq n_{p} \leq 2a_{p} \), and apply Theorem 2 to \( S \). Here

\[
f(a_{1}, \ldots, a_{p}) = \left( \frac{x}{a_{1} \ldots a_{p}} \right)^{1/p+1},
\]

\[
e_{s} = \frac{1}{r+1} \left( \frac{s}{a_{s} \ldots a_{p}} \right)^{1/p+1}, \quad j = 1, \ldots, p.
\]

\( S \) is a region (*) contained in

\[
\left\{ \frac{1}{r+1} \right\} \ldots \left\{ \frac{1}{r+1} \right\} \frac{s}{a_{s} \ldots a_{p}} \frac{1}{r+1} \leq \frac{s}{a_{s} \ldots a_{p}} \leq \frac{1}{r+1} \frac{s}{a_{s} \ldots a_{p}}, \quad j = 1, \ldots, p,
\]

where \( s = (a_{1} a_{2} \ldots a_{p})^{1/p+1} \geq a_{1} \geq 1 \) if the sum \( S \) is to be non-null. Here we find

\[
r = 2^{1/(p+1)}, \quad R = x^{-1/p}, \quad m = p/2,
\]

\[
f(a_{1}, \ldots, a_{p}) - \sum_{a_{1}^{r_{1}} a_{2}^{r_{2}} \ldots a_{p}^{r_{p}}} \leq \frac{p+1}{R^{p+1}} \left( \sum_{a_{1}^{r_{1}} a_{2}^{r_{2}} \ldots a_{p}^{r_{p}}} \right)^{1/(p+1)}
\]

\[
= \frac{p+1}{R^{p+1}} \left( \frac{x}{a_{1} \ldots a_{p}} \right)^{1/(p+1)} \left( \frac{x}{a_{1} \ldots a_{p}} \right)^{1/(p+1)}.
\]

(*) Throughout, we are considering only regions satisfying the general conditions we have imposed before.
Hence by Theorem 2, we have

\[
\begin{aligned}
S - e^{-\frac{\pi r}{p+1}} \left( \sum_{(n_1, \ldots, n_p) \in \mathcal{A}} \frac{1}{(n_1^{p+1} + \cdots + n_p^{p+1})^{1/(p+1)}} \times \prod_{j=1}^{p} \left( 1 + \frac{1}{n_j^{p+1}} \right) \right)
\end{aligned}
\]

Hence, by Theorem 3,

\[
\begin{aligned}
&\sum_{p=1}^{\infty} \left( \frac{1}{\sigma_1^{p+1}} + \left( \frac{1}{\sigma_2^{p+1}} \right)^{1/2} \right) \left( \frac{1}{\sigma_p^{p+1}} \right)^{1/2} \\
&\quad + \sum_{p=1}^{\infty} \left( \frac{1}{\sigma_1^{p+1}} \right)^{1/2} \left( \frac{1}{\sigma_2^{p+1}} \right)^{1/2} \\
&\quad + \sum_{p=1}^{\infty} \left( \frac{1}{\sigma_1^{p+1}} \right)^{1/2} \left( \frac{1}{\sigma_2^{p+1}} \right)^{1/2} \\
&\quad + \sum_{p=1}^{\infty} \left( \frac{1}{\sigma_1^{p+1}} \right)^{1/2} \left( \frac{1}{\sigma_2^{p+1}} \right)^{1/2}
\end{aligned}
\]

We can now assume without loss of generality that

\[
\begin{aligned}
\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 1.
\end{aligned}
\]

Then, in the above, we can choose \( \sigma_1 = \sigma_2 = \ldots = \sigma_p \) if \( p \neq 2 \). Hence, if

\[
\begin{aligned}
0 < \theta < \frac{p}{2},
\end{aligned}
\]

and the third term

\[
\begin{aligned}
\leq 2^{-\frac{p-2}{2}} (a_0 + 2a_1 + 2a_2 + \cdots + 2a_p) \leq 1.
\end{aligned}
\]

So the sixth term on the right-hand side of (53)

\[
\begin{aligned}
\leq 2^{-\frac{p-2}{2}} (a_0 + 2a_1 + 2a_2 + \cdots + 2a_p) \leq 1,
\end{aligned}
\]

and apply Theorem 3 to \( S' \). Here we easily find that

\[
\begin{aligned}
\omega = \omega' = 1,
\end{aligned}
\]

where

\[
\begin{aligned}
\sigma_s = \sigma_s^{(p+1)}
\end{aligned}
\]

the convention being \( \sigma_s = 0 \) if \( s \) is not an integer. Therefore

\[
\begin{aligned}
S - e^{-\frac{\pi r}{p+1}} \left( \sum_{(n_1, \ldots, n_p) \in \mathcal{A}} \frac{1}{(n_1^{p+1} + \cdots + n_p^{p+1})^{1/(p+1)}} \times \prod_{j=1}^{p} \left( 1 + \frac{1}{n_j^{p+1}} \right) \right)
\end{aligned}
\]
Hence by Lemma 8, for arbitrary $M > 0$ and $s > \text{Max} \alpha_s$,
\[
S_1 = \sum_{(m_1, \ldots, m_M) \in D} \left( \frac{\phi_1}{\phi_1 \cdots \phi_M} \right)^{1+\nu}
\leq \sum_{m_1 = 1}^{\infty} \sum_{m_2 = 1}^{m_1} \sum_{m_3 = 1}^{m_2} \cdots \sum_{m_M = 1}^{m_{M-1}} \frac{c_0}{M} \sum_{s \in \{0, \ldots, M\}} \sum_{p \in \mathbb{N}} \phi_1^{m_1} \cdots \phi_M^{m_M} m_1 \cdots m_M^{s+1} + c_p m_1 \cdots m_M^{s-1}
= \frac{c_0}{M} \sum_{s \in \{0, \ldots, M\}} \sum_{p \in \mathbb{N}} \phi_1^{m_1} \cdots \phi_M^{m_M} m_1 \cdots m_M^{s+1} + c_p m_1 \cdots m_M^{s-1}
\leq \frac{c_0}{M} \sum_{s \in \{0, \ldots, M\}} \sum_{p \in \mathbb{N}} \phi_1^{m_1} \cdots \phi_M^{m_M} m_1 \cdots m_M^{s+1} + c_p m_1 \cdots m_M^{s-1}
\]

Now by Lemma 3, there is $M > 0$ so that
\[
S_1 \leq \sum_{a} \left( \sum_{(m_1, \ldots, m_M) \in D} \left( \frac{\phi_1}{\phi_1 \cdots \phi_M} \right)^{1+\nu} \right)
\]

Hence, by Lemma 1,
\[
\sum_{a} \left( \sum_{(m_1, \ldots, m_M) \in D} \left( \frac{\phi_1}{\phi_1 \cdots \phi_M} \right)^{1+\nu} \right) \leq \sum_{a} \left( \sum_{(m_1, \ldots, m_M) \in D} \left( \frac{\phi_1}{\phi_1 \cdots \phi_M} \right)^{1+\nu} \right)
\]

Hence
\[
S_2 = \sum_{m_1 = 1}^{\infty} \sum_{m_2 = 1}^{m_1} \cdots \sum_{m_M = 1}^{m_{M-1}} \left( \frac{\phi_1}{\phi_1 \cdots \phi_M} \right)^{1+\nu}
= \sum_{h_1 \cdots h_M} \sum_{m_1 \cdots m_M} \left( \frac{\phi_1}{\phi_1 \cdots \phi_M} \right)^{1+\nu}
\]

Since $S_{h_1 \cdots h_M}$ is obviously invariant for interchange of any two $h$'s, we have
\[
S_4 \leq \sum_{m_1 \cdots m_M} \left( \frac{\phi_1}{\phi_1 \cdots \phi_M} \right)^{1+\nu}
\]

From (55), (56) and Lemma 9, we deduce
\[
S_4 \leq \sum_{p \in \mathbb{N}} \left( \frac{\log p}{\log 2} \right)^{p+1} \sum_{a} \phi_1^{(a)}
\]

where
\[
\phi_1^{(a)} = 1 + \frac{\phi_{p-1}}{\phi_{p+1}} \frac{\phi_{p-2}}{\phi_{p+1}} \cdots \frac{\phi_{p-1-a}}{\phi_{p+1-a}} \frac{1}{\phi_{p+1-a}}
\]

$\sum_{a=1}^{\infty} \phi_1^{(a)}$ denoting zero if $\lambda = 0$.

We now choose $k_1 = \ldots = k_{p-1} = 0$, $k_p = 1$ in (53). Writing down the various values of $\phi_1^{(a)}$, we find, for fixed $\lambda$, that $\phi_1^{(a)}$ is a monotonic function of the variables of summation $s$, $s'$ in (53), and so $\phi_1^{(a)}$ corresponding to the end-values of $s$, $s'$ alone matter. Substituting the resulting values of $\phi_1^{(a)}$ in the expression for $\phi_1^{(a)}$, we find that the values for $\phi_1^{(a)}$ are again monotonic functions of $\lambda$, and so here again $\phi_1^{(a)}$ and $\phi_1^{(a)}$ alone matter. We find the best possible choice of $k$ to be $k = 1$, if $p = 4$, $k = 2$, if $p = 2$ or $3$ and $k = 3$, if $p = 1$. Choosing these values of $k$, we find
\[
\phi_1^{(a)} = \min_{a} \phi_1^{(a)} = \frac{2(p+4)}{(p+2)(p+4) - 1} - \frac{16}{23}
\]

if $p = 2$, $p = 1$ respectively.

\[
\phi_1^{(a)} = \min_{a} \phi_1^{(a)} = \frac{2(p+1)(p+4)}{(p+2)(p+3) - 1} - \frac{23}{14} + \frac{55}{14}
\]

if $p = 4$, $p = 3, 2, 1$ respectively.

So finally we find that (57) reduces to
\[
S_4 \leq \frac{1}{\phi_{p+1} + \phi_{p+1}^{(a)}} + \frac{1}{\phi_{p+1}^{(a)}} \phi_1^{(a)}
\]

Noting that $\phi < 2$, we have from (7)
\[
\phi_1^{(a)} = \frac{1}{\phi_{p+1} + \phi_{p+1}^{(a)}} + \frac{1}{\phi_{p+1}^{(a)}} \phi_1^{(a)}
\]

(12) is now an immediate consequence of (59), (60) and (61).

In conclusion, I wish to record with great pleasure my sincere thanks to Professors V. S. Krishnam and C. R. Rajagopala for the keen interest they evinced in the preparation of this paper and last but not least, to Professor V. Granapathy for his valuable suggestions and criticism.
Another note on Hardy-Littlewood's theorem

by

S. Knapowski and W. Szaś (Poznań)

1. In this paper we return to the subject of [3], i.e. to the investigation of the behaviour of

\[ F(y) = \sum_{n=1}^{\infty} \frac{(A(n)-1)e^{-ny}}{n^s}, \quad y > 0, \]

as \( y \to 0^+ \). Unlike in [3] we shall be interested here in oscillatory properties of the function (1.1). Hardy and Littlewood showed [1] that on the Riemann hypothesis there is a constant \( K \) such that each of the inequalities

\[ F(y) < \frac{K}{y^{1/2}}, \quad F(y) > \frac{K}{y^{1/2}} \]

is satisfied for an infinity of values of \( y \) tending to zero. In connection with this result we shall suppose here inequalities similar, though somewhat weaker, to (1.2) holding however in an explicit form and without any hypothesis. In the proof we shall make use of the method of Turán (see [5]), particularly its development to the study of oscillatory questions in prime number theory (see [4]). Our result reads as follows:

**Theorem.** For \( 0 < \delta < \varepsilon \) \( (*) \) we have

\[ \max_{s \in \mathbb{C}^{+}} F(y) > \delta^{-1/2} \exp \left( -14 \frac{\log(1/\delta)\log\log(1/\delta)}{\log(1/\delta)} \right) \]

and

\[ \min_{s \in \mathbb{C}^{+}} F(y) < -\delta^{-1/2} \exp \left( -14 \frac{\log(1/\delta)\log\log(1/\delta)}{\log(1/\delta)} \right). \]

**Corollary.** Replacing the exponent \( \frac{1}{2} \) in (1.2) by \( \frac{1}{2} - \varepsilon, \varepsilon > 0 \) and arbitrary, the inequalities are satisfied (without any hypothesis!) for an infinity of values of \( y \) tending to zero.

(*) \( \varepsilon, \alpha_1, \ldots \) denote positive, numerically calculable constants.