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The lattice point problem of many-dimensional hyperboloids I

by

B. R. Srinivasan (Madras)

To the loving and respectful memory of Prof. Dr R. Vaidhyanathaswamy

We obtain here explicit asymptotic expressions for the sum

$$D_k^{(r,\varrho)}(x) = \binom{k}{r} \sum_{\substack{n_1\ldots n_{k-r} n_j^r \leqslant x \\ j=1,\ldots,k-r}} \left(\frac{x}{n_1 \ldots n_{k-r}}\right)^{1+\varrho}$$

(where n_j are positive integers, x, ϱ real, $x \geqslant 1$, $\varrho > -2$, $0 \leqslant r < k$) with error terms of as small an order as we please. In the case $\varrho = -1$ the above sum represents the number of lattice points in a region bounded by the coordinate hyperplanes and a certain number of hyperboloids in a $(k-r)$-dimensional Euclidean space. If, further, we take $r = 0$ the above sum reduces to the kth divisor sum, i.e. the sum-function of the kth divisor function $d_k(n)$ = the number of ways in which the positive integer n can be written as the product of k positive integers. The related problem of the lattice points in many-dimensional ellipsoids has been considered by Walfisz [1].

1. We begin by proving a fundamental inversion formula.

Theorem 1. *We have*

$$(1) \qquad F_k(x) = \sum_{r=0}^{k} \binom{k}{r} \sum_{\substack{n_1\ldots n_r n_j^{k-r} \leqslant x \\ j=1,\ldots,r}} f_{k-r}\left(\frac{x}{n_1 \ldots n_r}\right) \quad \textit{for} \quad 0 \leqslant k \leqslant K ,$$

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[1] For references and litterature on this subject, cf. [2].

if and only if

$$(2)\qquad f_k(x) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} \sum_{\substack{n_j^k \leqslant x\\ j=1,\dots,r}} F_{k-r}\left(\frac{x}{n_1 \dots n_r}\right) \quad \textit{for} \quad 0 \leqslant k \leqslant K$$

where

$$\sum_{\substack{n_1\dots n_r n_j^{k-r} \leqslant x\\ j=1,\dots,r}} f_{k-r}\left(\frac{x}{n_1 \dots n_r}\right) \quad \textit{and} \quad \sum_{\substack{n_j^k \leqslant x\\ j=1,\dots,r}} F_{k-r}\left(\frac{x}{n_1 \dots n_r}\right)$$

denote respectively $f_k(x)$ *and* $F_k(x)$ *when* $r = 0$.

To prove the theorem, we require the following lemmas:

Lemma 1. *Let* D, D_i, $i = 1, \dots, k$, *denote regions in a* k*-dimensional Euclidean space* E_k *and let* D_i', $i = 1, \dots, k$, *be the set-complement of the region* D_i *in* E_k. *Then*

$$(3)\qquad \sum_{D\cap D_1'\cap\dots\cap D_k'} f(n_1, \dots, n_k) = \Big\{\sum_{D} - \sum_{D\cap D_1} - \dots - \sum_{D\cap D_k} + \dots + \sum_{D\cap D_i\cap D_{i'}} + \dots - \sum_{D\cap D_i\cap D_{i'}\cap D_{i''}} - \dots + (-1)^k \sum_{D\cap D_1\cap\dots\cap D_k}\Big\} f(n_1, \dots, n_k)$$

where $f(n_1, \dots, n_k)$ *is an arbitrary function and* $\sum_{D}$ *means that the sum is taken over the lattice-points of* D.

Proof. Any lattice-point $(n_1, \dots, n_k)$ of the region D lie in r of the regions $D_1, \dots, D_k$ where $r \geqslant 0$. The number of times the point is counted on the right-hand side of equation (3) is

$$1 - \binom{r}{1} + \binom{r}{2} - \dots + (-1)^r = (1-1)^r = \begin{cases} 0 & \text{if} \quad r \geqslant 1, \\ 1 & \text{if} \quad r = 0. \end{cases}$$

Hence those lattice-points of D which lie in none of the regions D_i, $i = 1, \dots, k$, i.e. the lattice points of $D \cap D_1' \cap \dots \cap D_k'$, are the only points which are counted once in the summation on the right-hand side of (3). Hence the lemma.

Lemma 2. *Let us assume that the conditions of Lemma 1 are satisfied and further that* D_i *is the transform of the region* D_j *for the permutation* σ_{ij} *of the coordinate variables* x_i, x_j $(1 \leqslant i \neq j \leqslant k)$, D *is invariant for all permutations* σ_{ij} *and* $f(x_1, \dots, x_k)$ *is a symmetric function of* $x_1, \dots, x_k$. *Then*

$$\Big\{\sum_{D} - \binom{k}{1} \sum_{D\cap D_1} + \binom{k}{2} \sum_{D\cap D_1\cap D_2} - \dots + (-1)^k \sum_{D\cap D_1\cap\dots\cap D_k}\Big\} f(n_1, \dots, n_k) = \sum_{D\cap D_1'\cap\dots\cap D_k'} f(n_1, \dots, n_k).$$

Proof. Lemma 2 is an immediate consequence of Lemma 1.

In the following lemmas, g is an arbitrary function.

Lemma 3. *We have*

$$(4)\qquad \sum_{r=0}^{t} (-1)^r \binom{t}{r} \sum_{\substack{n_i^k \leqslant x,\ n_1\dots n_t n_j^{k-t} \leqslant x\\ i=1,\dots,r,\ j=r+1,\dots,t}} g(n_1, n_2, \dots, n_t) = \begin{cases} g(1) & \textit{if} \quad t = 0, \\ 0 & \textit{if} \quad t \geqslant 1. \end{cases}$$

Proof. The inequalities $n_i^k \leqslant x$, $n_1 \dots n_t n_j^{k-t} \leqslant x$, $i = 1, \dots, r$, $j = r+1, \dots, t$ imply

$$(n_1 \dots n_r)^{k-t} \prod_{j=r+1}^{t} (n_1 \dots n_t n_j^{k-t}) \leqslant x^{\frac{r}{k}(k-t)+t-r},$$

i.e. $n_1 \dots n_t \leqslant x^{t/k}$ and this implies in turn $n_j^{k-t} n_1 \dots n_t \leqslant x$ for $j = 1, \dots, r$. Hence the left-hand side of equation (4) is equal to

$$\sum_{r=0}^{t} (-1)^r \binom{t}{r} \sum_{\substack{n_i^k \leqslant x,\ n_1\dots n_t n_j^{k-t} \leqslant x\\ i=1,\dots,r,\ j=1,\dots,t}} g(n_1 \dots n_t) = 0 \quad \text{if} \quad t \geqslant 1,$$

by Lemma 2, where we take D to be the region $n_1 \dots n_t n_j^{k-t} \leqslant x$, $j = 1, \dots, t$, and D_i to be the region $n_i^k \leqslant x$ for $i = 1, 2, \dots, t$; so that $D \cap D_1' \cap D_2' \cap \dots \cap D_t' = \Phi$. Hence the lemma.

Lemma 4.

$$(5)\qquad \sum_{r=0}^{t} (-1)^r \binom{t}{r} \sum_{\substack{n_1\dots n_r n_j^{k-r} \leqslant x\\ j=1,\dots,t}} g(n_1 \dots n_t) = \begin{cases} g(1) & \textit{if} \quad t = 0, \\ 0 & \textit{if} \quad t \geqslant 1. \end{cases}$$

Proof. When $t = 1$, the left side of (5) is

$$\sum_{n_1^k \leqslant x} g(n_1) - \sum_{n_1^k \leqslant x} g(n_1) = 0.$$

Hence (5) holds when $t = 1$. We shall now prove (5) by induction on t. Let (5) be true for values of $t < T$ $(T \geqslant 1)$. We shall then prove (5) for $t = T$.

Taking D to be the region $n_1 \dots n_r n_j^{k-r} \leqslant x$, $j = 1, \dots, T$ and D_i to be the regions $n_i^k \leqslant x$, $i = 1, \dots, r$, so that $D \cap D_1' \cap \dots \cap D_r' = \Phi$ we have by Lemma 2

$$(6)\qquad \sum_{s=0}^{r} (-1)^s \binom{r}{s} \sum_{\substack{n_i^k \leqslant x,\ n_1\dots n_r n_j^{k-r} \leqslant x\\ i=1,\dots,s,\ j=1,\dots,T}} g(n_1 \dots n_T) = 0 \qquad (r \geqslant 1)$$

and hence

$$(7)\qquad \sum_{r=1}^{T}(-1)^r\binom{T}{r}\sum_{\substack{n_1\dots n_r n_j^{k-r}\leqslant x\\ j=1,\dots,T}} g(n_1\dots n_T)$$

$$=\sum_{r=1}^{T}(-1)^r\binom{T}{r}\sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}\sum_{\substack{n_i^k\leqslant x,\ n_1\dots n_r n_j^{k-r}\leqslant x\\ j=1,\dots,T,\ i=1,\dots,s}} g(n_1\dots n_T)\qquad \text{(by (6))}$$

$$=\sum_{s=1}^{T}\binom{T}{s}\sum_{s'=0}^{T-s}(-1)^{s'-1}\binom{T-s}{s'}\sum_{\substack{n_i^k\leqslant x,\ n_1\dots n_{s+s'} n_j^{k-s-s'}\leqslant x\\ i=1,\dots,s,\ j=1,\dots,T}} g(n_1\dots n_T)$$

(putting $s+s'=r$).

Now the inequalities $n_i^k\leqslant x$, $i=1,\dots,s$; $n_1\dots n_{s+s'}n_j^{k-s-s'}\leqslant x$, $j=s+1,\dots,T$, imply

$$(n_1\dots n_s)^{k-s-s'}(n_1\dots n_{s+s'})^{s'}(n_{s+1}\dots n_{s'})^{k-s-s'}\leqslant x^{\frac{s(k-s-s')}{k}+s'},$$

i.e. $n_1\dots n_{s+s'}\leqslant x^{(s+s')/k}$ and this in turn implies

$$n_1\dots n_{s+s'}n_j^{k-s-s'}\leqslant x \quad \text{for} \quad j=1,\dots,s \text{ also}.$$

Hence

$$\sum_{\substack{n_i^k\leqslant x,\ n_1\dots n_{s+s'} n_j^{k-s-s'}\leqslant x\\ i=1,\dots,s,\ j=1,\dots,T}} g(n_1\dots n_T)=\sum_{\substack{n_i^k\leqslant x,\ n_1\dots n_{s+s'} n_j^{k-s-s'}\leqslant x\\ i=1,\dots,s,\ j=s+1,\dots,T}} g(n_1\dots n_T).$$

This relation together with (7) gives

$$\sum_{r=1}^{T}(-1)^r\binom{T}{r}\sum_{\substack{n_1\dots n_r n_j^{k-r}\leqslant x\\ j=1,\dots,T}} g(n_1\dots n_T)$$

$$=\sum_{s=1}^{T}\binom{T}{s}\sum_{\substack{n_i^k\leqslant x\\ i=1,\dots,s}}\Bigg\{\sum_{s'=0}^{T-s}(-1)^{s'-1}\binom{T-s}{s'}\sum_{\substack{n_{s+1}\dots n_{s+s'} n_j^{k-s-s'}\leqslant \frac{x}{n_1\dots n_s}\\ j=s+1,\dots,T}} g(n_1\dots n_T)\Bigg\}.$$

The inner sum inside the double bracket above is zero by the induction hypothesis if $T-s\geqslant 1$ and equal to $-g(n_1\dots n_s)$ if $T-s=0$. Hence

$$\sum_{r=1}^{T}(-1)^r\binom{T}{r}\sum_{\substack{n_1\dots n_r n_j^{k-r}\leqslant x\\ j=1,\dots,T}} g(n_1\dots n_T)=-\sum_{\substack{n_i^k\leqslant x\\ i=1,\dots,T}} g(n_1\dots n_T),$$

i.e.

$$\sum_{r=0}^{T}(-1)^r\binom{T}{r}\sum_{\substack{n_1\dots n_r n_j^{k-r}\leqslant x\\ j=1,\dots,T}} g(n_1\dots n_T)=0.$$

The proof of Lemma 4 is complete.

Proof of Theorem 1. Suppose (1) holds. Then if $0\leqslant k\leqslant K$,

$$\sum_{r=0}^{k}(-1)^r\binom{k}{r}\sum_{\substack{n_j^k\leqslant x\\ j=1,\dots,r}} F_{k-r}\left(\frac{x}{n_1\dots n_r}\right)$$

$$=\sum_{r=0}^{k}(-1)^r\binom{k}{r}\sum_{\substack{n_j^k\leqslant x\\ j=1,\dots,r}}\sum_{s=0}^{k-r}\binom{k-r}{s}\sum_{\substack{n_1\dots n_r n_{r+1}\dots n_{r+s} n_{r+j}^{k-r-s}\leqslant x\\ j=1,\dots,s}} f_{k-r-s}\left(\frac{x}{n_1\dots n_{r+s}}\right)$$

(by (1))

$$=\sum_{t=0}^{k}\binom{k}{t}\Bigg\{\sum_{r=0}^{t}(-1)^r\binom{t}{r}\sum_{\substack{n_i^k\leqslant x,\ n_1\dots n_t n_j^{k-t}\leqslant x\\ i=1,\dots,r,\ j=r+1,\dots,t}} f_{k-t}\left(\frac{x}{n_1\dots n_t}\right)\Bigg\}$$

(putting $r+s=t$)

$$=f_k(x)\qquad \text{(by Lemma 3)}.$$

Hence (2) holds.

Conversely, suppose (2) holds. Hence if $0\leqslant k\leqslant K$, then

$$\sum_{r=0}^{k}\binom{k}{r}\sum_{\substack{n_1\dots n_r n_j^{k-r}\leqslant x\\ j=1,\dots,r}} f_{k-r}\left(\frac{x}{n_1\dots n_r}\right)$$

$$=\sum_{r=0}^{k}\binom{k}{r}\sum_{\substack{n_1\dots n_r n_j^{k-r}\leqslant x\\ j=1,\dots,r}}\sum_{s=0}^{k-r}(-1)^s\binom{k-r}{s}\sum_{\substack{n_1\dots n_r n_j^{k-r}\leqslant x\\ j=r+1,\dots,r+s}} F_{k-r-s}\left(\frac{x}{n_1\dots n_{r+s}}\right)$$

(by (2))

$$=\sum_{t=0}^{k}(-1)^t\binom{k}{t}\Bigg\{\sum_{r=0}^{t}(-1)^r\binom{t}{r}\sum_{\substack{n_1\dots n_r n_j^{k-r}\leqslant x\\ j=1,\dots,t}} F_{k-t}\left(\frac{x}{n_1\dots n_r}\right)\Bigg\}$$

(putting $r+s=t$)

$$=F_k(x)\qquad \text{(by Lemma 4)}.$$

Hence (1) holds and hence the theorem.

2. Defining the functions $\psi_n(x)$ by the equations

$$\psi_n(x) = -\sum_{\nu=-\infty}^{\infty}{}' \frac{e^{2\pi i\nu x}}{(2\pi i\nu)^n} \quad \text{if} \quad n \geqslant 1, \quad \psi_0(x) = 1, \tag{8}$$

where the dash denotes that the term corresponding to $\nu = 0$ is omitted, we have the well-known relations

$$\psi_1(x) = x - [x] - \tfrac{1}{2} \tag{9}$$

whenever x is not an integer, $[x]$ being the largest integer $\leqslant x$.

$$\psi_n'(x) = \psi_{n-1}(x) \quad \text{if} \quad n \geqslant 2. \tag{10}$$

$$\sum_{n=0}^{\infty} \psi_n(x) u^n = \frac{u e^{u\psi_1(x)}}{e^{u/2} - e^{-u/2}}. \tag{11}$$

We have from (11),

$$\Big\{\sum_{n=0}^{\infty} \psi_n(x) u^n\Big\} = \Big\{\sum_{n=0}^{\infty} \psi_n(\tfrac{1}{2}) u^n\Big\} e^{u\psi_1(x)};$$

comparing coefficients of u^n on both sides, we have

$$\psi_n(x) = \sum_{r=0}^{n} \psi_r(\tfrac{1}{2}) \frac{\psi_1^{n-r}(x)}{(n-r)!}. \tag{12}$$

It is the purpose of this section to prove the following two lemmas:

LEMMA 5.

$$\Big\{x^{1/k} + u\psi_1(x^{1/k}) - u(u+1)\int_1^\infty \psi_1(yx^{1/k})y^{-u-2}dy\Big\}^r -$$

$$- \sum_{0\leqslant\lambda\leqslant m-1} x^{(r-\lambda)/k} \text{coeff of } v^\lambda \text{ in } \Big\{\sum_{n=0}^{\lambda} v^n \psi_n(x^{1/k}) u(u+1)\dots(u+n-1)\Big\}^r$$

$$= O(x^{(r-m)/k}) \quad \textit{if} \quad u > -2 \text{ [1]},$$

or alternatively, in case $|u| < 1$,

$$= 2\nu x^{(r-m)/k} \text{ coeff of } v^{m-1} \text{ in } \Big\{\sum_{n=0}^{m} v^n a_{n+1} \frac{|u(u+1)\dots(u+n)|}{1-|u|/(n+1)}\Big\}\times$$

$$\times \sum_{0\leqslant s\leqslant r-1} \Big\{\sum_{n=0}^{m-1} v^n a_n |u(u+1)\dots(u+n-1)|\Big\}^{r-s-1} (1-|u|)^{-s}$$

where ν *(not necessarily the same at each occurrence) is independent of* u *and is such that* $|\nu| \leqslant 1$, $x \geqslant 1$, m *is an arbitrary integer* $\geqslant 1$, *and* $a_m = \operatorname{Max}|\psi_m(x)|$.

(1) The constant in O-term here, in general, depends on u also.

LEMMA 6.

$$\text{Coeff of } v^\lambda \text{ in } \Big\{\sum_{n=0}^{\lambda} v^n \psi_n(x) u(u+1)\dots(u+n-1)\Big\}^r$$

$$= -\frac{1}{ur+\lambda} \sum_{s=0}^{\lambda} \{1+(-1)^s\}\psi_{\lambda-s}(x) \binom{ur+\lambda}{\lambda-s} (\lambda-s)! \times$$

$$\times \text{coeff of } v^{s+1} \text{ in } \Big\{\sum_{n=0}^{s+1} v^n \psi_n(0) u(u+1)\dots(u+n-1)\Big\}^r.$$

Proof of Lemma 5. We have

$$1 + ux^{-1/k}\psi_1(x^{1/k}) - u(u+1)x^{-1/k}\int_1^\infty \psi_1(yx^{1/k})y^{-u-2}dy \tag{13}$$

$$= \sum_{n=0}^{m-1} x^{-n/k}\psi_n(x^{1/k}) u(u+1)\dots(u+n-1) -$$

$$- u(u+1)\dots(u+m-1)x^{-m/k}\int_1^\infty \psi_{m-1}(yx^{1/k})y^{-u-m}d(yx^{1/k})$$

by successive integration by parts. Now,

$$-\int_1^\infty \psi_{m-1}(yx^{1/k})y^{-u-m}d(yx^{1/k})$$

$$= \psi_m(x^{1/k}) + \int_1^\infty \psi_m(yx^{1/k})dy^{-u-m}$$

$$= \psi_m(x^{1/k}) + \int_1^\infty \psi_m(yx^{1/k}) \sum_{s=0}^{\infty} \frac{(-u)^s}{s!}\Big\{\frac{s(\log y)^{s-1}}{y^{m+1}} - \frac{m(\log y)^s}{y^{m+1}}\Big\}dy$$

$$= \psi_m(x^{1/k}) + \sum_{s=0}^{\infty} \frac{(-u)^s}{s!}\int_1^\infty \psi_m(yx^{1/k})\Big\{\frac{s(\log y)^{s-1}}{y^{m+1}} - \frac{m(\log y)^s}{y^{m+1}}\Big\}dy$$

and

$$\Big|\int_1^\infty \frac{\psi_m(yx^{1/k})(\log y)^s}{s!\,y^{m+1}}dy\Big| \leqslant a_m \int_1^\infty \frac{(\log y)^s}{s!} y^{-m-1}dy$$

$$= \frac{a_m}{s!}\int_0^\infty e^{-my}y^s dy = \frac{a_m}{m^{s+1}}.$$

So

$$\left|\int_1^\infty \psi_{m-1}(yx^{1/k})\,y^{-u-m}\,d(yx^{1/k})\right| = \left|\psi_m(x^{1/k}) + \int_1^\infty \psi_m(yx^{1/k})\,dy^{-u-m}\right|$$

$$\leqslant a_m + a_m + \sum_{s=1}^\infty \frac{|u|^s}{s!}\left(\frac{s!\,2a_m}{m^s}\right)$$

$$= 2a_m\left(1-\frac{|u|}{m}\right)^{-1} \quad \text{if} \quad m\geqslant 1,\ |u|<1.$$

Hence for every integer $m\geqslant 1$

(14) $$1+ux^{-1/k}\psi_1(x^{1/k}) - u(u+1)x^{-1/k}\int_1^\infty \psi_1(yx^{1/k})\,y^{-u-2}\,dy$$

$$=\sum_{n=0}^{m-1} x^{-n/k}\psi_n(x^{1/k})\,u(u+1)\dots(u+n-1) + 2\nu x^{-m/k}a_m\frac{|u(u+1)\dots(u+m-1)|}{1-|u|/m}$$

$$=\sum_{n=0}^m a_n^{(m)}x^{-n/k}$$

where

(15) $$a_n^{(m)} = a_n = \psi_n(x^{1/k})\,u(u+1)\dots(u+n-1) \quad \text{if} \quad n=0,1,\dots,m-1,$$
$$a_m^{(m)} = 2\nu a_m|u(u+1)\dots(u+m-1)|\left(1-\frac{|u|}{m}\right)^{-1} \quad \text{for} \quad m\geqslant 1.$$

Taking $m=1$, in the above, since $a_1=\frac{1}{2}$ and $x\geqslant 1$,

$$1+ux^{-1/k}\psi_1(x^{1/k})-u(u+1)x^{-1/k}\int_1^\infty\psi_1(yx^{1/k})\,y^{-u-2}\,dy$$
$$= 1+\frac{\nu|u|x^{-1/k}}{1-|u|}=\frac{\nu}{1-|u|}.$$

Hence (13), (14), (15) hold for $m=0$, where now we define $a_0^{(0)}$ by

(16) $$a_0^{(0)} = \nu/(1-|u|).$$

So

$$\left\{1+ux^{-1/k}\psi_1(x^{1/k})-u(u+1)x^{-1/k}\int_1^\infty\psi_1(yx^{1/k})\,y^{-u-2}\,dy\right\}^r$$

$$=\sum_{0\leqslant n_1\leqslant m} x^{-n_1/k}a_{n_1}^{(m)}\sum_{0\leqslant n_2\leqslant m-n_1}x^{-n_2/k}a_{n_2}^{(m-n_1)}\dots\sum_{0\leqslant n_r\leqslant m-(n_1+\dots+n_{r-1})}x^{-n_r/k}a_{n_r}^{(m-\overline{n_1+\dots+n_{r-1}})}$$

(for every integer $m\geqslant 0$)

$$=\sum_{0\leqslant\lambda\leqslant m-1}x^{-\lambda/k}\sum_{n_1+\dots+n_r=\lambda}\prod_{s=1}^r a_{n_s}+$$

$$+2\nu x^{-m/k}\sum_{\substack{n_1+\dots+n_{s+1}=m\\0\leqslant s\leqslant r-1,\ n_{s+1}\geqslant 1}}(1-|u|)^{-(r-s-1)}a_{n_{s+1}}\frac{|u(u+1)\dots(u+n_{s+1}-1)|}{1-|u|/n_{s+1}}\prod_{i=1}^s|a_{n_i}|$$

$$=\sum_{0\leqslant\lambda\leqslant m-1}x^{-\lambda/k}\,\text{coeff of }v^\lambda\text{ in }\left\{\sum_{n=0}^\lambda v^n\psi_n(x^{1/k})\,u(u+1)\dots(u+n-1)\right\}^r+$$

$$+2\nu x^{-m/k}\sum_{\substack{0\leqslant s\leqslant r-1\\1\leqslant n\leqslant m}}(1-|u|)^{-(r-s-1)}\frac{a_n|u(u+1)\dots(u+n-1)|}{1-|u|/n}\times$$

$$\times\ \text{coeff of }v^{m-n}\text{ in }\left\{\sum_{j=0}^{m-n}a_jv^j|u(u+1)\dots(u+j-1)|\right\}^s.$$

The second statement of Lemma 5, when $|u|<1$, now follows. To prove the first statement, we observe that, if $u>-2$, and $m\geqslant 2$

$$\left|\int_1^\infty\psi_{m-1}(yx^{1/k})\,y^{-u-m}\,d(yx^{1/k})\right| = \left|\psi_m(x^{1/k})+\int_1^\infty\psi_m(yx^{1/k})\,dy^{-u-m}\right|$$

$$\leqslant a_m+a_m\left|\int_1^\infty dy^{-u-m}\right| = 2a_m.$$

And so, if $m\geqslant 2$, we have from (13)

$$1+ux^{-1/k}\psi_1(x^{1/k})-u(u+1)x^{-1/k}\int_1^\infty\psi_1(yx^{1/k})\,y^{-u-2}\,dy$$
$$= \sum_{n=0}^{m-1}x^{-n/k}\psi_n(x^{1/k})\,u(u+1)\dots(u+n-1)+O(x^{-m/k})$$

and because $x\geqslant 1$, the above result is true trivially for $m=0,1$ also. If we carry out the above arguments starting from this result instead of (14), we have Lemma 5.

Proof of Lemma 6. We have

(17) $$g_\lambda(x,u,r)=g_\lambda(x)$$

$$=\text{coeff of }v^\lambda\text{ in }\left\{\sum_{n=0}^\lambda v^n\psi_n(x)\,u(u+1)\dots(u+n-1)\right\}^r$$

$$=\text{coeff of }v^\lambda\text{ in }\left\{\sum_{0\leqslant s\leqslant n\leqslant\lambda}v^n\psi_s(\tfrac{1}{2})\frac{\psi_1^{n-s}(x)}{(n-s)!}\,u(u+1)\dots(u+n-1)\right\}^r$$

(by (12))

$$= \text{coeff of } v^\lambda \text{ in } \Big\{\sum_{0\leqslant s\leqslant\lambda} v^s\psi_s(\tfrac12)\,u(u+1)\dots(u+s-1)\overline{1-v\psi_1(x)}^{-(u+s)}\Big\}^r$$

$$= \text{coeff of } v^\lambda \text{ in } \{1-v\psi_1(x)\}^{-ur}\times$$

$$\times\sum_{0\leqslant n\leqslant\lambda}\Big\{\frac{v}{1-v\psi_1(x)}\Big\}^n\sum_{s_1+\dots+s_r=n}\prod_{i=1}^{r}\psi_{s_i}(\tfrac12)\,u\dots(u+s_i-1)$$

$$=\sum_{0\leqslant n\leqslant\lambda}\psi_1^{\lambda-n}(x)\frac{(ur+n)(ur+n+1)\dots(ur+\lambda-1)}{(\lambda-n)!}\times$$

$$\times\sum_{s_1+\dots+s_r=n}\prod_{i=1}^{r}\psi_{s_i}(\tfrac12)\,u(u+1)\dots(u+s_i-1)$$

$$= \text{coeff of } v^\lambda \text{ in } \Big\{\sum_{s=0}^{\lambda} v^s\psi_s(\tfrac12)\,u(u+1)\dots(u+s-1)\Big\}^r\sum_{n=0}^{\lambda}\binom{ur+\lambda-1}{\lambda-n}\{v\psi_1(x)\}^{\lambda-n}$$

$$=\sum_{n=0}^{\lambda}\binom{ur+\lambda-1}{\lambda-n}\psi_1^{\lambda-n}(x)\,g_n(\tfrac12)\,.$$

Hence

$$G(x,v,u,r)=G(x,v)=\sum_{\lambda=0}^{\infty}\frac{g_\lambda(x)v^\lambda}{ur(ur+1)\dots(ur+\lambda-1)}$$

$$=\Big\{\sum_{n=0}^{\infty}\frac{g_n(\frac12)v^n}{ur(ur+1)\dots(ur+n-1)}\Big\}e^{v\psi_1(x)}\,.$$

So we have

(18) $$G(x,v)=G(\tfrac12,v)\,e^{v\psi_1(x)}$$

(19) $$=G(\tfrac12,v)\frac{e^{v/2}-e^{-v/2}}{v}\Big(\sum_{n=0}^{\infty}v^n\psi_n(x)\Big)\qquad\text{(by (11))}\,.$$

Again from (11) it follows that $\psi_n(\frac12)=0$ if n is odd, and so $g_\lambda(\frac12)=0$ if λ is odd.

Hence

$$G(\tfrac12,v)=G(\tfrac12,-v)\,.$$

From (18),

$$G(0,v)=G(\tfrac12,v)e^{-v/2}\,,$$

and so

$$G(0,-v)=G(\tfrac12,v)e^{v/2}\,.$$

Hence from (18) we get

$$G(x,v)=\frac{G(0,-v)-G(0,v)}{v}\Big\{\sum_{n=0}^{\infty}v^n\psi_n(x)\Big\}$$

$$=-\Big\{\sum_{\lambda=0}^{\infty}\frac{g_\lambda(0)v^{\lambda-1}\overline{1+(-1)^{\lambda-1}}}{ur(ur+1)\dots(ur+\lambda-1)}\Big\}\Big\{\sum_{n=0}^{\infty}v^n\psi_n(x)\Big\}.$$

Comparing the coefficients of v^λ on both sides, we have

$$g_\lambda(x)=-\frac{1}{ur+\lambda}\sum_{n=0}^{\lambda}\psi_{\lambda-n}(x)\{1+(-1)^n\}\binom{ur+\lambda}{\lambda-n}(\lambda-n)!\,g_{n+1}(0)\,,$$

thus proving Lemma 6.

3. THEOREM 2. *If in Theorem* 1

$$F_k(x)=\frac{1}{\varrho}\binom{k}{r}\sum_{s=0}^{k-r}(-1)^s\binom{k-r}{s}u_s^{-(k-r-1)}x^{1+u_s}\times$$

$$\times\Big\{1+\tfrac12u_s-u_s(u_s+1)\int_1^\infty\psi_1(y)y^{-u_s-2}dy\Big\}^{k-r-s}$$

where

$$u_s=\frac{\varrho r}{r+s},\quad \varrho r\neq0\,,\quad \varrho>-2\,,\quad k\geqslant r\geqslant0\,,\quad \textit{and}\quad F_k(x)=0\ \textit{ if }\ 0\leqslant k<r\,,$$

then

$$-f_k(x)=$$

$$=x^{r(1+\varrho)/k}\sum_{p=0}^{k-r}\{1+(-1)^p\}k^{k-r-p-1}\psi_{k-r-p}(x^{1/k})\ \text{coeff of } u^{p+1}\text{ in }\Big(\frac{u}{e^u-1}\Big)^k+$$

$$+\frac{1}{\varrho}\binom{k}{r}\sum_{1\leqslant\lambda\leqslant r+m-k-1}x^{(r(1+\varrho)-\lambda)/k}\sum_{p=0}^{\lambda+k-r}\{1+(-1)^p\}\psi_{\lambda+k-r-p}(x^{1/k})\times$$

$$\times\Big\{\sum_{s=0}^{k-r}(-1)^s\binom{k-r}{s}u_s^{-(k-r-1)}\binom{u_s\overline{k-r-s}+\lambda+k-r}{\lambda+k-r-p}\times$$

$$\times\frac{(\lambda+k-r-p)!}{u_s\overline{k-r-s}+\lambda+k-r}\,g_{p+1}(0,u_s,k-r-s)\Big\}+O\big(x^{(r\varrho+k-m)/k}\big)$$

where $g_\lambda(x,u,r)$ *is defined as in* (17) *and* m *is an arbitrary integer* $\geqslant1$.

Proof. We have

$$f_k(x)=\sum_{s=0}^{k}(-1)^s\binom{k}{s}\sum_{\substack{n_j^k\leqslant x\\ j=1,\dots,s}}F_{k-s}\left(\frac{x}{n_1\dots n_s}\right)$$

$$=\sum_{s=0}^{k}(-1)^s\binom{k}{s}\frac{1}{\varrho}\binom{k-s}{r}\sum_{p=0}^{k-r-s}(-1)^p\binom{k-r-s}{p}u_p^{-(k-r-s-1)}x^{1+u_p}\times$$

$$\times\Big\{1+\tfrac12 u_p-u_p(u_p+1)\int_1^\infty\psi_1(y)y^{-u_p-2}dy\Big\}^{k-r-s-p}\Big(\sum_{n\leqslant x^{1/k}}n^{-1-u_p}\Big)^s$$

$$=\frac{1}{\varrho}\binom{k}{r}\sum_{p=0}^{k-r}(-1)^p\binom{k-r}{p}u_p^{-(k-r-1)}x^{1+u_p}\Big\{1+\tfrac12 u_p-u_p(u_p+1)\int_1^\infty\psi_1(y)y^{-u_p-2}dy-$$

$$-u_p\sum_{n\leqslant x^{1/k}}n^{-1-u_p}\Big\}^{k-r-p}$$

$$=\frac{1}{\varrho}\binom{k}{r}\sum_{p=0}^{k-r}(-1)^p\binom{k-r}{p}u_p^{-(k-r-1)}x^{(1+u_p)(r+p)/k}\times$$

$$\times\Big\{x^{1/k}+u_p\psi_1(x^{1/k})-u_p(u_p+1)\int_1^\infty\psi_1(yx^{1/k})y^{-u_p-2}dy\Big\}^{k-r-p}$$

(by Euler's summation formula (cf. p. 25 of [2]) applied to $\sum_{n\leqslant x^{1/k}}n^{-1-u_p}$)

$$(20)\qquad =\frac{1}{\varrho}\binom{k}{r}\sum_{p=0}^{k-r}(-1)^p\binom{k-r}{p}u_p^{-(k-r-1)}\sum_{0\leqslant\lambda\leqslant m-1}x^{(k+\varrho r-\lambda)/k}\times$$

$$\times\text{ coeff of } v^\lambda \text{ in }\Big\{\sum_{n=0}^{\lambda}v^n\psi_n(x^{1/k})u_p\dots(u_p+n-1)\Big\}^{k-r-p}+O\big(x^{(k+\varrho r-m)/k}\big)$$

by the first statement of Lemma 5, for every integer $m\geqslant 1$. Now

$$\sum_{0\leqslant\lambda\leqslant k-r}x^{(k+\varrho r-\lambda)/k}\times$$

$$\times\text{coeff of } v^\lambda\text{ in }\sum_{p=0}^{k-r}(-1)^p\binom{k-r}{p}u_p^{-(k-r-1)}\Big\{\sum_{n=0}^{\lambda}v^n\psi_n(x^{1/k})u_p\dots(u_p+n-1)\Big\}^{k-r-p}$$

$$=\sum_{0\leqslant\lambda\leqslant k-r}x^{(k+\varrho r-\lambda)/k}\text{ coeff of } v^\lambda\text{ in }\sum_{p=0}^{k-r}(-1)^p\binom{k-r}{p}\sum_{s=0}^{\lambda}\Big(\frac{r+p}{\varrho r}\Big)^{k-r-s-1}\times$$

$$\times\text{ coeff of } u^s\text{ in }\Big\{\sum_{n=0}^{\lambda}v^n\psi_n(x^{1/k})u\dots(u+n-1)\Big\}^{k-r-p}$$

$$=\sum_{\substack{0\leqslant\lambda\leqslant k-r\\ 0\leqslant s\leqslant\min(\lambda,k-r-1)}}x^{(k+\varrho r-\lambda)/k}\times$$

$$\times\text{coeff of }\frac{v^\lambda u^s w^{k-r-s-1}}{(k-r-s-1)!}\text{ in } e^{w/\varrho}\Big\{\sum_{n=0}^{\lambda}v^n\psi_n(x^{1/k})u\dots(u+n-1)-e^{w/\varrho r}\Big\}^{k-r}+$$

$$+x^{r(1+\varrho)/k}\text{ coeff of }(uv)^{k-r}\text{ in }\times$$

$$\times\sum_{p=0}^{k-r}(-1)^p\binom{k-r}{p}\frac{\varrho r}{r+p}\Big\{\sum_{n=0}^{k-r}v^n\psi_n(x^{1/k})u\dots(u+n-1)\Big\}^{k-r-p}$$

$$=x^{r(1+\varrho)/k}\text{ coeff of } u^{k-r}\text{ in }\sum_{p=0}^{k-r}(-1)^p\binom{k-r}{p}\frac{\varrho r}{r+p}\Big\{\sum_{n=0}^{\infty}\psi_n(x^{1/k})u^n\Big\}^{k-r-p}.$$

(The λ, s-sum in the previous expression vanishing, since each term in the expansion of $\big\{\sum_{n=0}^{\lambda}v^n\psi_n(x^{1/k})u\dots(u+n-1)-e^{w/\varrho r}\big\}^{k-r}$ is of combined degree *at least* $k-r$ in u and w while the required coefficient is of combined degree $k-r-1$ in u and w.)

$$(21)\qquad =r\varrho x^{r(1+\varrho)/k}\text{ coeff of } u^{k-r}\text{ in }\int_0^1\Big(\frac{ue^{u\psi_1(x^{1/k})}}{e^{u/2}-e^{-u/2}}-y\Big)^{k-r}y^{r-1}dy$$

$$=\frac{\varrho}{\binom{k}{r}}x^{r(1+\varrho)/k}\text{ coeff of } u^{k-r}\text{ in } I_r\,,\ \text{ say}\,,$$

where

coeff of u^{k-s} in I_r

$$=\text{coeff of } u^{k-s}\text{ in } r\binom{k}{r}\int_0^1(a-y)^{k-r}y^{r-1}dy\,,\qquad a=\frac{ue^{u\psi_1(x^{1/k})}}{e^{u/2}-e^{-u/2}}$$

$$=\text{coeff of } u^{k-s}\text{ in }\Big\{-r\binom{k}{r}\frac{(a-1)^{k-r+1}}{(k-r+1)}+\frac{r(r-1)}{k-r+1}\binom{k}{r}\int_0^1(a-y)^{k-r+1}y^{r-2}dy\Big\}$$

$$\text{if}\quad r\geqslant 2\,,$$

$$=\text{coeff of } u^{k-s}\text{ in }\Big\{(r-1)\binom{k}{r-1}\int_0^1(a-y)^{k-r+1}y^{r-2}dy=I_{r-1}\Big\}\quad\text{if}\quad r\leqslant s\,.$$

Hence we have, if $2\leqslant r\leqslant s$,

$$\text{coeff of } u^{k-s}\text{ in } I_r=\text{coeff of } u^{k-s}\text{ in }\{I_1=a^k-(a-1)^k\}\,.$$

So

$$\text{(22)} \qquad \text{coeff of } u^{k-r} \text{ in } I_r = \text{coeff of } u^{k-r} \text{ in } \left(\frac{u e^{u\psi_1(x^{1/k})}}{e^{u/2}-e^{-u/2}}\right)^k \quad \text{since} \quad r \geqslant 1 .$$

Using (21) and (22) in (20) we have

$$\text{(23)} \qquad f_k(x) = x^{r(1+\varrho)/k} \text{ coeff of } u^{k-r} \text{ in } \left(\frac{u e^{u\psi_1(x^{1/k})}}{e^{u/2}-e^{-u/2}}\right)^k +$$

$$+\frac{1}{\varrho}\binom{k}{r}\sum_{p=0}^{k-r}(-1)^p\binom{k-r}{p}u_p^{-(k-r-1)}\sum_{1\leqslant\lambda\leqslant r+m-k-1} x^{(r(1+\varrho)-\lambda)/k} g_{\lambda+k-r}(x^{1/k}, u_p, k-r-p)+$$

$$+O(x^{(r\varrho+k-m)/k}) .$$

Now,

$$\text{(24)} \qquad \text{coeff of } u^{k-r} \text{ in } \left(\frac{u e^{u\psi_1(x^{1/k})}}{e^{u/2}-e^{-u/2}}\right)^k$$

$$= -\text{coeff of } u^{k-r} \text{ in } \left\{\sum_{n=0}^{\infty}(uk)^n\psi_n(x^{1/k})\right\}\frac{1}{uk}\left\{\left(\frac{u}{e^u-1}\right)^k - \left(\frac{-u}{e^{-u}-1}\right)^k\right\}$$

$$= -\sum_{p=0}^{k-r}\{1+(-1)^p\}k^{k-r-p-1}\psi_{k-r-p}(x^{1/k}) \text{ coeff of } u^{p+1} \text{ in } \left(\frac{u}{e^u-1}\right)^k .$$

Theorem 2 now follows from (23), (24) and Lemma 6.

THEOREM 3. *If in Theorem* 1,

$$F_k(x) = \text{coeff of } u^{k-r} \text{ in } x^{1+u}\left\{1+\tfrac{1}{2}u-u(u+1)\int_1^{\infty}\psi_1(y)\,y^{-u-2}\,dy\right\}^k$$

where $r \geqslant 0$, *and* $|u| < 1$, *then*

$$-f_k(x) = x^{r/k}\sum_{p=0}^{k-r}\{1+(-1)^p\}k^{k-r-p-1}\psi_{k-r-p}(x^{1/k}) \times$$

$$\times \text{ coeff of } u^{p+1} \text{ in } \left(\frac{u}{e^u-1}\right)^k +$$

$$+\sum_{1\leqslant\lambda\leqslant r+m-k-1} x^{(r-\lambda)/k}\sum_{p=0}^{\lambda+k-r}\{1+(-1)^p\}\psi_{\lambda+k-r-p}(x^{1/k}) \times$$

$$\times \text{ coeff of } u^{k-r} \text{ in } \binom{uk+\lambda+k-r}{\lambda+k-r-p}\frac{(\lambda+k-r-p)!}{uk+\lambda+k-r}\, g_{p+1}(0, u, k)+$$

$$+O(x^{(k-m)/k})$$

for arbitrary integer $m \geqslant 1$.

Proof. We have

$$f_k(x) = \sum_{s=0}^{k}(-1)^s\binom{k}{s}\sum_{\substack{n_j^k\leqslant x\\ j=1,\ldots,s}} F_{k-s}\left(\frac{x}{n_1\ldots n_s}\right)$$

$$= \text{coeff of } u^{k-r} \text{ in} \times$$

$$\times\sum_{s=0}^{k}(-1)^s\binom{k}{s}u^s x^{1+u}\Big(\sum_{n\leqslant x^{1/k}} n^{-1-u}\Big)^s\Big(1+\tfrac{1}{2}u-u(u+1)\int_1^{\infty}\psi_1(y)\,y^{-u-2}dy\Big)^{k-s}$$

$$= \text{coeff of } u^{k-r} \text{ in } x^{1+u}\Big(1+\tfrac{1}{2}u-u(u+1)\int_1^{\infty}\psi_1(y)\,y^{-u-2}dy-u\sum_{n\leqslant x^{1/k}} n^{-1-u}\Big)^k$$

$$= \text{coeff of } u^{k-r} \text{ in } \Big\{x^{1/k}+u\psi_1(x^{1/k})-u(u+1)\int_1^{\infty}\psi_1(yx^{1/k})\,y^{-u-2}dy\Big\}^k$$

$$= \sum_{0\leqslant\lambda\leqslant m-1} x^{(k-\lambda)/k} \text{ coeff of } u^{k-r}v^{\lambda} \text{ in}\Big\{\sum_{n=0}^{\lambda}v^n\psi_n(x^{1/k})\,u(u+1)\ldots(u+n-1)\Big\}^k+$$

$$+O(x^{(k-m)/k})$$

by the second statement of Lemma 5, provided $|u| < 1$. Now

$$\text{(26)} \qquad \sum_{0\leqslant\lambda\leqslant k-r} x^{(k-\lambda)/k} \text{ coeff of } u^{k-r}v^{\lambda} \text{ in } \Big\{\sum_{n=0}^{\lambda}v^n\psi_n(x^{1/k})\,u(u+1)\ldots(u+n-1)\Big\}^k$$

$$= x^{r/k} \text{ coeff of } (uv)^{k-r} \text{ in } \Big\{\sum_{n=0}^{k-r}v^n\psi_n(x^{1/k})\,u(u+1)\ldots(u+n-1)\Big\}^k$$

$$= x^{r/k} \text{ coeff of } u^{k-r} \text{ in } \left(\frac{u e^{u\psi_1(x^{1/k})}}{e^{u/2}-e^{-u/2}}\right)^k .$$

Theorem 3 now follows from (24), (25), (26) and Lemma 6.

THEOREM 4. *If in Theorem* 1,

$$F_k(x) = \text{coeff of } u^{k-1} \text{ in } \frac{x^{1+u}}{u-\varrho}\Big\{1+\tfrac{1}{2}u-u(u+1)\int_1^{\infty}\psi_1(y)\,y^{-u-2}dy\Big\}^k+$$

$$+\frac{x^{1+\varrho}}{\varrho^k}\Big\{1+\tfrac{1}{2}\varrho-\varrho(\varrho+1)\int_1^{\infty}\psi_1(y)\,y^{-\varrho-2}dy\Big\}^k$$

where

$$\varrho \neq 0 , \qquad \varrho > -2 \qquad \textit{and} \qquad |u| < \min(|\varrho|, 1) ,$$

then, for $k \geqslant 1$ and arbitrary integer $m \geqslant 1$,

$$-f_k(x) = \sum_{p=0}^{k} \{1+(-1)^p\} k^{k-p-1} \psi_{k-p}(x^{1/k}) \text{ coeff of } u^{p+1} \text{ in } \left(\frac{u}{e^u-1}\right)^k +$$

$$+ \sum_{1\leqslant\lambda\leqslant m-k-1} x^{-\lambda/k} \sum_{p=0}^{\lambda+k} \{1+(-1)^p\} \psi_{\lambda+k-p}(x^{1/k}) \times$$

$$\times \Big\{\text{coeff of } u^{k-1} \text{ in } \binom{uk+k+\lambda}{k+\lambda-p} \frac{(k+\lambda-p)!}{uk+k+\lambda} g_{p+1}(0, u, k) +$$

$$+ \frac{1}{\varrho^k} \binom{\varrho k+k+\lambda}{k+\lambda-p} \frac{(k+\lambda-p)!}{\varrho k+k+\lambda} g_{p+1}(0, \varrho, k)\Big\} +$$

$$+ O(x^{(k-m)/k}) .$$

Proof. We have, if $k \geqslant 1$,

$$(27) \qquad f_k(x) = \sum_{s=0}^{k} (-1)^s \binom{k}{s} \sum_{\substack{n_j^k \leqslant x \\ j=1,\ldots,s}} F_{k-s}\left(\frac{x}{n_1 \ldots n_s}\right)$$

$$= \text{coeff of } u^{k-1} \text{ in } \frac{x^{1+u}}{u-\varrho} \left(1+\tfrac{1}{2}u - u(u+1) \int_1^\infty \psi_1(y) y^{-u-2} dy - u \sum_{n\leqslant x^{1/k}} n^{-1-u}\right)^k +$$

$$+ \frac{x^{1+\varrho}}{\varrho^k} \left\{1+\tfrac{1}{2}\varrho - \varrho(\varrho+1) \int_1^\infty \psi_1(y) y^{-\varrho-2} dy - \varrho \sum_{n\leqslant x^{1/k}} n^{-1-\varrho}\right\}^k$$

$$= \text{coeff of } u^{k-1} \text{ in } \frac{1}{u-\varrho} \left\{x^{1/k} + u\psi_1(x^{1/k}) - u(u+1) \int_1^\infty \psi_1(yx^{1/k}) y^{-u-2} dy\right\}^k$$

$$+ \frac{1}{\varrho^k} \left\{x^{1/k} + \varrho\psi_1(x^{1/k}) - \varrho(\varrho+1) \int_1^\infty \psi_1(yx^{1/k}) y^{-\varrho-2} dy\right\}^k$$

$$= \text{coeff of } u^{k-1} \text{ in } \sum_{0\leqslant\lambda\leqslant m-1} x^{(k-\lambda)/k} \text{ coeff of } v^\lambda \text{ in } \frac{1}{u-\varrho} \times$$

$$\times \left[\left\{\sum_{n=0}^{\lambda} v^n \psi_n(x^{1/k}) u \ldots (u+n-1)\right\}^k - \left\{\sum_{n=0}^{\lambda} v^n \psi_n(x^{1/k}) \varrho \ldots (\varrho+n-1)\right\}^k\right] +$$

$$+ O(x^{(k-m)/k})$$

by the second statement of Lemma 5, since $|u| < 1$.

Now, because $u-\varrho$ divides the expression in the square bracket above, if $\lambda < k$, the coeff of v^λ in $\frac{1}{u-\varrho} \times$ square bracket term is of degree

at most $k-2$ in u, and so the corresponding terms of the sum for $\lambda < k$ vanish. So

(28)

$$\sum_{0\leqslant\lambda\leqslant k} x^{(k-\lambda)/k} \text{ coeff of } u^{k-1} v^\lambda \text{ in } \frac{1}{u-\varrho} \left[\left\{\sum_{n=0}^{\lambda} v^n \psi_n(x^{1/k}) u(u+1) \ldots (u+n-1)\right\}^k - \right.$$

$$\left. - \left\{\sum_{n=0}^{\lambda} v^n \psi_n(x^{1/k}) \varrho(\varrho+1) \ldots (\varrho+n-1)\right\}^k\right]$$

$$= \text{coeff of } u^{k-1} v^k \text{ in } \frac{1}{u-\varrho} \left[\left\{\sum_{n=0}^{k} v^n \psi_n(x^{1/k}) u(u+1) \ldots (u+n-1)\right\}^k - \right.$$

$$\left. - \left\{\sum_{n=0}^{k} v^n \psi_n(x^{1/k}) \varrho(\varrho+1) \ldots (\varrho+n-1)\right\}^k\right]$$

$$= \text{coeff of } u^{k-1} \text{ in } \frac{1}{u-\varrho v} \left[\left\{\sum_{n=0}^{k} \psi_n(x^{1/k}) u(u+v) \ldots (u+\overline{n-1}v)\right\}^k - \right.$$

$$\left. - \left\{\sum_{n=0}^{\lambda} \psi_n(x^{1/k}) \varrho(\varrho+1) \ldots (\varrho+n-1) v^n\right\}^k\right]$$

(replacing u by u/v so that now $|u| < |\varrho||v|, |v|$)

$$= \text{coeff of } u^k \text{ in } \left\{\left(\frac{ue^{u\psi_1(x^{1/k})}}{e^{u/2}-e^{-u/2}}\right)^k - 1\right\}.$$

Theorem 4 follows now from (27), (28), (24) and Lemma 6.

4. We are now in a position to prove our main theorems on

$$(29) \qquad D_k^{(r,\varrho)}(x) = \binom{k}{r} \sum_{\substack{n_1 \ldots n_{k-r} n_j^r \leqslant x \\ j=1,\ldots,k-r}} \left(\frac{x}{n_1 \ldots n_{k-r}}\right)^{1+\varrho},$$

$$x \geqslant 1, \quad \varrho > -2, \quad 0 \leqslant r \leqslant k .$$

We require the following

LEMMA 7. *If $\varrho < 0$, then $D_k^{(r,\varrho)}(x) = O(x^{1+\varrho r/k})$.*

We prove the result by induction on k. The result is obviously true for $k = r$ and $k = r+1$.

Assume the result for all k such that $r+1 \leqslant k < K$. We have by Theorem 1

$$\sum_{k=0}^{K-r} (-1)^k \binom{K}{k} \sum_{\substack{n_j^K \leqslant x \\ j=1,\ldots,k}} D_{K-k}^{(r,\varrho)}\left(\frac{x}{n_1 \ldots n_k}\right) = 0 .$$

So

$$D_K^{(r,\varrho)}(x) = \sum_{k=1}^{K-r} (-1)^{k-1} \binom{K}{k} \sum_{\substack{n_j^K \leqslant x \\ j=1,\ldots,k}} D_{K-k}^{(r,\varrho)} \left(\frac{x}{n_1 \ldots n_k}\right)$$

$$= O\left\{\sum_{k=1}^{K-r} \sum_{\substack{n_j^k \leqslant x \\ j=1,\ldots,k}} \left(\frac{x}{n_1 \ldots n_k}\right)^{1+\frac{\varrho r}{K-k}}\right\}$$

(by the induction hypothesis)

$$= O\left\{\sum_{k=1}^{K-r} x^{1+\frac{\varrho r}{K-k}-\frac{\varrho r k}{K(K-k)}}\right\}$$

$$= O(x^{1+\frac{\varrho r}{K}}) .$$

Lemma 7 now follows.

THEOREM 5.

$$D_k^{(r,\varrho)}(x) = P_k^{(r,\varrho)}(x) + \Delta_k^{(r,\varrho)}(x)$$

where

$$P_k^{(r,\varrho)}(x) = \frac{1}{\varrho}\binom{k}{r} \sum_{s=0}^{k-r} (-1)^s \binom{k-r}{s} u_s^{-(k-r-1)} x^{1+u_s} \times$$

$$\times \left\{1+\tfrac{1}{2}u_s - u_s(u_s+1)\int_1^\infty \psi_1(y)\, y^{-u_s-2} dy\right\}^{k-r-s}$$

$$\textit{if} \quad u_s = \varrho r/(r+s) \quad \textit{and} \quad \varrho r \neq 0;$$

$$= \text{coeff of } u^{k-1} \text{ in } \frac{x^{1+u}}{u-\varrho}\left\{1+\tfrac{1}{2}u - u(u+1)\int_1^\infty \psi_1(y)\, y^{-u-2}dy\right\}^k +$$

$$+ \frac{x^{1+\varrho}}{\varrho^k}\left\{1+\tfrac{1}{2}\varrho - \varrho(\varrho+1)\int_1^\infty \psi_1(y)\, y^{-u-2}dy\right\}^k \quad \textit{if} \quad r=0,\ \varrho\neq 0;$$

$$= \text{coeff of } u^{k-r} \text{ in } x^{1+u}\left\{1+\tfrac{1}{2}u - u(u+1)\int_1^\infty \psi_1(y)\, y^{-u-2}dy\right\}^k \quad \textit{if} \quad \varrho = 0.$$

$$\Delta_k^{(r,\varrho)}(x) = \sum_{s=0}^{k-r-1} \binom{k}{s} \sum_{\substack{n_1\ldots n_s n_j^{k-s} \leqslant x \\ j=1,\ldots,s}} \delta_{k-s}^{(r,\varrho)}\left(\frac{x}{n_1 \ldots n_s}\right) + O(x^{(r\varrho+k-m)/k}) ,$$

$$\delta_k^{(r,\varrho)}(x) = \sum_{0\leqslant\lambda\leqslant r+m-k-1} x^{(r(1+\varrho)-\lambda)/k} \sum_{p=0}^{\lambda+k-r} \{1+(-1)^p\}\psi_{\lambda+k-r-p}(x^{1/k})\, a_{k,\lambda,p}^{(r,\varrho)} ,$$

$$a_{k,0,p}^{(r,\varrho)} = k^{k-r-p-1} \text{ coeff of } u^{p+1} \text{ in } \left(\frac{u}{e^u-1}\right)^k ,$$

when $\lambda \geqslant 1$

$$a_{k,\lambda,p}^{(r,\varrho)} = \begin{cases} \dfrac{1}{\varrho}\dbinom{k}{r}\displaystyle\sum_{s=0}^{k-r}(-1)^s\dbinom{k-r}{s} u_s^{-(k-r-1)}\dbinom{u_s\overline{k-r-s}+\lambda+k-r}{\lambda+k-r-p} \times \\ \quad \times \dfrac{(\lambda+k-r-p)!}{u_s(k-r-s)+\lambda+k-r}\, g_{p+1}(0, u_s, k-r-s) \quad \textit{if} \quad r, \varrho\neq 0 , \\ \text{coeff of } u^{k-1} \text{ in } \dfrac{1}{u-\varrho}\dbinom{uk+k+\lambda}{k+\lambda-p}\dfrac{(k+\lambda-p)!}{uk+k+\lambda}\, g_{p+1}(0, u, k) + \\ \quad + \dfrac{1}{\varrho^k}\dbinom{\varrho k+k+\lambda}{k+\lambda-p}\dfrac{(k+\lambda-p)!}{\varrho k+k+\lambda}\, g_{p+1}(0, \varrho, k) \quad \textit{if} \quad \varrho\neq 0 ,\ r=0 , \\ \text{coeff of } u^{k-r} \text{ in } \dbinom{uk+k+\lambda-r}{k+\lambda-r-p}\dfrac{(k+\lambda-r-p)!}{uk+k+\lambda-r}\, g_{p+1}(0, u, k) \\ \hfill \textit{if} \quad \varrho = 0; \end{cases}$$

g being the function defined in (17), *and* m *is an integer such that* $m\geqslant 1$ *and* $m > r\varrho$.

Proof. First, if in Theorem 1 we take $f_r(x) = x^{1+\varrho}$, $f_k(x) = 0$, $k\neq r$, the corresponding $F_k(x) = D_k^{(r,\varrho)}(x)$. Secondly, if in Theorem 1 we take $F_k(x) = P_k^{(r,\varrho)}(x)$, then by Theorems 2, 3, and 4, the corresponding $f_k(x)$ is given by

$$f_k(x) = -\delta_k^{(r,\varrho)}(x) + O(x^{1-\frac{m-r\varrho}{k}}) .$$

Hence if we take

$$F_k(x) = D_k^{(r,\varrho)}(x) - P_k^{(r,\varrho)}(x) = \Delta_k^{(r,\varrho)}(x)$$

in Theorem 1, the corresponding $f_k(x)$ is obviously given by

$$f_k(x) = \begin{cases} \delta_k^{(r,\varrho)}(x) + O(x^{1-\frac{m-r\varrho}{k}}) & \text{if} \quad k\geqslant r+1 , \\ 0 \quad \text{if} \quad k\leqslant r , & \text{since} \quad \delta_k^{(r,\varrho)}(x) = 0 \text{ if } k<r , \\ & \text{and} \quad \delta_k^{(r,\varrho)}(x) = -x^{1+\varrho} \text{ if } k=r . \end{cases}$$

Hence

$$\Delta_k^{(r,\varrho)}(x) = \sum_{s=0}^{k-r-1}\binom{k}{s}\sum_{\substack{n_1\ldots n_s n_j^{k-s}\leqslant x \\ j=1,\ldots,s}}\left\{\delta_{k-s}^{(r,\varrho)}\left(\frac{x}{n_1\ldots n_s}\right) + O\left(\frac{x}{n_1\ldots n_s}\right)^{1+\frac{r\varrho-m}{k-s}}\right\}$$

$$= \sum_{s=0}^{k-r-1}\binom{k}{s}\sum_{\substack{n_1\ldots n_s n_j^{k-s}\leqslant x \\ j=1,\ldots,s}} \delta_{k-s}^{(r,\varrho)}\left(\frac{x}{n_1\ldots n_s}\right) + O\left\{\sum_{s=0}^{k-r-1} D_k^{\left(k-s, \frac{r\varrho-m}{k-s}\right)}(x)\right\} .$$

Since $r\varrho - m < 0$ we have by Lemma 7

$$\sum_{s=0}^{k-r-1} D_k^{\left(k-s,\frac{r\varrho-m}{k-s}\right)} = O\left(x^{1+\frac{r\varrho-m}{k}}\right).$$

Theorem 5 is now immediate. The following theorem results at once from Theorem 5 when we take $m = 2$.

THEOREM 6. *If $r\varrho < 2$, using the notations of Theorem 5,*

$$\Delta_k^{(r,\varrho)}(x) = -(r+1)\binom{k}{r+1} \sum_{\substack{n_1\dots n_{k-r-1} n_j^{r+1} \leqslant x \\ j=1,\dots,k-r-1}} \left(\frac{x}{n_1\dots n_{k-r-1}}\right)^{\frac{r(1+\varrho)}{r+1}} \psi_1\left(\frac{x}{n_1\dots n_{k-r-1}}\right)^{\frac{1}{r+1}} + O\left(x^{(k+r\varrho-2)/k}\right).$$

Particular cases:

1. If we take $r = 0$, $\varrho = -1$, $k = 2$ in Theorem 6, we get

$$\Delta_2^{(0,-1)}(x) = -2 \sum_{n \leqslant x^{1/2}} \psi_1(x/n) + O(1), \tag{30}$$

a result due to Landau [1], which was the starting point of Van der Corput's investigations of the Dirichlet's divisor problem.

2. Taking $r = 0$, $\varrho = -1$, $k = 3$, in Theorem 6, we get

$$\Delta_3^{(0,-1)}(x) = -3 \sum_{n_1^2 n_2,\, n_1 n_2^2 \leqslant x} \psi_1\left(\frac{x}{n_1 n_2}\right) + O\left(x^{1/3}\right). \tag{31}$$

We have, from Theorem 6, trivially

$$\begin{aligned} \Delta_k^{(r,\varrho)}(x) &= O\left\{D_k^{\left(r+1,\frac{r\varrho-1}{r+1}\right)}(x)\right\} + O\left(x^{(k+r\varrho-2)/k}\right) \\ &= O\left(x^{(k+r\varrho-1)/k}\right) \quad \text{if} \quad r\varrho < 1, \text{ by Lemma 7}. \end{aligned}$$

I shall return to the general problem of the order of $\Delta_k^{(r,\varrho)}(x)$ in a subsequent paper.

References

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The lattice point problem of many-dimensional hyperboloids II

by

B. R. SRINIVASAN (Madras)

To the loving and respectful memory of Prof. Dr R. Vaidhyanathaswamy

1. In many problems in the analytic theory of numbers, it is necessary to obtain non-trivial inequalities for exponential sums of the form

$$\sum_n e^{2\pi i f(n)} \tag{1}$$

where $f(n)$ is a real function. An important method of obtaining such inequalities is due to Van der Corput (1). Titchmarch ([10], [11]) has extended Van der Corput's method to two-dimensional sums of the type

$$\sum_{m,n} e^{2\pi i f(m,n)}. \tag{2}$$

We consider here sums of the type

$$\sum_{n_1,\dots,n_p} e^{2\pi i f(n_1,\dots,n_p)} \tag{3}$$

for arbitrary positive integer p and extend, step by step, Van der Corput's theory in one dimension to these p-dimensional sums. In the case $p = 1$ the present method reduces completely to Van der Corput's method. In the case $p = 2$ the present method includes (and in fact, slightly refines) Titchmarsh's method (cf. [8] also).

The method seems to be of general importance, but in each application there are considerable difficulties of detail. As a straightforward illustration, I consider here the lattice point problem of certain many-dimensional hyperboloids which I have considered elsewhere.

(1) For an account of the method and references, cf. [12].