ACTA ARITHMETICA
LXXIX.3 (1997)

Small zeros of quadratic forms over algebraic function fields

by

ALBRECHT PFISTER (Mainz)

Dedicated to J. W. S. Cassels on his 75th birthday

1. Introduction. About 40 years ago J. W. S. Cassels [1] proved the following theorem:

Let $q(x) = \sum_{1\le i\le j\le n} q_{ij}x_ix_j$ *be a quadratic form with integer coefficients* $q_{ij} \in \mathbb{Z}$. *Assume that* q *is isotropic over* $\mathbb{Q}$. *Then there is an* $0 \ne a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ *with* $q(a) = 0$ *such that*

$$(0)\qquad |a| = \max |a_i| \le c_n\, Q^{(n-1)/2}$$

where c_n *is a constant which depends only on* n *and where* $Q = \sum_{i,j} |q_{ij}|$.

In the addendum to [1] he showed that the exponent $(n-1)/2$ is best possible by giving an example which was found by M. Kneser. In his book [3, 6.8] he gave a better proof and estimate, in particular one can take $c_n = 3^{(n-1)/2}$.

For a generalization to totally isotropic subspaces of higher dimension see [8].

Several years later Cassels [2] published his theorem on the representation of a polynomial $f(x)$ in one variable as a sum of n squares: If such a representation is possible over the rational function field $k(x)$ then it is already possible over the polynomial ring $k[x]$. He remarked that the underlying geometrical idea for the proof was essentially the same as in his first paper [1]. This idea is as follows:

Given $a \ne 0$ with $q(a) = 0$, intersect the "quadric" $q = 0$ with a "line" $l = 0$ passing through a. If l is chosen carefully then the second intersection point of q and l may be "smaller" than the original point a. For the choice of l one has to use the fact that $\mathbb{Z}$ resp. $k[x]$ are euclidean domains.

These results have been generalized in the following directions:

(1) In 1965 I generalized Cassels' representation theorem (= Darstellungssatz) to an arbitrary quadratic form q over k instead of the "unit form" $q = \langle 1, \ldots, 1\rangle$. See [4, Ch. 1] for a proof which includes the case $\operatorname{char} k = 2$.

(2) In 1975 Raghavan [6] generalized Cassels' zero theorem (= Nullstellensatz) to the ring of integers in an algebraic number field K. His estimate for $|a|$ is of the same shape as in (0) but the constant c_n now depends on $n = \dim q$ and the degree and the discriminant of K. Except for the precise value of the constant c_n this paper essentially finishes the number theoretic case.

(3) In 1987 Prestel [5] stated and proved the zero theorem for a *rational* function field $k(x)$. It reads as follows: There exists $0 \neq a = (a_1, \ldots, a_n)$ with $a_i \in k[x]$ and $q(a) = 0$ such that

$$\deg a \leq \frac{n-1}{2} \deg Q$$

where $\deg a = \max_i(\deg a_i)$, $\deg Q = \max_{i,j}(\deg q_{ij})$.

This may be considered as an additive version of (0) with $c_n = 1$ for all n. The strengthening of the estimate is due to the fact that the valuation on $k(x)$ which is induced by the degree is non-archimedean (= ultrametric).

In the same paper Prestel constructs an example of (a sequence of) isotropic quadratic forms over $\mathbb{R}(x, y)$ in $n = 4$ variables with coefficients q_{ij} of degree 2 such that the minimal degree of a non-trivial solution a is unbounded. This proves that one cannot expect results about "small" zeros for function fields in more than one variable.

The aim of the present paper is to prove the Nullstellensatz (Theorem 1) and the Darstellungssatz (Theorem 4) in the remaining open case where K is an *algebraic* function field in one variable over an arbitrary field k. The main difficulty is to find an argument which replaces the euclidean algorithm for $k[x]$. In my proofs this will be the theorem of Riemann–Roch.

I found the main breakthrough three years ago when I spent my sabbatical in Cambridge and enjoyed the privilege of being a Visiting Fellow Commoner of Trinity College.

I have been informed that Dorothea Diers (Münster) has obtained very similar results in her thesis under W. Scharlau but I have not seen any details of her work.

Notation. k is an arbitrary field, K/k is an algebraic function field in one variable. As usual we assume that K/k is finitely generated and that k is algebraically closed in K. Divisors of K/k are denoted by latin capitals $A, B, \ldots$, prime divisors by P. Ω denotes the set of all places P of K/k, $S = \{P_1, \ldots, P_s\}$ is an arbitrary non-empty finite subset of Ω. We write $A = \sum_{P\in\Omega} v_P(A)P$. For $P \in \Omega$ the valuation ring of P, residue field of P

and residue degree of P are denoted by R_P, k_P and f_P respectively. $v_P : K \to \mathbb{Z} \cup \infty$ denotes the normalized discrete valuation of K/k corresponding to P, $\pi = \pi_P$ is a prime element for P. It is well known that $R := R(S) := \bigcap_{P \notin S} R_P$ is a Dedekind ring in K with $\operatorname{quot}(R) = K$. For abbreviation we write $R_\sigma, k_\sigma, f_\sigma$ etc. instead of $R_{P_\sigma}, k_{P_\sigma}, f_{P_\sigma}$ $(\sigma = 1, \ldots, s)$. Finally, let $f = f(S) = \max\{f_1, \ldots, f_s\}$ and let g be the genus of K/k.

We consider quadratic forms

$$q = q(x) = \sum_{\substack{i,j=1 \\ i \le j}}^{n} q_{ij} x_i x_j \in K[x_1, \ldots, x_n].$$

Since $\operatorname{char} k = 2$ is not excluded q is not supposed to be in diagonal form nor to be non-degenerate. The only standard assumptions about q are:

$$q \neq 0, \qquad \dim q = n \geq 1.$$

In Section 2, q is an isotropic form over K. This implies $n \geq 2$ since $q = q_{11} x_1^2$ with $q_{11} \neq 0$ cannot be isotropic. The symmetric bilinear form corresponding to q is given by

$$q(x, y) := q(x + y) - q(x) - q(y) = \sum_{i \le j} q_{ij}(x_i y_j + x_j y_i).$$

We look for isotropic vectors $0 \neq a = (a_1, \ldots, a_n) \in R^n$ of q. The *pole divisor* A of a is the smallest non-negative divisor $A \geq 0$ such that $(a_i) + A \geq 0$ for $i = 1, \ldots, n$. For $a_i \in R$ we see that A is a linear combination of the prime divisors $P_1, \ldots, P_s \in S$ with coefficients from $\mathbb{N}_0$: $A = \sum_{\sigma=1}^{s} v_\sigma(A) P_\sigma$. Similarly Q denotes the *pole divisor of* q, i.e. the smallest non-negative divisor such that

$$(q_{ij}) + Q \geq 0 \qquad (1 \leq i \leq j \leq n).$$

S and q are called *compatible* if all $q_{ij} \in R = R(S)$.

In Section 3 we use a slightly different notation. Here q is a form over k such that the extended form $q \otimes K$ represents a given element $t \in K^*$ (over K). We then work with the $(n+1)$-dimensional isotropic form $q \otimes K \perp \langle -t \rangle$ over K.

2. The Nullstellensatz. With the terminology introduced in Section 1 let q be an isotropic quadratic form over K.

Definition. A vector $a = (a_1, \ldots, a_n) \in R^n$ with pole divisor A is called a *minimal vector of q over R* if

(i) $0 \neq a \in R^n$,
(ii) $q(a) = 0$,
(iii) $\deg A \in \mathbb{N}_0$ is minimal (under conditions (i), (ii)).

Clearly every isotropic form q has at least one minimal vector a over $R = R(S)$ and then $\deg A$ depends only on $K/k, q$ and S but not on a.

The first main result of this article is the following:

Theorem 1 (Nullstellensatz). *For every isotropic quadratic form*

$$0 \neq q = q(x) = \sum_{1 \leq i \leq j \leq n} q_{ij} x_i x_j \in R[x_1, \ldots, x_n]$$

and every minimal vector $0 \neq a = (a_1, \ldots, a_n) \in R^n$ *of* q *with pole divisor* A *we have*

$$\deg A \leq n(f + g - 1) + \frac{n-1}{2} \deg Q.$$

Proof. 1. First we consider the case $n = 2$. Here we have

$$q(x) = q_{11} x_1^2 + q_{12} x_1 x_2 + q_{22} x_2^2.$$

Without loss of generality we assume $q_{11} q_{22} \neq 0$. [Otherwise $(1,0)$ or $(0,1)$ is an isotropic vector with pole divisor 0.] Then

$$q(x) = q_{11}(x_1 - c x_2)(x_1 - c' x_2)$$

with $c, c' \in K$ since q is isotropic. We have

$$q_{11} c^2 + q_{12} c + q_{22} = 0, \quad c + c' = -q_{12}/q_{11}, \quad cc' = q_{22}/q_{11}.$$

By the definition of Q we have

$$(q_{ij}) = Q_{ij} - Q$$

with non-negative divisors $Q_{ij} \geq 0$ (provided $q_{ij} \neq 0$) $(1 \leq i \leq j \leq n)$. This implies $v_P(q_{22}) - v_P(q_{11}) = v_P(Q_{22}) - v_P(Q_{11})$ for every prime divisor P. Consider the principal divisor

$$(c^2) = 2(c) = 2 \sum_P v_P(c) P$$

and define $\gamma_P \in \mathbb{Z}$ by

$$2 v_P(c) = v_P(Q_{22}) - v_P(Q_{11}) + \gamma_P.$$

Then $(cc')^2 = (q_{22}/q_{11})^2$ implies

$$2 v_P(c') = v_P(Q_{22}) - v_P(Q_{11}) - \gamma_P$$

for all P.

Let C, C' be the pole divisors of c, c'. We have

$$\begin{aligned} v_P(C) + v_P(C') = \tfrac{1}{2} & \max\{v_P(Q_{11}) - v_P(Q_{22}) - \gamma_P, 0\} \\ & + \tfrac{1}{2} \max\{v_P(Q_{11}) - v_P(Q_{22}) + \gamma_P, 0\}. \end{aligned}$$

We have to distinguish three cases for P:

(i) $v_P(Q_{11}) - v_P(Q_{22}) + |\gamma_P| \leq 0$. Then $v_P(C) = v_P(C') = 0$.

(ii) $v_P(Q_{11}) - v_P(Q_{22}) - |\gamma_P| \le 0 < v_P(Q_{11}) - v_P(Q_{22}) + |\gamma_P|$. Then $v_P(c) \ne v_P(c')$, hence

$$\begin{aligned}\tfrac{1}{2}(v_P(Q_{22}) - v_P(Q_{11}) - |\gamma_P|) &= \min\{v_P(c), v_P(c')\} = v_P(c+c') \\ &= v_P(q_{12}) - v_P(q_{11}) = v_P(Q_{12}) - v_P(Q_{11})\end{aligned}$$

and

$$v_P(C) + v_P(C') = v_P(Q_{11}) - v_P(Q_{12}) > 0.$$

[Note that this case can only occur if $q_{12} \ne 0$.]

(iii) $0 < v_P(Q_{11}) - v_P(Q_{22}) - |\gamma_P|$. Then

$$\begin{aligned}v_P(C) + v_P(C') &= \tfrac{1}{2}(v_P(Q_{11}) - v_P(Q_{22}) - \gamma_P) \\ &\quad + \tfrac{1}{2}(v_P(Q_{11}) - v_P(Q_{22}) + \gamma_P) \\ &= v_P(Q_{11}) - v_P(Q_{22}) > 0.\end{aligned}$$

With an obvious notation this implies

$$\begin{aligned}\deg C + \deg C' &= \sum_{2} f_P(v_P(Q_{11}) - v_P(Q_{12})) \\ &\quad + \sum_{3} f_P(v_P(Q_{11}) - v_P(Q_{22})) \\ &\le \sum_{2+3} f_P v_P(Q_{11}) \le \deg Q_{11} = \deg Q.\end{aligned}$$

Hence we may assume without loss of generality that $\deg C \le \frac{1}{2}\deg Q$.

It is now easy to find a "small" isotropic vector $a = (a_1, a_2) \in R^2$ by solving the linear equation $a_1 = ca_2$ with $a_1, a_2 \in R \setminus \{0\}$, $(a_i) + A \ge 0$, $A \ge 0$, $\deg A$ minimal. We need $(a_2) + A - C \ge 0$. Then $(a_1) + A = (c) + (a_2) + A = (c) + C + (a_2) + A - C \ge 0$.

For any divisor D let $L(D) = \{d \in K : (d) + D \ge 0\}$. By the theorem of Riemann–Roch this is a finite-dimensional k-vector space of dimension

$$l(D) = \deg D + 1 - g + i(D)$$

where $i(D) \ge 0$ is the index of speciality of D.

We want: $\operatorname{supp} A \subset S$, $l(A - C) \ge 1$. Then there is some $0 \ne a_2 \in L(A - C)$ with $a_2 \in R$, $a_1 = ca_2 \in R$. It is sufficient to find $A \ge 0$ with $\operatorname{supp} A \subset S$ and $\deg(A - C) \ge g$. Since all multiples of f occur as degrees of divisors which are supported by S we can find such an A with

$$g \le \deg(A - C) \le g + f - 1.$$

Then $\deg A \le f + g - 1 + \deg C \le f + g - 1 + \frac{1}{2}\deg Q$.

N o t e. In the case $n = 2$ we have shown that

$$\deg A \le f + g - 1 + \tfrac{1}{2}\deg Q.$$

This estimate is better than the estimate of the theorem unless $f = 1$ and $g = 0$. Another easy estimate is obtained by taking $a_2 = q_{11}$ and $a_1 = ca_2$. Then

$$a_1^2 + q_{12}a_1 + q_{11}q_{22} = 0,$$
$$v_P(a_1) \geq \min\{v_P(q_{11}), v_P(q_{12}), v_P(q_{22})\} \geq -v_P(Q)$$

for all $P \in \Omega$, hence

$$(a_1) + Q \geq 0, \quad (a_2) + Q \geq 0, \quad A \leq Q, \quad \deg A \leq \deg Q.$$

This estimate is better than the above estimate if $\frac{1}{2}\deg Q < f + g - 1$.

2. From now on we assume $n \geq 3$. Furthermore, we can assume that either $s = 1$ or that S is the exact set of poles of q, i.e. $v_P(Q) > 0$ for all $P \in S$. To see this let $S' = \{P \in S : v_P(Q) > 0\} \subset S$. Then $f' = \max\{f_P : P \in S'\} \leq f$. If $|S'| \geq 1$ then application of Theorem 1 for S' instead of S yields an isotropic vector $0 \neq a = (a_1, \ldots, a_n) \in R(S')^n$ with

$$\deg A \leq n(f' + g - 1) + \frac{n-1}{2}\deg Q \leq n(f + g - 1) + \frac{n-1}{2}\deg Q.$$

If $S' = \emptyset$, i.e. $Q = 0$, i.e. $q_{ij} \in k$ for all i, j, then S can be replaced by any one-point subset $S'' \subset S$.

The proof will be by contradiction. Hence we start with the

Hypothesis. $0 \neq a \in R^n$ is a *minimal* vector of q with

$$\deg A > n(f + g - 1) + \frac{n-1}{2}\deg Q.$$

Let $A = \sum_{\sigma=1}^{s} v_\sigma(A)P_\sigma$. Since $\deg A > 0$ we can fix a $\sigma \in \{1, \ldots, s\}$ with $v_\sigma(A) > 0$. Then $B := A - P_\sigma \geq 0$ and $B < A$, $\deg B = \deg A - f_\sigma \geq \deg A - f$. By our hypothesis we have

$$\deg A > 3(f + g - 1), \qquad \deg A \geq 3f + 3g - 2,$$
$$\deg B \geq 2f + 3g - 2 > 2g - 2.$$

Therefore

$$\dim L(B) = \deg B + 1 - g \geq 2f + 2g - 1 \geq 1.$$

Put

$$V := \{b = (b_1, \ldots, b_n) \in K^n : (b_i) + B \geq 0 \text{ for } i = 1, \ldots, n\}.$$

Then V is a k-vector space, $\dim V = n(\deg B + 1 - g) \geq n > 0$. For $0 \neq b \in V$ we clearly have $q(b) \neq 0$ since the pole divisor of b is $\leq B < A$.

3. The main idea of the proof is as follows: Join the point $a \in R^n$ on the quadric $q = 0$ to the point $b \in R^n$ off the quadric $q = 0$ by a line l and intersect l with this quadric. This yields a second point of intersection

$$a^* := q(b)a - q(a, b)b \in R^n.$$

For a "good choice" of $b \in V$ we can show that the pole divisor A^* of a^* satisfies $\deg A^* < \deg A$, which contradicts the minimality of a.

Let us check that $a^* \neq 0$ and $q(a^*) = 0$ for any $0 \neq b \in V$. Since $q(b) \neq 0$ and $q(a) = 0$ we know that a, b are linearly independent (over R), hence $a^* \neq 0$. Further,

$$q(a^*) = q(b)^2 q(a) + q(a,b)^2 q(b) - q(b)q(a,b)q(a,b) = 0.$$

4. In order to compute $v_\sigma(a_j^*)$ for $\sigma \in \{1, \ldots, s\}$ choose $i(\sigma) \in \{1, \ldots, n\}$ such that

$$v_\sigma(a_j) \geq v_\sigma(a_{i(\sigma)}) = -v_\sigma(A), \qquad j = 1, \ldots, n.$$

Put

$$c^{(\sigma)} := b - \frac{b_{i(\sigma)}}{a_{i(\sigma)}} a \in K^n.$$

Then

$$\begin{aligned} a^* &= q\Big(\frac{b_{i(\sigma)}}{a_{i(\sigma)}} a + c^{(\sigma)}\Big) a - q(a, c^{(\sigma)}) \Big(\frac{b_{i(\sigma)}}{a_{i(\sigma)}} a + c^{(\sigma)}\Big) \\ &= \frac{b_{i(\sigma)}}{a_{i(\sigma)}} q(a, c^{(\sigma)}) a + q(c^{(\sigma)}) a - \frac{b_{i(\sigma)}}{a_{i(\sigma)}} q(a, c^{(\sigma)}) a - q(a, c^{(\sigma)}) c^\sigma \\ &= q(c^{(\sigma)}) a - q(a, c^{(\sigma)}) c^{(\sigma)}. \end{aligned}$$

Hence

$$\begin{aligned} v_\sigma(a_h^*) &\geq \min_{i,j} v_\sigma(q_{ij}) + \min_j v_\sigma(a_j) + 2 \min_j v_\sigma(c_j^{(\sigma)}) \\ &\geq -v_\sigma(Q) - v_\sigma(A) + 2 \min_j v_\sigma(c_j^{(\sigma)}) \end{aligned}$$

for all $h = 1, \ldots, n$. We want to make $v_\sigma(c_h^{(\sigma)})$ as large as possible. A priori we have

$$\begin{aligned} v_\sigma(c_h^{(\sigma)}) &= v_\sigma(b_h a_{i(\sigma)} - b_{i(\sigma)} a_h) - v_\sigma(a_{i(\sigma)}) \\ &\geq \min_j v_\sigma(b_j) \geq -v_\sigma(B) \qquad \text{for all } h. \end{aligned}$$

By suitable choice of $b \in V$ we want to arrange that

$$(*) \qquad v_\sigma(c_j^{(\sigma)}) \geq -v_\sigma(B) + \gamma_\sigma \qquad \text{for all } j \neq i(\sigma)$$

where the numbers $\gamma_\sigma \in \mathbb{N}_0$ $(\sigma = 1, \ldots, s)$ are chosen later. [Note that $c_{i(\sigma)}^{(\sigma)} = 0$, $v_\sigma(c_{i(\sigma)}^{(\sigma)}) = \infty$.]

Fix for a moment σ, $j \neq i(\sigma)$ and $\pi = \pi_\sigma$. In the completion $K_\sigma \cong k_\sigma((\pi))$ of K with respect to P_σ the element $c_j^{(\sigma)}$ has a Laurent series

$$c_j^{(\sigma)} = \pi^{-v_\sigma(B)}(c_0 + c_1\pi + \ldots) \qquad \text{with } c_0, c_1, \ldots \in k_\sigma.$$

$(*)$ is fulfilled for $c_j^{(\sigma)}$ iff $c_0 = c_1 = \ldots = c_{\gamma_\sigma - 1} = 0$. Let

$$W := \bigoplus_{\sigma=1}^{s} \bigoplus_{j \neq i(\sigma)} W_{\sigma,j} \qquad \text{with } W_{\sigma,j} := k_\sigma^{\gamma_\sigma}.$$

Then W is a k-vector space of dimension $\dim W = (n-1)\sum_{\sigma=1}^{s} f_\sigma \gamma_\sigma$. The map

$$\alpha : V \to W, \qquad b \mapsto \bigoplus_{\sigma} \bigoplus_{j} (c_0, \ldots, c_{\gamma_\sigma - 1})_{\sigma,j},$$

is clearly k-linear.

$(*)$ is fulfilled for all σ and j iff $0 \neq b \in \ker \alpha$. Therefore we impose the condition $\dim V > \dim W$. This gives some upper bound for $\sum f_\sigma \gamma_\sigma$.

5. The above estimate for $v_\sigma(a_h^*)$ leads to a useful estimate for $v_\sigma(A^*)$ only in the case

$$-v_\sigma(Q) - v_\sigma(A) + 2(-v_\sigma(B) + \gamma_\sigma) \le 0.$$

Therefore we impose the following conditions on the numbers $\gamma_\sigma \in \mathbb{N}_0$:

(1) $\gamma_\sigma \le \gamma_\sigma^* := v_\sigma(B) + \left[\frac{v_\sigma(Q) + v_\sigma(A)}{2}\right]$ for each $\sigma = 1, \ldots, s$,

(2) $\sum_{\sigma=1}^{s} f_\sigma \gamma_\sigma < \frac{n}{n-1}(\deg B + 1 - g)$, i.e. $\dim W < \dim V$.

We compute $\sum f_\sigma \gamma_\sigma^*$:

(i) For $s \ge 2$ we have $v_\sigma(Q) \ge 1$ for all σ by our a priori assumption. This gives

$$\begin{aligned} \gamma_\sigma^* &\ge v_\sigma(B) + \tfrac{1}{2} v_\sigma(A), \\ \sum f_\sigma \gamma_\sigma^* &\ge \deg B + \tfrac{1}{2}\deg A \ge \deg B + \tfrac{1}{2}(\deg B + 1) \\ &= \tfrac{3}{2}(\deg B + 1) - 1 \ge \frac{n}{n-1}(\deg B + 1 - g) - 1 \\ &= \frac{1}{n-1}\dim V - 1 \end{aligned}$$

since $n \ge 3$.

(ii) For $s = 1$ we have $A = mP$ with $m > 0$, $B = (m-1)P$. Then

$$\begin{aligned} \gamma_1^* &\ge m - 1 + \left[\frac{m}{2}\right] \ge m - 1 + \frac{m-1}{2} = \tfrac{3}{2}(m-1), \\ f_1\gamma_1^* &= f\gamma_1^* \ge \tfrac{3}{2} f(m-1) = \tfrac{3}{2}\deg B. \end{aligned}$$

Hence again (for $n \ge 3$)

$$\frac{1}{n-1}\dim V - 1 \le \tfrac{1}{2}\dim V - 1 = \tfrac{1}{2}(\deg B + 1 - g) - 1 \le \tfrac{3}{2}\deg B \le f_1\gamma_1^*$$

since $-\frac{1}{2}(1+g) \le 0 \le \deg B$.

Since $\sum f_\sigma \gamma_\sigma^* \in \mathbb{N}_0$ this shows that $\sum f_\sigma \gamma_\sigma^*$ is greater than or equal to the largest integer *below* $\dim V/(n-1)$ in both cases.

Let now $(\gamma_1, \ldots, \gamma_s)$ be any s-tuple with $0 \le \gamma_\sigma \le \gamma_\sigma^*$ and $\sum_{\sigma=1}^s f_\sigma \gamma_\sigma > 0$. Replacing a fixed $\gamma_\tau > 0$ by $\gamma_\tau - 1$ reduces this sum to $\sum f_\sigma \gamma_\sigma - f_\tau$ where $1 \le f_\tau \le f$. This shows that for any closed interval $I \subset \big[0, \frac{1}{n-1} \dim V\big)$ of length $\ge f$ there exists a system $(\gamma_1, \ldots, \gamma_s) \in \mathbb{N}_0^s$ with $\gamma_\sigma \le \gamma_\sigma^*$ for all σ and $\sum f_\sigma \gamma_\sigma \in I$.

6. We are now able to derive an estimate for $\deg A^*$. From our Hypothesis we have

$$\deg B \ge \deg A - f > n(g-1) + (n-1)f + \frac{n-1}{2} \deg Q,$$

$$\tfrac{1}{2} \deg Q + f < \frac{1}{n-1}(\deg B + n(1-g)),$$

$$\tfrac{1}{2} \deg Q + \deg B + f < \frac{n}{n-1}(\deg B + 1 - g) = \frac{1}{n-1} \dim V,$$

say

$$\tfrac{1}{2} \deg Q + \deg B + f + \varepsilon = \frac{1}{n-1} \dim V \qquad \text{with } \varepsilon > 0.$$

Choose $I = \big[\frac{1}{2} \deg Q + \deg B + \varepsilon/2,\ \frac{1}{2} \deg Q + \deg B + f + \varepsilon/2\big]$. Then we find a system $(\gamma_1, \ldots, \gamma_s)$ as above such that $\sum f_\sigma \gamma_\sigma \in I$, i.e.

$$(**) \qquad \tfrac{1}{2} \deg Q + \deg B < \sum f_\sigma \gamma_\sigma < \frac{1}{n-1} \dim V.$$

Put $C := \sum_{\sigma=1}^s \gamma_\sigma P_\sigma$. Let W be the k-vector space corresponding to $(\gamma_1, \ldots, \gamma_s)$. Then $\dim W = (n-1) \sum f_\sigma \gamma_\sigma < \dim V$.

Choose

$$0 \ne b \in \ker\{\alpha : V \to W\}, \qquad a^* = q(b)a - q(a,b)b \in R^n$$

and let A^* denote the pole divisor of a^*. The estimates of part 4 imply:

$$\begin{aligned}
v_\sigma(a_h^*) &\ge -v_\sigma(Q) - v_\sigma(A) + 2 \min_j v_\sigma(c_j^{(\sigma)}) \\
&\ge -v_\sigma(Q) - v_\sigma(A) - 2v_\sigma(B) + 2\gamma_\sigma, \\
v_\sigma(A^*) &\le v_\sigma(Q) + v_\sigma(A) + 2v_\sigma(B) - 2\gamma_\sigma \in \mathbb{N}_0, \\
\deg A^* &= \sum_{\sigma=1}^s f_\sigma v_\sigma(A^*) \le \deg Q + \deg A + 2 \deg B - 2 \deg C < \deg A,
\end{aligned}$$

since $\deg Q + 2 \deg B - 2 \deg C < 0$ by $(**)$. This contradicts the minimality of the isotropic vector a and proves the theorem. ∎

From Theorem 1 we can easily derive the following more general but slightly weaker

THEOREM 2. *Let $q \neq 0$ be an isotropic quadratic form over K of dimension $n \geq 2$. Let Q be the pole divisor of q. For $Q \neq 0$ let S be the exact support of Q, and $f = f(S)$. For $Q = 0$ choose $S = \{P\}$ and $f = f(S) = f_P$ with an arbitrary prime divisor $P \in \Omega$, e.g. such that $f_P = f_0 := \min\{f_P : P \in \Omega\}$. In addition, let D be any divisor of K such that*

$$\deg D \geq n(f+g-1) + \frac{n-1}{2}\deg Q + g.$$

Then there exists a non-trivial vector $b = (b_1, \ldots, b_n)$ with $q(b) = 0$ and $(b_i) + D \geq 0$ $(i = 1, \ldots, n)$.

Proof. Define $R = R(S)$ and choose a minimal vector $a \in R^n$ of q with pole divisor $A \geq 0$. By Theorem 1 we have

$$\deg A \leq n(f+g-1) + \frac{n-1}{2}\deg Q.$$

We put $b = t \cdot a$ with $t \in K^*$ and try to choose t such that $(b_i) + D \geq 0$ $(i = 1, \ldots, n)$. Clearly $q(b) = 0$. Since $(a_i) + A \geq 0$, $b_i = ta_i$ we need

$$(a_i) + (t) + D \geq 0.$$

This is true if $(t) + D \geq A$, i.e. $t \in L(D-A)$. So we need $\dim L(D-A) > 0$. This is certainly the case if

$$\deg D - \deg A + 1 - g > 0, \qquad \deg D \geq \deg A + g.$$

By our assumption on D this inequality is true. ∎

Note. If we write $D = D_1 - D_2$ with (disjoint) non-negative divisors D_1, D_2 then $(b_i) + D_1 \geq 0$. This means $b_i \in R_1 := R(S_1)$ where $S_1 = \operatorname{supp}(D_1)$, $0 \neq b \in R_1^n$, $q(b) = 0$. Here S_1 and q are not compatible in general.

Instead of scaling the vector a we can also scale the quadratic form q without changing the equation $q(a) = 0$. Thereby we can derive

THEOREM 1′. *Every isotropic form q admits an isotropic vector $a \neq 0$ such that*

$$\deg A \leq \frac{3n-1}{2}(f_0+g-1) + \frac{n-1}{2}\deg Q$$

where $f_0 = \min\{f_P : P \in \Omega\}$.

Proof. Let $P_0 \in \Omega$ be such that $f_{P_0} = f_0$. We try to find $q_0 \in K^*$ such that $(q_0 q_{ij}) + mP_0 \geq 0$ for suitable $m \in \mathbb{N}_0$. We need $(q_0) + mP_0 - Q \geq 0$. For this it is enough that

$$g \leq mf_0 - \deg Q \leq g + f_0 - 1, \quad \text{or} \quad m = \left[\frac{\deg Q + f_0 + g - 1}{f_0}\right].$$

We apply Theorem 1 to the isotropic form $q' = q_0 q$ with pole divisor $Q' \le mP_0$. This leads to an isotropic vector $a \in R_0^n$ (where $R_0 = R(\{P_0\})$) such that

$$\begin{aligned}\deg A &\le n(f_0+g-1) + \frac{n-1}{2}\deg Q' \le n(f_0+g-1) + \frac{n-1}{2} m f_0 \\ &\le \frac{3n-1}{2}(f_0+g-1) + \frac{n-1}{2}\deg Q. \ \blacksquare\end{aligned}$$

Note. Depending on the special values of f_0, f, g and n, Theorem 1′ or Theorem 1 may give a "smaller" isotropic vector a for q.

EXAMPLE 1. For the rational function field $K = k(x)$ and the set $S = \{\infty\}$ with $v_\infty(u) = -\deg u$ for all $u \in K$ we have $R = k[x]$, $f = 1$, $g = 0$. If then $q_{ij} \in R$ $(1 \le i \le j \le n)$, Theorem 1 coincides with the theorem of Prestel [5]. The estimate is then best possible for all n.

EXAMPLE 2. Let $\operatorname{char} k \ne 2$ and let $q = \langle 1, q_2, q_3\rangle$ be an anisotropic ternary quadratic form over k. Consider for $g \in \mathbb{N}_0$ the function field

$$K = k(t,u) \quad \text{with } -u^2 = q_2 t^{2g+2} + q_3.$$

It has genus g. All prime divisors P of K have even degree, since otherwise $q \otimes k_P$ would be isotropic over the odd-degree extension k_P/k, which contradicts Springer's theorem. There is one place $P = \infty$ over the infinite place of $k(t)$, it has $k_\infty = k(\sqrt{-q_2})$, $f_\infty = 2$. For $S = \{\infty\}$ we get $R = R(S) = k[t,u]$ and $Q = 0$ since q is a "constant" form. The vector $a = (u, t^{g+1}, 1)$ is a minimal vector of q over R with $\deg A = 2(g+1)$. The minimality follows since u must occur in at least one component of every isotropic vector $a \in R^3$ and since u has pole divisor $U = (g+1)\infty$ with $\deg U = 2(g+1)$. The estimate of Theorem 1 gives the weaker estimate

$$\deg A \le n(f+g-1) = 3(g+1).$$

[For $g = 0$ Theorem 1 is essentially sharp since $\deg A \le 3$ automatically implies $\deg A \le 2$.]

EXAMPLE 3. Let $k = \mathbb{Q}_{\text{pyth}}$ be the pythagorean closure of $\mathbb{Q}$. Then k is real with Pythagoras number $p(k) = 1$. It is known (see e.g. [4, Ch. 7]) that $p(k(t)) \ge 3$ and that there exists a polynomial $h = h(t)$ of degree 4 which is a sum of 3 but not of 2 squares in $k[t]$. Put $K = k(t,u)$ with $u^2 = -h$. Then K is non-real. In particular, all prime divisors of K/k have even degree, the place ∞ has degree $f_\infty = 2$, t has pole divisor ∞, u has pole divisor 2∞. Let $S = \{\infty\}$ and $R = [t,u]$ as above. Then $f = 2$ and $g = 1$ (since h has degree 4).

We take the constant ternary quadratic form $q = \langle 1,1,1\rangle$ with pole divisor $Q = 0$. From the equation $u^2 + h = 0$ it follows that the form $\langle 1\rangle \oplus q = 4 \times \langle 1\rangle$ is isotropic *over* K. As is well known this implies that

$q \otimes K$ is isotropic as well. Let $a = (a_1, a_2, a_3)$ be a minimal vector of q over R. Then $a_i = b_i + c_i u$ $(i = 1, 2, 3)$ with $b_i, c_i \in k[t]$, and a_i has pole divisor $m_i \cdot \infty$ with

$$m_i = \max\{\deg b_i, \deg c_i + 2\}$$

where deg is the ordinary degree on $k[t]$. Theorem 1 gives the estimate $\deg A \le 3(f + g - 1) = 6$ for the pole divisor A of a. We want to show that this estimate is sharp (recall that $\deg A$ is even):

Assume $\deg A \le 4$. Then c_1, c_2, c_3 must be constants from k (not all zero). The equation

$$0 = \sum_{i=1}^{3} (b_i + c_i u)^2 = \sum b_i^2 - h \sum c_i^2 + 2u \sum b_i c_i$$

implies:

1) $\sum b_i c_i = 0$, i.e. b_1, b_2, b_3 are linearly dependent over k, say $b_3 = \lambda b_1 + \mu b_2$ with $\lambda, \mu \in k$,

2) $\sum_{i=1}^{3} b_i^2 = h \sum_{i=1}^{3} c_i^2 = h \cdot c^2$ with $0 \ne c \in k$.

By normalizing we can arrange $c = 1$. Then

$$h = b_1^2 + b_2^2 + (\lambda b_1 + \mu b_2)^2 \quad \text{in } k[t].$$

Put

$$\tau := \sqrt{\lambda^2 + \mu^2 + 1} \in k, \quad \nu = \frac{\lambda^2 \mu + \tau}{\lambda^2 + 1}, \quad \omega = \frac{\lambda(\mu - \tau)}{\lambda^2 + 1}.$$

An easy computation yields

$$h = (b_1 + \omega b_2)^2 + (\lambda b_1 + \nu b_2)^2,$$

i.e. h is a sum of 2 squares in $k[t]$: Contradiction.

Final remarks to Theorems 1, 1′, 2

1. Since the estimates for $\deg A$ grow with the number $n = \dim q$ they can only be sharp if all proper subforms of q are anisotropic over K. For certain fields this ensures an a priori upper bound for n. If e.g. K is a C_i-field then every quadratic form of dimension $> 2^i$ over K is isotropic, which means that we can suppose $n \le 2^i + 1$.

2. If $g \ne 1$ then there is a well-known upper bound for the greatest common divisor δ of all possible divisor degrees (δ is called the index of K/k) and for the minimal prime divisor degree f_0, namely $\delta \le f_0 \le |2g - 2|$. Then f_0 or $f = f(S)$ could be eliminated from the estimates in many cases.

3. If $g = 1$ then $f_0 = \delta$ and δ can be any natural number. For concrete examples where k is a p-adic field see e.g. [7, Cor. 13a].

3. The Darstellungssatz. Let now $q \ne 0$ be a quadratic form over k and suppose that $q \otimes K$ represents a given element $t \in K^*$. We look for a representation

$$q(u) = q(u_1, \ldots, u_n) = t \tag{1}$$

for which the pole divisor U of u has small degree. The result will be slightly different depending on whether $q \otimes K$ is isotropic or not. For the anisotropic case we first need a variant of Theorem 1. Here a non-trivial solution in K of the equation

$$q(a_1, \ldots, a_n) - ta_0^2 = 0 \tag{2}$$

automatically satisfies $a_0 \neq 0$, $q(a_1, \ldots, a_n) \neq 0$, and leads to a solution of (1) by putting $u_i = a_i/a_0$. Since then a_0 is (up to sign) completely determined by q, t and the vector $a = (a_1, \ldots, a_n)$ we measure the "smallness" of a solution of (2) be the degree of the pole divisor A of a (as in Section 2).

Let $N \geq 0$ be the divisor of zeros of t, let $T \geq 0$ be the pole divisor of t, i.e. $(t) = N - T$. We have unique decompositions $N = 2N_0 + N_1$, $T = 2T_0 + T_1$ with $v_P(N_1)$, $v_P(T_1) \in \{0, 1\}$ for all prime divisors P. Since $\deg N = \deg T$ we have

$$\begin{aligned} 2\deg(T_0 + T_1 - N_0) &= \deg T + \deg T_1 - 2\deg N_0 \\ &= \deg T_1 + \deg N - 2\deg N_0 \\ &= \deg T_1 + \deg N_1 \geq 0. \end{aligned}$$

If $\operatorname{supp} T \neq \emptyset$ let $S = \operatorname{supp} T$, otherwise let $S = \{P\}$ be any one-point set from Ω. Put $R = R(S)$ and $f = f(S)$. Call a non-trivial solution $(a_1, \ldots, a_n, a_0)$ of (2) resp. its vector $a = (a_1, \ldots, a_n)$ *minimal* if $\deg A$ is as small as possible.

THEOREM 3. *Under the above assumptions there exists a non-trivial solution of* (2) *with* $a \in R^n$. *If* a *is minimal then*

$$\deg A \leq (n+1)(f + g - 1) + \deg(T_0 + T_1 - N_0).$$

Proof. The first statement is clear since $q \otimes K$ represents t and $K = \operatorname{quot}(R)$. Suppose now that a is minimal but $\deg A$ does not satisfy the estimate of Theorem 3. Then $A = \sum_{\sigma=1}^{s} v_\sigma(A)\, P_\sigma$ with $P_\sigma \in S$, $s = |S|$, and $v_{\sigma_0}(A) > 0$ for at least one σ_0 since $\deg A > 0$ by our assumption. Fix σ_0 and put $B := A - P_{\sigma_0}$. Then $0 \leq B < A$ and $\deg B = \deg A - f_0 \geq \deg A - f$. As in the proof of Theorem 1 we shall construct a vector $0 \neq a^* \in R^n$ and an element $a_0^* \neq 0$ such that $q(a^*) = ta_0^{*2}$ but $\deg A^* < \deg A$. This contradicts the minimality of a and proves the theorem.

From (2) we have

$$(a_i) + A \geq 0 \quad \text{for } i = 1, \ldots, n,$$

hence

$$(t) + 2(a_0) + 2A \geq 0,$$
$$N_1 + 2N_0 - 2T_0 + 2(a_0) + 2A \geq 0,$$
$$2(N_0 - T_0 + (a_0) + A) \geq 0,$$

and finally,

$$(a_0) + A + N_0 - T_0 \geq 0. \tag{3}$$

We look for vectors $b = (b_1, \ldots, b_n) \in K^n$ and elements $b_0 \in K$ such that

$$\begin{aligned} (b_i) + B &\geq 0 \quad \text{for } i = 1, \ldots, n, \\ (b_0) + B + N_0 - T_0 - T_1 &\geq 0. \end{aligned} \tag{4}$$

The k-vector space V of all $(n+1)$-tuples $(b_1, \ldots, b_n, b_0)$ satisfying (4) has dimension

$$\dim V \geq (n+1)(\deg B + 1 - g) + \deg(N_0 - T_0 - T_1). \tag{5}$$

From (2) and (4) we conclude

$$(ta_0^2) + 2A \geq 0, \quad (tb_0^2) + T - N + 2B + 2(N_0 - T_0 - T_1) \geq 0,$$

i.e. $(tb_0^2) + 2B - T_1 - N_1 \geq 0$, hence

$$(tb_0^2) + 2B \geq 0 \quad \text{and} \quad (ta_0b_0) + A + B \geq 0. \tag{6}$$

For all $i = 0, \ldots, n$ define

$$a_i^* := (q(b) - tb_0^2)a_i - (q(a,b) - 2ta_0b_0)b_i,$$
$$a^* := (a_1^*, \ldots, a_n^*), \quad A^* := \text{pole divisor of } a^*.$$

It follows that

$$q(a^*) = ta_0^{*2}, \tag{7}$$

$$\begin{aligned} (a_i^*) + A + 2B &\geq 0 \quad \text{for } i = 1, \ldots, n, \\ (a_0^*) + A + 2B + N_0 - T_0 &\geq 0. \end{aligned} \tag{8}$$

In particular, we have $a^* \in R^n$.

Since $B < A$ we cannot have $q(b) - tb_0^2 = 0$ unless $b = 0$ and $b_0 = 0$. If $0 \neq (b_1, \ldots, b_n, b_0) \in V$ this implies that (b, b_0) is independent of (a, a_0) over K, hence $a^* \neq 0$. We have to choose (b, b_0) in such a way that $\deg A^* < \deg A$. As in the proof of Theorem 1 we choose for each $\sigma \in \{1, \ldots, s\}$ an index $i(\sigma) \in \{1, \ldots, n\}$ such that $v_\sigma(A) = -v_\sigma(a_{i(\sigma)})$ and put

$$c_j^{(\sigma)} := b_j - \frac{b_{i(\sigma)}}{a_{i(\sigma)}} a_j \quad (j = 0, \ldots, n).$$

Then we get

$$a_j^* = (q(c^{(\sigma)}) - tc_0^{(\sigma)2})\, a_j - (q(a, c^{(\sigma)}) - 2ta_0c_0^{(\sigma)})\, c_j^{(\sigma)}$$

with $c^{(\sigma)}_{i(\sigma)} = 0$,

$$v_\sigma(c_j^{(\sigma)}) + v_\sigma(B) \geq 0 \quad (j = 1, \ldots, n),$$
$$v_\sigma(t_0 c_0^{(\sigma)2}) + 2v_\sigma(B) \geq 0.$$

By a suitable choice of $(b, b_0) \in V$ we want to arrange for stronger inequalities

$$(9) \qquad \begin{aligned} &v_\sigma(c_j^{(\sigma)}) + v_\sigma(B) - \gamma_\sigma \geq 0 \quad \text{for } i(\sigma) \neq j \in \{1, \ldots, n\}, \\ &v_\sigma(tc_0^{(\sigma)2}) + 2v_\sigma(B) - 2\gamma_\sigma \geq 0 \end{aligned}$$

where the $\gamma_\sigma \in \mathbb{N}_0$ are chosen later. If (9) holds then a^* satisfies

$$(10) \qquad v_\sigma(a_j^*) + v_\sigma(A) + 2v_\sigma(B) - 2\gamma_\sigma \geq 0 \quad (j = 1, \ldots, n).$$

For fixed σ condition (9) puts $nf_\sigma\gamma_\sigma$ k-linear equations on the vector $(b, b_0) \in V$, that is, $n\sum_{\sigma=1}^{s} f_\sigma\,\gamma_\sigma$ equations altogether. We take

$$\gamma_\sigma = v_\sigma(B) + \begin{cases} 0 & \text{for } \sigma \neq \sigma_0, \\ 1 & \text{for } \sigma = \sigma_0. \end{cases}$$

Then $n\sum f_\sigma\,\gamma_\sigma = n(\deg B + f_0) < \dim V$, since

$$\begin{aligned} &\dim V - n(\deg B + f_0) \\ &\quad \geq (n+1)\deg B + (n+1)(1-g) + \deg(N_0 - T_0 - T_1) - n(\deg B + f_0) \\ &\quad = \deg B - nf_0 - (n+1)(g-1) - \deg(T_0 + T_1 - N_0) \\ &\quad \geq \deg A - (n+1)(f+g-1) - \deg(T_0 + T_1 - N_0) > 0 \end{aligned}$$

by our assumption. Therefore there exists $0 \neq (b, b_0) \in V$ with (9). By (10) the pole divisor A^* of a^* satisfies $v_\sigma(A^*) \leq v_\sigma(A)$ for $\sigma \neq \sigma_0$ and

$$v_{\sigma_0}(A^*) \leq \begin{cases} v_{\sigma_0}(A) - 2 & \text{for } v_{\sigma_0}(A) \geq 2, \\ 0 & \text{for } v_{\sigma_0}(A) = 1. \end{cases}$$

In any case we get $v_{\sigma_0}(A^*) < v_{\sigma_0}(A)$, hence $\deg A^* < \deg A$: Contradiction. ∎

N o t e s. 1. For a prime divisor $P \in \operatorname{supp}(N_1 + T_1)$ the validity of equation (1), when read over the completion K_P of K, implies that $q \otimes K_P$, hence $q \otimes k_P$ must be isotropic. By Springer's theorem this implies that $\deg P = [k_P : k]$ is even.

2. For $t \in k^*$, i.e. $T = N = 0$, Theorem 3 coincides with Theorem 1 for the constant quadratic form $q \perp \langle -t \rangle$.

EXAMPLE 4. Let $K = k(x)$ be the rational function field, and let $t = f(x) \in k[x]$ be a squarefree polynomial. If the anisotropic form q (which remains anisotropic over K) represents t over K then $\deg f = d = 2m$ is even and the pole divisor T of t is given by $T = 2T_0$ with $T_0 = m \cdot \infty$, $f =$

$f_\infty = 1$, $g = 0$. Theorem 3 gives us a representation $q(a_1, \ldots, a_n) = ta_0^2$ with

$$a_i \in k[x] = R(\infty), \quad \deg a_i \le 0 + \deg T_0 = m \quad (i = 1, \ldots, n).$$

Then $0 \ne q(a) \in k[x]$ with $\deg q(a) \le 2m$. Since f is squarefree we get $0 \ne a_0 \in k[x]$ with $\deg a_0 \le 0$, i.e. $a_0 \in k^*$. This proves the original representation theorem (for anisotropic form q).

We are now ready to state and prove the representation theorem for an arbitrary algebraic function field K/k with genus g.

Theorem 4 (Darstellungssatz). *Let q be a quadratic form over k, $\dim q = n$. Suppose that $t \in K^*$ is represented by the form $q \otimes K$. Let $N = 2N_0 + N_1$, $T = 2T_0 + T_1$ and $S \subset \Omega$, $f = f(S)$ be as before. Then there exists a vector $u = (u_1, \ldots, u_n) \in K^n$ with pole divisor U such that $q(u) = t$ and*

$$\deg U \le (n+1)(f+g-1) + \deg(T_0 + T_1) + \tfrac{1}{2}\deg N_1$$

or

$$\deg U \le n(f+g-1) + \deg T.$$

[*In the "reduced" case the first* (*second*) *estimate holds if $q \otimes K$ is anisotropic* (*isotropic*).]

Proof. Since the estimates for $\deg U$ grow with n we can assume that t is not represented by $\widetilde{q} \otimes K$ for any proper k-subform $\widetilde{q}$ of q (otherwise replace q by $\widetilde{q}$ with $\dim \widetilde{q} < \dim q$). We then call q *reduced* with respect to t. In particular, q is non-defective, i.e. $q = q' \perp q''$ with $q' = \operatorname{rad} q = \langle q_1, \ldots, q_r \rangle$ anisotropic over k and q'' regular over k. (See e.g. [4, Ch. 1]; for $\operatorname{char} k \ne 2$ one has $q' = 0, q$ regular.)

First case: *$q \otimes K$ anisotropic*. Here we can start with a non-trivial solution of $q(a_1, \ldots, a_n) - ta_0^2 = 0$ according to Theorem 3 and put $u_i = a_i/a_0$ $(i = 1, \ldots, n)$, $u = a/a_0 \in K^n$.

A pole P of u_i either comes from a pole of a_i or from a zero of a_0. In the first case we have $P \in S$. Let $U = U_1 + U_2$ with $U_1 \ge 0$, $U_2 \ge 0$, $\operatorname{supp} U_1 \subset S$, $S \cap \operatorname{supp} U_2 = \emptyset$. The equation $q(a) = ta_0^2$ shows:

1) For $P \notin S$ we have $v_P(a_i) \ge 0$, $v_P(q(a)) \ge 0$, $v_P(t) \ge 0$ and $v_P(q(a)) = v_P(t) + 2v_P(a_0)$. Hence

$$v_P\left(\frac{a_i}{a_0}\right) + \tfrac{1}{2}v_P(q(a)) \ge -v_P(a_0) + \tfrac{1}{2}v_P(q(a)) \ge 0.$$

This implies $v_P(U_2) \le \frac{1}{2}v_P(q(a))$ for every $P \notin S$, i.e.

$$\deg U_2 \le \tfrac{1}{2}\sum_{P \notin S} f_P v_P(q(a)).$$

2) For $P \in S$ we have $v_P(a_i) + v_P(A) \geq 0$, $v_P(t) + v_P(T) \geq 0$ and again $2v_P(a_0) = v_P(q(a)) - v_P(t)$. Hence

$$v_P\left(\frac{a_i}{a_0}\right) + v_P(A) + \tfrac{1}{2}v_P(q(a)) - \tfrac{1}{2}v_P(t) \geq 0,$$

where $v_P(A) + \frac{1}{2}v_P(q(a)) \geq 0$. This implies $v_P(U_1) \leq v_P(A) + \frac{1}{2}v_P(q(a)) + \frac{1}{2}v_P(T)$, hence

$$\deg U_1 \leq \deg A + \tfrac{1}{2}\deg T + \tfrac{1}{2}\sum_{P\in S} f_P v_P(q(a)).$$

Since $\sum_{\text{all } P} f_P v_P(q(a)) = \deg(q(a)) = 0$ the two estimates imply

$$\deg U = \deg U_1 + \deg U_2 \leq \deg A + \tfrac{1}{2}\deg T = \deg A + \deg N_0 + \tfrac{1}{2}\deg N_1.$$

With the estimate of Theorem 3 for $\deg A$ this gives

$$\deg U \leq (n+1)(f+g-1) + \deg(T_0 + T_1) + \tfrac{1}{2}\deg N_1.$$

Second case: $q \otimes K$ *isotropic*. Here we first show that $q' \otimes K$ must be anisotropic (if q is reduced). Suppose that $q' \otimes K$ is isotropic, say $q_r = \sum_{i=1}^{r-1} q_i c_i^2$ with $c_i \in K$. Let $t = q(u_1, \ldots, u_n) = q'(u_1, \ldots, u_r) + q''(u_{r+1}, \ldots \ldots, u_n)$ be any representation with $u_i \in K$. Then

$$q'(u_1, \ldots, u_r) = \sum_{i=1}^{r} q_i u_i^2 = \sum_{i=1}^{r-1} q_i(u_i^2 + c_i^2 u_r^2) = \sum_{i=1}^{r-1} q_i(u_i + c_i u_r)^2.$$

Hence t is represented by $\widetilde{q} \otimes K$ where $\widetilde{q} = \langle q_1, \ldots, q_{r-1}\rangle \perp q''$ is a proper subform of q (over k): Contradiction.

Let now $0 \neq a \in R(S)^n$ be a solution of $q(a) = 0$ which satisfies the condition of Theorem 1, i.e. a has pole divisor A with $\deg A \leq n(f+g-1)$. [Note that q is constant, i.e. $Q = 0$.] Assume for a moment that $q(a,b) = 0$ for all vectors $b \in k^n$. By linearity this would imply $q(a,b) = 0$ for all $b \in K^n$, i.e. $0 \neq a \in \operatorname{rad}(q \otimes K) = q' \otimes K$. In other words, $q'(a) = 0$, which is a contradiction. Hence we can choose a vector $0 \neq b \in k^n$ such that $a_0 := q(a,b) \in K^*$. Clearly $(a_0) + A \geq 0$, in particular $a_0 \in R = R(S)$.

We can now find a good representation of t. Put $u := b + \lambda a$ with some $\lambda \in K$. Then

$$q(u) = q(b) + \lambda q(a,b) = q(b) + \lambda a_0,$$

i.e.

$$q(u) = t \Leftrightarrow \lambda = \frac{t - q(b)}{a_0} \Leftrightarrow u = b + (t - q(b)) \cdot \frac{a}{a_0}.$$

Let C be the pole divisor of $c := a/a_0$. Since $b \in k^n$ and $q(b) \in k$ this shows that the pole divisor U of u satisfies $U \leq T + C$. It remains to estimate

$\deg C$. For every $P \in \Omega$ we have

$$v_P(a_0) + v_P(A) \geq 0 \quad \text{and} \quad v_P\left(\frac{a_i}{a_0}\right) + v_P(a_0) + v_P(A) \geq 0 \quad (i = 1, \ldots, n),$$

hence $v_P(c_i) + (v_P(a_0) + v_P(A)) \geq 0$ for $c_i = a_i/a_0$, i.e. $v_P(C) \leq v_P(a_0) + v_P(A)$. This gives

$$\deg C = \sum_P f_P v_P(C) \leq \sum_P f_P v_P(a_0) + \sum_P f_P v_P(A) = \deg A,$$

since $\sum f_P v_P(a_0) = \deg(a_0) = 0$. Therefore

$$\deg U \leq \deg C + \deg T \leq \deg A + \deg T \leq n(f + g - 1) + \deg T. \ \blacksquare$$

Note. If $q(u) = t$ then the poles $P \in \operatorname{supp} T$ must turn up with suitable multiplicity among the poles of u_i for at least one $i \in \{1, \ldots, n\}$. This shows that $U = T_0 + T_1 + U'$ with a non-negative divisor $U' \geq 0$ of degree

$$\deg U' \leq (n+1)(f + g - 1) + \tfrac{1}{2} \deg N_1 \text{ or } \deg U' \leq n(f + g - 1) + \deg T_0.$$

It does not seem possible to prescribe the prime divisors $P \in \operatorname{supp} U'$ since this would amount to knowing the zeros of a_0.

References

[1] J. W. S. Cassels, *Bounds for the least solutions of homogeneous quadratic equations*, Proc. Cambridge Philos. Soc. 51 (1955), 262–264. *Addendum*, ibid. 52 (1956), 604.

[2] —, *On the representation of rational functions as sums of squares*, Acta Arith. 9 (1964), 79–82.

[3] —, *Rational Quadratic Forms*, London Math. Soc. Monographs 13, Academic Press, London, 1978.

[4] A. Pfister, *Quadratic Forms with Applications to Algebraic Geometry and Topology*, London Math. Soc. Lecture Note Ser. 217, Cambridge Univ. Press, 1995.

[5] A. Prestel, *On the size of zeros of quadratic forms over rational function fields*, J. Reine Angew. Math. 378 (1987), 101–112.

[6] S. Raghavan, *Bounds for minimal solutions of Diophantine equations*, Nachr. Akad. Wiss. Göttingen 1975 (9), 109–114.

[7] P. Roquette, *Analytic theory of elliptic functions over local fields*, Hamburg. Math. Einzelschr. 1, Vandenhoeck & Ruprecht, Göttingen, 1970.

[8] H. P. Schlickewei and W. M. Schmidt, *Quadratic forms which have only large zeros*, Monatsh. Math. 105 (1988), 295–311.

Fachbereich Mathematik
Universität Mainz
55099 Mainz, Germany
E-mail: pfister@mat.mathematik.uni-mainz.de

Received on 3.7.1996 (3017)