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On the Mahler measure of polynomials in many variables

by

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To Professor J. W. S. Cassels on his 75th birthday

For $F \in \mathbb{C}[z_1, z_1^{-1}, \ldots, z_s, z_s^{-1}]$ the *Mahler measure* $M(F)$ is given by the formula

$$M(F) = \exp \int\limits_{[0,1)^s} \!\!\ldots\!\! \int \log |F(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_s})| \, d\theta_1 \ldots d\theta_s,$$

while

$$\|F\| = \Big(\int\limits_{[0,1)^s} \!\!\ldots\!\! \int |F(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_s})|^2 \, d\theta_1 \ldots d\theta_s \Big)^{1/2}.$$

Let $F = \sum_{i=1}^{l} a_i \prod_{\sigma=1}^{s} z_\sigma^{\alpha_{i\sigma}}$, where $a_i \in \mathbb{C}^*$ and $\boldsymbol{\alpha}_i = \langle \alpha_{i1}, \ldots, \alpha_{is} \rangle \in \mathbb{Z}^s$ are distinct. We call two terms of F

$$a_j \prod_{\sigma=1}^{s} z_\sigma^{\alpha_{j\sigma}} \quad \text{and} \quad a_k \prod_{\sigma=1}^{s} z_\sigma^{\alpha_{k\sigma}} \quad (j \neq k)$$

opposite extreme if there exists a vector $\boldsymbol{r} \in \mathbb{R}^s$ such that

$$\boldsymbol{r}\boldsymbol{\alpha}_j < \boldsymbol{r}\boldsymbol{\alpha}_i < \boldsymbol{r}\boldsymbol{\alpha}_k \quad \text{for all } i \neq j, k.$$

Moreover, we put

$$JF = F \prod_{\sigma=1}^{s} z_\sigma^{-\min_{1 \le i \le l} \alpha_{i\sigma}}$$

and for $F \in \mathbb{C}[\boldsymbol{z}]$ we denote by ∂F the maximal degree of F with respect to z_σ $(1 \le \sigma \le s)$. We note that

$$\|F\|^2 = \sum_{i=1}^{l} |a_i|^2.$$

We shall show

THEOREM. *If* $F \in \mathbb{C}[z_1, z_1^{-1}, \ldots, z_s, z_s^{-1}]$ *and* a_1, a_2 *are the coefficients of opposite extreme terms of* F *then*

$$M(F)^2 + |a_1a_2|^2 M(F)^{-2} \le \|F\|^2. \tag{1}$$

Equality occurs if and only if $F(z_1, \ldots, z_s)\overline{F}(z_1^{-1}, \ldots, z_s^{-1})$ *has just three non-zero coefficients.*

For $s = 1$ the first part of the theorem was proved by Vicente Gonçalvez [1], the second part by the writer [5]. The proof of the first part of Lemma 1 below is Ostrowski's proof [4] of Vicente Gonçalvez's theorem, rediscovered by Mignotte [3].

LEMMA 1. *The Theorem holds for* $s = 1$ *and if* c *is the coefficient of* z^m $(m \ne 0, \pm\partial JF)$ *in* $F(z)\overline{F}(z^{-1})$ *then*

$$M(F)^2 + |a_1a_2|^2M(F)^{-2} + \sqrt{(M(F)^2 + |a_1a_2|^2M(F)^{-2})^2 + 2|c|^2} \le 2\|F\|^2.$$

Proof. Replacing if necessary F by JF or $JF(z^{-1})$ which changes neither $\|F\|$ nor $M(F)$ nor the set $\{F(z)\overline{F}(z^{-1}), \overline{F}(z)F(z^{-1})\}$ we may assume that $F \in \mathbb{C}[z]$, $F(0) \ne 0$ and a_1 is the leading coefficient of F, $a_2 = F(0)$.

Let

$$a_1^{-1}F(z) = \prod_{i=1}^{n}(z - \alpha_i) = G(z)H(z),$$

$$G(z) = \prod_{\substack{i=1\\|\alpha_i|\ge 1}}^{n}(z-\alpha_i), \qquad H(z) = \prod_{\substack{i=1\\|\alpha_i|<1}}^{n}(z-\alpha_i)$$

and compute

$$\begin{aligned}|a_1|^{-2}F(z)\overline{F}(z^{-1}) &= G(z)H(z)\overline{G}(z^{-1})\overline{H}(z^{-1})\\ &= (G(z)\overline{H}(z^{-1}))(\overline{G}(z^{-1})H(z)).\end{aligned} \tag{2}$$

The constant term on the left is $\|a_1^{-1}F\|^2$, on the right $\|E\|^2$, where

$$E = z^{\partial H}G(z)\overline{H}(z^{-1}) = \prod_{\substack{i=1\\|\alpha_i|\ge 1}}^{n}(z-\alpha_i)\prod_{\substack{i=1\\|\alpha_i|<1}}^{n}(1-\overline{\alpha}_i z).$$

Let us put

$$E = \sum_{i=0}^{n} e_i z^i. \tag{3}$$

We have

$$e_0 = \prod_{\substack{i=1\\|\alpha_i|<1}}^{n}(-\alpha_i), \qquad e_n = \prod_{\substack{i=1\\|\alpha_i|\ge 1}}^{n}(-\overline{\alpha}_i). \tag{4}$$

Hence

$$\|a_1^{-1}F\|^2 \geq \prod_{\substack{i=1\\|\alpha_i|<1}}^{n} |\alpha_i|^2 + \prod_{\substack{i=1\\|\alpha_i|\geq 1}}^{n} |\alpha_i|^2,$$

which gives (1) since by Jensen's formula

$$M(F) = |a_1| \prod_{\substack{i=1\\|\alpha_i|\geq 1}}^{n} |\alpha_i|. \tag{5}$$

Equality in (1) is attained if and only if E has just two non-zero coefficients. If this condition is satisfied then $F(z)\overline{F}(z^{-1}) = |a_1|^2 E(z)\overline{E}(z^{-1})$ has just three non-zero coefficients.

Conversely, if the latter condition holds we have

$$\begin{aligned} z^n F(z)\overline{F}(z^{-1}) &= a_1\overline{F}(0)z^{2n} + \|F\|^2 z^n + \overline{a}_1 F(0) \\ &= a_1\overline{F}(0)\left(z^n + \frac{\|F\|^2 + \sqrt{\|F\|^4 - 4|a_1F(0)|^2}}{2a_1\overline{F}(0)}\right) \\ &\quad \times \left(z^n + \frac{\|F\|^2 - \sqrt{\|F\|^4 - 4|a_1F(0)|^2}}{2a_1\overline{F}(0)}\right). \end{aligned}$$

All zeros of the first, respectively second, bracketed factor are in absolute value greater, respectively less than 1, hence E equals the first factor multiplied by a constant and thus has just two non-zero coefficients.

Assume that $F(z)\overline{F}(z^{-1})$ has a term cz^m, where $m \neq 0, \pm n$ and $c \neq 0$. Replacing if necessary cz^m by $\overline{c}z^{-m}$ we may assume $m > 0$. By (2) and (3) we obtain

$$e_m\overline{e}_0 + \ldots + e_n\overline{e}_{n-m} = |a_1|^{-2}c.$$

Now, by the Schwarz inequality

$$\begin{aligned} &(|e_m|^2 + |e_{m+1}|^2 + \ldots + |e_{n-1}|^2 + |e_{n-m}|^2) \\ &\qquad \times (|e_0|^2 + |e_1|^2 + \ldots + |e_{n-m-1}|^2 + |e_n|^2) \geq |a_1|^{-4}|c|^2. \end{aligned}$$

However, the first factor does not exceed $2(\|E\|^2 - |e_0|^2 - |e_n|^2)$, and the second factor does not exceed $\|E\|^2$. Thus we obtain

$$|a_1|^4\|E\|^4 - (|a_1e_0|^2 + |a_1e_n|^2)|a_1|^2\|E\|^2 - \frac{1}{2}|c|^2 \geq 0$$

and

$$2\|F\|^2 = 2|a_1|^2\|E\|^2 \geq |a_1e_0|^2 + |a_1e_n|^2 + \sqrt{(|a_1e_0|^2 + |a_1e_n|^2)^2 + 2|c|^2},$$

which completes the proof, since by (4) and (5),

$$|a_1e_0| = \frac{|a_1F(0)|}{M(F)}, \qquad |a_1e_n| = M(F).$$

Lemma 2. *For every* $F \in \mathbb{C}[z_1, z_1^{-1}, \ldots, z_s, z_s^{-1}]$ *we have*

$$M(F) = \lim_{q(\boldsymbol{r})\to\infty} M(F(z^{r_1}, \ldots, z^{r_s})), \qquad \|F\| = \lim_{q(\boldsymbol{r})\to\infty} \|F(z^{r_1}, \ldots, z^{r_s})\|,$$

where

$$q(\boldsymbol{r}) = \min\{h(\boldsymbol{a}) : \boldsymbol{a} \in \mathbb{Z}^s\backslash\{\boldsymbol{0}\} \text{ and } \boldsymbol{a}r = 0\}, \qquad h(\boldsymbol{a}) \text{ is the height of } \boldsymbol{a}.$$

Proof. The first equality is a result of Lawton [2], the second is trivial.

Proof of the Theorem. Let

$$(6)\qquad \begin{aligned} &F = \sum_{\boldsymbol{j}\in\mathbb{Z}^s} a(\boldsymbol{j}) \prod_{\sigma=1}^{s} z_\sigma^{j_\sigma}, \qquad a_\nu = a(\boldsymbol{j}_\nu) \neq 0 \ (\nu = 1, 2), \qquad \boldsymbol{j}_1 \neq \boldsymbol{j}_2;\\ &J = \{\boldsymbol{j} \in \mathbb{Z}^s : a(\boldsymbol{j}) \neq 0\}.\end{aligned}$$

Since a_1, a_2 are the coefficients of opposite extreme terms of F there exists a vector $\boldsymbol{r}_0 \in \mathbb{R}^s$ such that

$$\boldsymbol{r}_0\boldsymbol{j}_1 < \boldsymbol{r}_0\boldsymbol{j} < \boldsymbol{r}_0\boldsymbol{j}_2 \quad \text{ for all } \boldsymbol{j} \in J\backslash\{\boldsymbol{j}_1, \boldsymbol{j}_2\}$$

and thus

$$-m_1 := \max_{\boldsymbol{j}\in J\backslash\{\boldsymbol{j}_1\}} \frac{\boldsymbol{r}_0(\boldsymbol{j}_1 - \boldsymbol{j})}{h(\boldsymbol{j}_1 - \boldsymbol{j})} < 0 < \min_{\boldsymbol{j}\in J\backslash\{\boldsymbol{j}_2\}} \frac{\boldsymbol{r}_0(\boldsymbol{j}_2 - \boldsymbol{j})}{h(\boldsymbol{j}_2 - \boldsymbol{j})} =: m_2.$$

Hence for every vector $\boldsymbol{r} \in \mathbb{R}^s$ such that

$$(7)\qquad h(\boldsymbol{r} - \boldsymbol{r}_0) < \frac{\min\{m_1, m_2\}}{s}$$

we have

$$(8)\qquad \boldsymbol{r}\boldsymbol{j}_1 < \boldsymbol{r}\boldsymbol{j} < \boldsymbol{r}\boldsymbol{j}_2 \quad \text{ for all } \boldsymbol{j} \in J\backslash\{\boldsymbol{j}_1, \boldsymbol{j}_2\}.$$

Now (7) is satisfied by s linearly independent vectors $\boldsymbol{r} \in \mathbb{Q}^s$. Since (8) is homogeneous with respect to $\boldsymbol{r}$ it is satisfied by s linearly independent vectors $\boldsymbol{r} \in \mathbb{Z}^s$. Hence for every $Q \in \mathbb{R}$ there exists an $\boldsymbol{r} \in \mathbb{Z}^s$ satisfying (8) with $q(\boldsymbol{r}) > Q$. However, (6) and (8) imply that a_1 and a_2 are the coefficients of opposite extreme terms of $F(z^{r_1}, \ldots, z^{r_s})$, hence by Lemma 1,

$$M(F(z^{r_1}, \ldots, z^{r_s}))^2 + |a_1a_2|^2 M(F(z^{r_1}, \ldots, z^{r_s}))^{-2} \leq \|F(z^{r_1}, \ldots, z^{r_s})\|^2.$$

Passing to the limit as $q(\boldsymbol{r}) \to \infty$ we obtain, by Lemma 2,

$$M(F)^2 + |a_1a_2|^2 M(F)^{-2} \leq \|F\|^2.$$

If $q(\boldsymbol{r}) > 2\partial JF$ the system (i.e. the set with multiplicities) of all non-zero coefficients of $F(z^{r_1}, \ldots, z^{r_s})\overline{F}(z^{-r_1}, \ldots, z^{-r_s})$ coincides with the system of all non-zero coefficients of $F(z_1, \ldots, z_s)\overline{F}(z_1^{-1}, \ldots, z_s^{-1})$. Hence if $F(z_1, \ldots, z_s)\overline{F}(z_1^{-1}, \ldots, z_s^{-1})$ has just three non-zero coefficients the same is true for $F(z^{r_1}, \ldots, z^{r_s})\overline{F}(z^{-r_1}, \ldots, z^{-r_s})$ and, by Lemma 1, (8) implies

$$M(F(z^{r_1}, \ldots, z^{r_s}))^2 + |a_1a_2|^2 M(F(z^{-r_1}, \ldots, z^{-r_s}))^{-2} = \|F(z^{r_1}, \ldots, z^{r_s})\|^2.$$

Passing to the limit as $q(\boldsymbol{r}) \to \infty$ we obtain, by Lemma 2, (1) with the equality sign.

If $F(z_1,\ldots,z_s)\overline{F}(z_1^{-1},\ldots,z_s^{-1})$ has at least four non-zero coefficients c_i $(1 \le i \le 4)$ and $q(\boldsymbol{r}) > 2\partial JF$, then there is an $m \ne 0, \pm\partial JF(z^{r_1},\ldots,z^{r_s})$ and an $i \le 4$ such that z^m occurs in $F(z^{r_1},\ldots,z^{r_s})\,\overline{F}(z^{-r_1},\ldots,z^{-r_s})$ with the coefficient c_i. Therefore, by Lemma 1, (8) implies

$$M(F(z^{r_1},\ldots,z^{r_s}))^2 + |a_1a_2|^2 M(F(z^{r_1},\ldots,z^{r_s}))^{-2} + \sqrt{(M(F(z^{r_1},\ldots,z^{r_s}))^2 + |a_1a_2|^2 M(F(z^{r_1},\ldots,z^{r_s}))^{-2})^2 + \min_{1\le i\le 4}|c_i|^2} \le 2\|F(z^{r_1},\ldots,z^{r_s})\|^2$$

and passing to the limit as $q(\boldsymbol{r}) \to \infty$ we obtain, by Lemma 2, (1) with the strict inequality sign.

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