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Burnside's uniformization

by

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To Ian Cassels on the occasion of his 75th birthday

1. Introduction. In 1893 the distinguished mathematician William Burnside (1852–1927) gave the first explicit uniformization of an algebraic equation of genus greater than unity [1]. This was the hyperelliptic equation

$$y^2 = x(x^4 - 1), \tag{1.1}$$

which has genus 2. This he accomplished by taking

$$x = \frac{\wp(\omega/2) - \wp(\omega)}{\wp(\omega'/2) - \wp(\omega)}, \tag{1.2}$$

where $\wp$ is the Weierstrass elliptic function with primitive periods 2ω and $2\omega'$. Further, he displayed y as a complicated quotient, whose numerator and denominator contained respectively five and four factors, each factor involving a value of $\wp$ or its derivative $\wp'$.

Burnside's work was based to some extent on results stated, but not proved in Klein and Fricke's treatise [2]. The object of the present paper is to examine his work closely, proving all results stated by him, and stating them in a form more readily appreciated, using theta functions in place of the Weierstrass function $\wp$.

THEOREM. *The equation* (1.1) *can be uniformized by taking*

$$x = -\vartheta_3(\tau/2)/\vartheta_4(\tau/2) \tag{1.3}$$

and

$$y = i\vartheta_3^{1/2}(\tau/2)\vartheta_2^2(\tau/2)\vartheta_4^{-5/2}(\tau/2). \tag{1.4}$$

These are elliptic modular functions belonging to a subgroup Γ of index 2 *in the principal congruence group $\Gamma(4)$ of level* 4.

It may be noted that we could simplify these results by omitting the factor i in (1.4) and replacing x by $-x$. Further (with this simplification), Jacobi's functions k and k' can be used to replace the theta functions, giving

$$x = \{k'(\tau/2)\}^{-1/2}, \qquad y = k(\tau/2)\{k'(\tau/2)\}^{-5/4}. \tag{1.5}$$

We shall make extensive use of results in Tannery and Molk's treatise [4], and so shall conform to the notation of that work by writing ω_1 and ω_3 in place of ω and ω', so that $\tau = \omega_3/\omega_1$ and $\operatorname{Im}\tau > 0$.

By (1.2) and Table XVI (p. 251) of [4], we have

$$x = \frac{\wp(\omega_1/2) - e_1}{\wp(\omega_3/2) - e_1} = \frac{\sqrt{e_1 - e_2}}{\sqrt{e_2 - e_3} - \sqrt{e_1 - e_2}}. \tag{1.6}$$

Formula (4) of Table XXXVI (p. 257) then gives

$$x = \frac{\vartheta_2^2 - \vartheta_3^2}{\vartheta_4^2}. \tag{1.7}$$

Here, as usual,

$$\vartheta_3 = \vartheta_3(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau}, \tag{1.8}$$

with ϑ_2 and ϑ_4 defined similarly. This simplifies, by Theorem 7.1.8 of [3], to give (1.3), and (1.1) then leads to (1.4), since $\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4$.

2. The functions f_{ij}. We now prepare to investigate the groups to which x and y belong. Write

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2.1}$$

for any matrix belonging to the modular group $\Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$, and put, in particular,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}. \tag{2.2}$$

Then $W = UVU$ and we write

$$T\tau = \frac{a\tau + b}{c\tau + d}. \tag{2.3}$$

We also write

$$J = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \tag{2.4}$$

so that

$$J\tau = \tau/2 \quad \text{and} \quad JTJ^{-1} = \begin{bmatrix} a & b/2 \\ 2c & d \end{bmatrix}. \tag{2.5}$$

The stroke operator $|$ is, as usual, defined by

$$f(\tau) \mid T = f(T\tau), \qquad f(\tau) \mid J = f(\tau/2). \tag{2.6}$$

Write also, for any positive integer n,

$$\Gamma(n) = \{T \in \Gamma(1) : T \equiv I \pmod{n}\}, \tag{2.7}$$

the principal homogeneous congruence subgroup of level n. The only values of n that we shall need are $n = 2, 4$ and 8.

Let s be any real number and write

$$\varrho = \exp(\pi i s/4). \tag{2.8}$$

The only values of s that will arise are $s = 1/2$ and $s = 2$. Put

$$f_{ij}(\tau) = \{\vartheta_i(\tau)/\vartheta_j(\tau)\}^s, \tag{2.9}$$

where i and j are different integers in the interval [2, 4].

The results stated in the following lemma can be found in [4], or by use of Theorem 7.1.2 of [3].

Lemma.

$$\begin{array}{lll}
f_{23} \mid U = \varrho f_{24}, & f_{34} \mid U = f_{43}, & f_{24} \mid U = \varrho f_{23}, \\
f_{23} \mid U^2 = \varrho^2 f_{23}, & f_{34} \mid U^2 = f_{34}, & f_{24} \mid U^2 = \varrho^2 f_{24}, \\
f_{23} \mid V = f_{43}, & f_{34} \mid V = f_{32}, & f_{24} \mid V = f_{42}, \\
f_{23} \mid W = f_{32}, & f_{34} \mid W = \varrho f_{24}, & f_{24} \mid W = \varrho f_{34}, \\
f_{23} \mid W^2 = f_{23}, & f_{34} \mid W^2 = \varrho^2 f_{34}, & f_{24} \mid W^2 = \varrho^2 f_{24}.
\end{array}$$

Since $\Gamma(2)$ is generated by U^2 and W^2, it follows that f_{ij} is a modular function belonging to $\Gamma(2)$ with a certain multiplier system. However, we are more interested in $\Gamma(4)$.

3. The action of the group $\Gamma(4)$. The group $\Gamma(4)$ is of rank 5 and is generated by the following matrices (see p. 355 of [2]):

$$v_1 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -3 & 4 \\ -4 & 5 \end{bmatrix}, \tag{3.1}$$

$$v_4 = \begin{bmatrix} -7 & 16 \\ -4 & 9 \end{bmatrix}, \quad v_5 = \begin{bmatrix} -11 & 36 \\ -4 & 13 \end{bmatrix}, \tag{3.2}$$

which are expressible as follows:

$$v_1 = U^4, \quad v_2 = W^{-4}, \quad v_3 = WU^4W^{-1}, \tag{3.3}$$

$$v_4 = U^2W^{-4}U^{-2}, \quad v_5 = U^3W^{-4}U^{-3}. \tag{3.4}$$

Note that

$$v_2 \equiv v_4 \pmod{8}, \quad v_3 \equiv v_5 \pmod{8}, \tag{3.5}$$

and

$$v_n^2 \equiv I \pmod{8} \quad (n = 1, \ldots, 5). \tag{3.6}$$

Since the independent variable of the theta functions in (1.3) and (1.4) is $\tau/2$ and not τ, we need to evaluate the following matrices (see (2.5)):

$$V_1 = Jv_1J^{-1} = U^2, \quad V_2 = Jv_2J^{-1} = W^{-8}, \tag{3.7}$$

$$V_3 = Jv_3J^{-1} = W^2U^2W^{-2}, \quad V_4 = Jv_4J^{-1} = UW^{-8}U^{-1}, \tag{3.8}$$

and, surprisingly,

$$V_5 = Jv_5J^{-1} = UW^2U^2W^{-2}U^{-1}. \tag{3.9}$$

The case $s = 1/2$. In this case we have $\varrho^8 = -1$, and write

$$g(\tau) = f_{34}(\tau/2) = \{\vartheta_3(\tau/2)/\vartheta_4(\tau/2)\}^{1/2}, \tag{3.10}$$

so that $g = f_{34} \mid J$ and we have

$$g \mid v_1 = f_{34} \mid Jv_1 = f_{34} \mid V_1J = f_{34}U^2J = g, \tag{3.11}$$

and we find, similarly, that

$$g \mid v_2 = -g, \quad g \mid v_3 = g, \quad g \mid v_4 = -g, \tag{3.12}$$

and

$$\begin{aligned} g \mid v_5 &= f_{34} \mid Jv_5 = f_{34} \mid V_5J = f_{34} \mid UW^2U^2W^{-2}U^{-1}J \\ &= f_{43} \mid W^2U^2W^{-2}U^{-1}J = \overline{\varrho}^2 f_{43} \mid U^2W^{-2}U^{-1}J \\ &= \overline{\varrho}^2 f_{43} \mid W^{-2}U^{-1}J = f_{43} \mid U^{-1}J = f_{34} \mid J = g. \end{aligned} \tag{3.13}$$

In particular, for $T \in \Gamma(4)$,

$$x \mid T = -g^2 \mid T = -g^2 = x, \tag{3.14}$$

so that x is a modular function for the group $\Gamma(4)$ with multiplier system 1.

The case $s = 2$. Take

$$h = \vartheta_2^2(\tau/2)/\vartheta_4^2(\tau/2) = f_{24}(\tau) \mid J, \tag{3.15}$$

so that $\varrho = i$. We find that

$$h \mid v_1 = f_{24} \mid Jv_1 = f_{24} \mid V_1J = f_{24} \mid U^2J = -h, \tag{3.16}$$

$$h \mid v_2 = f_{24} \mid Jv_2 = f_{24} \mid W^{-8}J = f_{24} \mid J = h, \tag{3.17}$$

$$h \mid v_3 = f_{24} \mid Jv_3 = f_{24} \mid W^2U^2W^{-2}J = -h, \tag{3.18}$$

$$h \mid v_4 = f_{24} \mid Jv_4 = f_{24} \mid UW^{-8}U^{-1}J = h, \tag{3.19}$$

$$\begin{aligned} h \mid v_5 &= f_{24} \mid Jv_5 = f_{24} \mid UW^2U^2W^{-2}U^{-1}J \\ &= \varrho f_{23} \mid W^2U^2W^{-2}U^{-1}J = \varrho f_{23} \mid U^2W^{-2}J \\ &= \varrho^3 f_{23} \mid W^{-2}U^{-1}J = \varrho^3 f_{23} \mid U^{-1}J = \varrho^2 f_{24} \mid J = -h. \end{aligned} \tag{3.20}$$

It follows that

$$gh \mid v_n = -gh \quad (n = 1, \ldots, 5). \tag{3.21}$$

Accordingly, $y = igh$ is a modular function belonging to $\Gamma(4)$ with a multiplier system χ such that

(3.22) $$\chi(v_n) = -1 \quad (n = 1, \ldots, 5).$$

Define

(3.23) $$\Gamma = \{T \in \Gamma(4) : \chi(T) = 1\}.$$

Hence Γ is a subgroup of $\Gamma(4)$ of index 2. It has index 96 in $\Gamma(1)$ and contains $\Gamma(8)$ as a subgroup of index 4.

Now any element T of Γ is a product of elements of the form

(3.24) $$v_i v_j, \quad v_i^{-1} v_j, \quad v_i v_j^{-1}, \quad v_i^{-1} v_j^{-1}.$$

Modulo 8 each of these is congruent to $v_i v_j$, and by (3.5) and (3.6) each is therefore congruent modulo 8 to one of

(3.25) $$I, \quad v_1 v_2 = A, \quad v_2 v_3 = B, \quad v_1 v_3 = C,$$

where

(3.26) $$A \equiv \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}, \quad B \equiv \begin{bmatrix} 5 & 4 \\ 0 & 5 \end{bmatrix}, \quad C \equiv \begin{bmatrix} 5 & 0 \\ 4 & 5 \end{bmatrix}$$

modulo 8; note that $v_i v_j \equiv v_j v_i \pmod 8$. These four elements I, A, B, C, constitute the four-group F modulo 8, and it is easily seen that

(3.27) $$\Gamma = \Gamma(8)F \quad \text{and} \quad \Gamma(4) = \Gamma \cup \Gamma U^4.$$

Accordingly, both x and y are modular functions (with multiplier system 1) belonging to the group Γ, which consists of all matrices $T \in \Gamma(1)$ that satisfy

(3.28) $$T \equiv I,\ A,\ B \text{ or } C \pmod 8,$$

as stated on p. 652 of [2].

References

[1] W. Burnside, *Note on the equation* $y^2 = x(x^4 - 1)$, Proc. London Math. Soc. 24 (1893), 17–20.

[2] F. Klein und R. Fricke, *Vorlesungen über die Theorie der elliptischen Modulfunktionen*, Band 1, Leipzig, 1890.

[3] R. A. Rankin, *Modular Forms and Functions*, Cambridge University Press, 1977.

[4] J. Tannery et L. Molk, *Eléments de la Théorie des Fonctions*, Tome 1, Paris, 1897.

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