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On limit distribution of the Matsumoto zeta-function

by

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In honour of Professor J. W. S. Cassels on his 75th birthday

In [5] K. Matsumoto considered a certain zeta-function $\varphi(s)$ and proved limit theorems in the complex plane $\mathbb{C}$ for it. Let

$$A_m(x) = \prod_{j=1}^{g(m)} (1 - a_m^{(j)} x^{f(j,m)}).$$

Here $g(m)$ is a positive integer, $a_m^{(j)}$ are complex numbers and $f(j,m)$ natural numbers, $1 \le j \le g(m)$, $m \in \mathbb{N}$, where $\mathbb{N}$ stands for the set of all natural numbers. Moreover, let $s = \sigma + it$ be a complex variable, and let p_m denote the mth prime number. Define

$$\varphi(s) = \prod_{m=1}^{\infty} A_m^{-1}(p_m^{-s}).$$

In [5] it is assumed that

$$g(m) \le c p_m^{\alpha}, \quad |a_m^{(j)}| \le p_m^{\beta}$$

with a positive constant c and non-negative constants α and β. The paper [5] contains two limit theorems for $\log \varphi(\sigma_0 + it)$. The first of them examines the case $\sigma_0 > \alpha + \beta + 1$, and the second, under some additional conditions on $\varphi(s)$, concerns the case $\varrho < \sigma_0 \le \alpha + \beta + 1$, where ϱ is a constant with $\alpha + \beta + 1/2 \le \varrho < \alpha + \beta + 1$. It is an interesting problem to study the limit distribution of $\varphi(s)$. This was done in [6], where the upper and lower bounds for this limit distribution were obtained.

In [3] we have given a generalization of the results from [5]. We have proved two functional limit theorems with weight for $\varphi(s)$. Let G be a region of the complex plane, and let $\mathbb{C}_\infty$ stand for the Riemann sphere with spherical metric d. Denote by $H(G)$ the space of analytic functions $f : G \to (\mathbb{C}_\infty, d)$ equipped with the topology of uniform convergence on compacta. Moreover, let T_0 be a fixed positive number, and let $w(\tau)$ be a positive function of bounded variation on $[T_0, \infty)$. Let $D_1 = \{s \in \mathbb{C} : \sigma > \alpha + \beta + 1\}$. Define

$$U = U(T, w) = \int_{T_0}^{T} w(\tau)\, d\tau,$$

suppose that $\lim_{T\to\infty} U(T, w) = \infty$ and define a probability measure

$$P_{T,w}(A) = \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau:\varphi(s+i\tau)\in A\}}\, d\tau, \qquad A \in \mathcal{B}(H(D_1)).$$

Here I_A denotes the indicator function of the set A, and $\mathcal{B}(S)$ stands for the class of Borel sets of the space S. In [3] the following assertion has been obtained.

Theorem A. *There is a probability measure P_w on $(H(D_1), \mathcal{B}(H(D_1)))$ such that the measure $P_{T,w}$ converges weakly to P_w as $T \to \infty$.*

Theorem 2 of [3] concerns the behaviour of $\varphi(s)$ in the half-plane $\sigma > \alpha + \beta + 1/2$.

It is of interest to find the explicit form of the limit measure in Theorem A. B. Bagchi [1] applied ergodic theory to identify limit measures. Unfortunately, we do not know an ergodic theorem with weight $w(\tau)$. Therefore we must introduce some additional condition on the function $w(\tau)$. Denote by $E\xi$ the mean of the random variable ξ. Let $X(\tau, \omega)$ be an ergodic process, $\tau \in \mathbb{R}$, $\omega \in \widetilde{\Omega}$, with $E|X(\tau, \omega)| < \infty$, and with sample paths integrable almost surely in the Riemann sense over every finite interval. Suppose that the function $w(\tau)$ satisfies the relation

$$\frac{1}{U} \int_{T_0}^{T} w(\tau) X(\tau, \omega)\, d\tau = EX(0, \omega) + o(1) \tag{1}$$

almost surely as $T \to \infty$. The latter relation is a generalization of the classical Birkhoff–Khinchin theorem which asserts that

$$\lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} X(\tau, \omega)\, d\tau = EX(0, \omega)$$

almost surely.

Denote by γ the unit circle on the complex plane, that is, $\gamma = \{s \in \mathbb{C} : |s| = 1\}$. Moreover, let

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. With the product topology and pointwise multiplication the infinite-dimensional torus Ω is a compact topological group. Therefore there exists a probability Haar measure m on $(\Omega, \mathcal{B}(\Omega))$. Thus we obtain the probability space $(\Omega, \mathcal{B}(\Omega), m)$. Let $\omega(p)$ stand for the projection of $\omega \in \Omega$ to the coordinate space γ_p. Then, setting

$$\omega(k) = \prod_{p^\alpha \parallel k} \omega^\alpha(p),$$

where $p^\alpha \parallel k$ means that $p^\alpha \mid k$ but $p^{\alpha+1} \nmid k$, we obtain an extension of the function $\omega(p)$ to the set of all natural numbers as a completely multiplicative unimodular function.

As noted in [5], $\varphi(s)$ is a holomorphic function in the half-plane $\sigma > \alpha+\beta+1$ with no zeros, and it is represented there by an absolutely convergent Dirichlet series

$$\varphi(s) = \sum_{k=1}^{\infty} \frac{b(k)}{k^s},$$

where $b(k) = Bk^{\alpha+\beta}$. Let $D = \{s \in \mathbb{C} : \sigma > \alpha + \beta + 1/2\}$, and for $s \in D$ and $\omega \in \Omega$, set

$$\varphi(s, \omega) = \sum_{k=1}^{\infty} \frac{b(k)\omega(k)}{k^s}.$$

Since, as $N \to \infty$,

$$\sum_{k \le N} |b(k)|^2 = BN^{2(\alpha+\beta+1/2)},$$

by Lemma 3.4.3 of [1] the series $\varphi(s, \omega)$ converges uniformly on compact subsets of D, and $\varphi(s, \omega)$ is an $H(D)$-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m)$. Let P_φ denote the distribution of $\varphi(s, \omega)$, and let $P_{1,\varphi}$ be the restriction of P_φ to $(H(D_1), \mathcal{B}(H(D_1)))$. The aim of this paper is to prove the following result.

THEOREM. *Under the assumption* (1) *the measure $P_{T,w}$ converges weakly to $P_{1,\varphi}$ as $T \to \infty$.*

This theorem shows that the limit measure is independent of the weight function $w(\tau)$. For its proof we will apply the method of [1].

First we state a lemma for trigonometric polynomials

$$p_n(s) = \sum_{k=1}^{n} \frac{a(k)}{k^s}, \qquad p_n(s, g) = \sum_{k=1}^{n} \frac{a(k)g(k)}{k^s},$$

where $g(k)$ is a unimodular completely multiplicative function. Let G be a region in $\mathbb{C}$, and

$$P_{T,p_n,w}(A) = \frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau : p_n(s+i\tau)\in A\}}\,d\tau, \qquad A \in \mathcal{B}(H(G)),$$

$$\widetilde{P}_{T,p_n,w}(A) = \frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau : p_n(s+i\tau,g)\in A\}}\,d\tau, \qquad A \in \mathcal{B}(H(G)).$$

LEMMA 1. *The probability measures* $P_{T,p_n,w}$ *and* $\widetilde{P}_{T,p_n,w}$ *both converge weakly to the same measure as* $T \to \infty$.

Proof. This is Lemma 2 of [4].

Now we will prove a similar assertion to Lemma 1 for the function $\varphi(s)$ in D_1. For convenience of the reader we recall some probabilistic results.

Let S be a separable metric space with a metric ϱ, and let Y_n, X_{1n}, $X_{2n}, \ldots$ be S-valued random elements defined on $(\Omega_1, \mathcal{F}, \mathbb{P})$. The following assertion is Theorem 4.2 of [2].

LEMMA 2. *Suppose that* $X_{kn} \xrightarrow[n\to\infty]{\mathcal{D}} X_k$ *for each* k *and also* $X_k \xrightarrow[k\to\infty]{\mathcal{D}} X$. *If for every* $\varepsilon > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}(\varrho(X_{kn}, Y_n) \ge \varepsilon) = 0,$$

then $Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X$.

Now let P_n and P be probability measures on $(S, \mathcal{B}(S))$.

LEMMA 3. P_n *converges weakly to* P *as* $n \to \infty$ *if and only if any subsequence* $\{P_{n'}\}$ *contains another subsequence* $\{P_{n''}\}$ *such that* $P_{n''} \to P$ *as* $n'' \to \infty$.

Proof. This is Theorem 2.3 of [2].

Let S and S_1 be two metric spaces, and let $h : S \to S_1$ be a measurable function. Then every probability measure P on $(S, \mathcal{B}(S))$ induces a unique probability measure Ph^{-1} on $(S_1, \mathcal{B}(S_1))$ defined by $Ph^{-1}(A) = P(h^{-1}A)$, $A \in \mathcal{B}(S_1)$.

LEMMA 4. *Let* $h : S \to S_1$ *be a continuous function, and let* P_n *and* P *be probability measures on* $(S, \mathcal{B}(S))$. *Suppose that* P_n *converges weakly to* P *as* $n \to \infty$. *Then* $P_n h^{-1}$ *converges weakly to* Ph^{-1} *as* $n \to \infty$.

Proof. This is a particular case of Theorem 5.1 of [2].

For $\omega \in \Omega$, let

$$\widetilde{P}_{T,w}(A) = \frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau : \varphi(s+i\tau,\omega)\in A\}}\,d\tau, \qquad A \in \mathcal{B}(H(D_1)).$$

LEMMA 5. *There exists a probability measure P_w on $(H(D_1), \mathcal{B}(H(D_1)))$ such that the measures $P_{T,w}$ and $\widetilde{P}_{T,w}$ both converge weakly to P_w as $T \to \infty$.*

Proof. Let

$$\varphi_n(s) = \sum_{k=1}^{n} \frac{b(k)}{k^s}$$

and, for $\omega \in \Omega$,

$$\varphi_n(s, \omega) = \sum_{k=1}^{n} \frac{b(k)\omega(k)}{k^s}.$$

Define two probability measures

$$P_{T,\varphi_n,w}(A) = \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau : \varphi_n(s+i\tau) \in A\}}\, d\tau, \qquad A \in \mathcal{B}(H(D_1)),$$

$$\widetilde{P}_{T,\varphi_n,w}(A) = \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau : \varphi_n(s+i\tau,\omega) \in A\}}\, d\tau, \qquad A \in \mathcal{B}(H(D_1)).$$

Then by Lemma 1 both $P_{T,\varphi_n,w}$ and $\widetilde{P}_{T,\varphi_n,w}$ converge weakly to the same measure $P_{\varphi_n,w}$, say, as $T \to \infty$. We will prove that the family $\{P_{\varphi_n,w} : n \in \mathbb{N}\}$ of probability measures is tight. Let η be a random variable on $(\Omega_1, \mathcal{F}, \mathbb{P})$ with distribution

$$\mathbb{P}(\eta \in A) = \frac{1}{U} \int_{T_0}^{T} w(\tau) I_A \, d\tau, \qquad A \in \mathcal{B}(\mathbb{R}).$$

We set $X_{T,\varphi_n}(s) = \varphi_n(s + i\eta)$. Then

$$X_{T,\varphi_n} \xrightarrow[T \to \infty]{\mathcal{D}} X_{\varphi_n}, \tag{2}$$

where X_{φ_n} is an $H(D_1)$-valued random element with distribution $P_{\varphi_n,w}$. Since, for $\sigma > \alpha+\beta+1$, the Dirichlet series for $\varphi(s)$ is absolutely convergent, it follows that

$$\sup_{n\ge1} \limsup_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) \sup_{s \in K_l} |\varphi_n(s+i\tau)|\, d\tau \le R_l < \infty,$$

where $\{K_l\}$ is a sequence of compact subsets of D_1 such that $D_1 = \bigcup_{l=1}^{\infty} K_l$, $K_l \subset K_{l+1}$, $l = 1, 2, \ldots$, and if K is a compact set and $K \subset D_1$ then $K \subseteq K_l$ for some l. Let $\varepsilon > 0$. Then, setting $M_l = R_l 2^l \varepsilon^{-1}$, we find that

$$\begin{aligned} &\limsup_{T\to\infty} \mathbb{P}(\sup_{s \in K_l} |X_{T,\varphi_n}(s)| > M_l) \\ &\quad \le \frac{1}{M_l} \sup_{n\ge1} \limsup_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) \sup_{s\in K_l} |\varphi_n(s+i\tau)|\, d\tau \le \varepsilon/2^l \end{aligned} \tag{3}$$

for all $l \in \mathbb{N}$. Define $h : H(D_1) \to \mathbb{R}$ by

$$h(f) = \sup_{s \in K_l} |f(s)|, \qquad f \in H(D_1).$$

Then h is continuous, and thus by (2) and Lemma 4,

$$\sup_{s \in K_l} |X_{T,\varphi_n}(s)| \xrightarrow[T\to\infty]{\mathcal{D}} \sup_{s \in K_l} |X_{\varphi_n}(s)|.$$

This together with (3) yields

$$\mathbb{P}(\sup_{s \in K_l} |X_{\varphi_n}(s)| > M_l) \le \varepsilon/2^l \tag{4}$$

for all $l \in \mathbb{N}$. Define

$$H_\varepsilon = \{f \in H(D_1) : \sup_{s \in K_l} |f(s)| \le M_l,\ l \ge 1\}.$$

Then H_ε is a set of functions analytic on D_1 and uniformly bounded on every compact $K \subset D_1$, and therefore, by the compactness principle, it is a compact subset of $H(D_1)$. The inequality (4) implies $\mathbb{P}(X_{\varphi_n}(s) \in H_\varepsilon) \ge 1-\varepsilon$ for all $n \ge 1$, or, since $P_{\varphi_n,w}$ is the distribution of X_{φ_n}, $P_{\varphi_n,w}(H_\varepsilon) \ge 1-\varepsilon$ for all $n \ge 1$. So we have proved that the family $\{P_{\varphi_n,w}\}$ is tight. Hence by the Prokhorov theorem it is relatively compact.

Let

$$\varrho_1(f_1, f_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\varrho_{1,l}(f_1, f_2)}{1 + \varrho_{1,l}(f_1, f_2)},$$

where

$$\varrho_{1,l}(f_1, f_2) = \sup_{s \in K_l} |f_1(s) - f_2(s)|, \qquad f_1, f_2 \in H(D_1).$$

Then $\varrho_1(f_1, f_2)$ is a metric on $H(D_1)$. Since $\varphi_n(s) \to \varphi(s)$ as $n \to \infty$ uniformly on compact subsets of D_1, we have, for every $\varepsilon > 0$,

$$\begin{aligned} &\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau : \varrho_1(\varphi(s+i\tau), \varphi_n(s+i\tau)) \ge \varepsilon\}}\, d\tau \\ &\le \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon U} \int_{T_0}^{T} w(\tau) \varrho_1(\varphi(s+i\tau), \varphi_n(s+i\tau))\, d\tau = 0. \end{aligned} \tag{5}$$

Now set

$$X_T(s) = \varphi(s + i\eta).$$

Then (5) can be written as

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}(\varrho_1(X_{T,\varphi_n}(s), X_T(s)) \ge \varepsilon) = 0. \tag{6}$$

Since the family $\{P_{\varphi_n,w}\}$ is relatively compact, there exists a subsequence $\{P_{\varphi_{n'},w}\}$ which converges weakly to P_w, say, as $n\to\infty$. Then, obviously,

$$X_{\varphi_{n'}} \xrightarrow[n'\to\infty]{\mathcal{D}} P_w.$$

Hence and from (6) and (2), using Lemma 2, we obtain

$$X_T \xrightarrow[T\to\infty]{\mathcal{D}} P_w. \tag{7}$$

This means that there is a probability measure P_w on $(H(D_1),\mathcal{B}(H(D_1)))$ such that $P_{T,w}$ converges weakly to P_w as $T\to\infty$. Moreover, (7) shows that P_w is independent of the choice of the subsequence $\{P_{n',w}\}$. Thus by Lemma 3,

$$X_{\varphi_n} \xrightarrow[n\to\infty]{\mathcal{D}} P_w. \tag{8}$$

Now, for $\omega\in\Omega$, let

$$\widetilde{X}_{T,\varphi_n}(s,\omega)=\varphi_n(s+i\eta,\omega),\qquad \widetilde{X}_T(s,\omega)=\varphi(s+i\eta,\omega).$$

Then, reasoning as above and using (8), we conclude that the measure $\widetilde{P}_{T,w}$ also converges weakly to P_w as $T\to\infty$.

We precede the proof of the Theorem by some remarks on ergodic theory.

Let $\mathcal{P}$ denote the set of all prime numbers, and let $a_\tau=\{p^{-i\tau}:p\in\mathcal{P}\}$ for $\tau\in\mathbb{R}$. Then $\{a_\tau:\tau\in\mathbb{R}\}$ is a one-parameter group. Define a one-parameter family $\{g_\tau:\tau\in\mathbb{R}\}$ of measurable transformations of Ω by $g_\tau(\omega)=a_\tau\omega$ for $\omega\in\Omega$. A set $A\in\mathcal{B}(\Omega)$ is called *invariant* with respect to the group $\{g_\tau:\tau\in\mathbb{R}\}$ if for each τ the sets A and $A_\tau=g_\tau(A)$ differ by a set of zero m-measure. In other words, $m(A\bigtriangleup A_\tau)=0$, where $\bigtriangleup$ denotes a symmetric difference. All invariant sets form a σ-field. A one-parameter group $\{g_\tau:\tau\in\mathbb{R}\}$ is called *ergodic* if its σ-field of invariant sets consists only of sets having m-measure equal to 0 or 1.

LEMMA 6. *The one-parameter group $\{g_\tau:\tau\in\mathbb{R}\}$ is ergodic.*

Proof. This is Lemma 3.4.2 of [1].

Proof of Theorem. By Lemma 5 the measures $P_{T,w}$ and $\widetilde{P}_{T,w}$ converge weakly to the same measure P_w as $T\to\infty$. It remains to prove that $P_w=P_{1,\varphi}$.

Let $A\in\mathcal{B}(H(D_1))$ be a continuity set of P_w. Then, by Lemma 5,

$$\lim_{T\to\infty}\frac{1}{U}\int_{T_0}^{T} w(\tau)I_{\{\tau:\varphi(s+i\tau,\omega)\in A\}}\,d\tau=P_w(A). \tag{9}$$

Fix A and define a random variable θ on $(\Omega, \mathcal{B}(\Omega))$ by

$$\theta(\omega) = \begin{cases} 1 & \text{if } \varphi(s,\omega) \in A, \\ 0 & \text{if } \varphi(s,\omega) \notin A. \end{cases}$$

Clearly,

$$E(\theta) = \int_{\Omega} \theta\, dm = m\{\omega : \varphi(s,\omega) \in A\} = P_{1,\varphi}(A) < \infty. \tag{10}$$

It follows from Lemma 6 that $\theta(g_\tau(\omega))$ is an ergodic process. Therefore, by (1)

$$\lim_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(\tau)\theta(g_\tau(\omega))\, d\tau = E(\theta) \tag{11}$$

for almost all $\omega \in \Omega$. On the other hand,

$$\begin{aligned} \frac{1}{U} \int_{T_0}^{T} w(\tau)\theta(g_\tau(\omega))\, d\tau &= \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau : \varphi(s, g_\tau(\omega)) \in A\}}\, d\tau \\ &= \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau : \varphi(s+i\tau, \omega) \in A\}}\, d\tau. \end{aligned}$$

From this, (10), and (11) we obtain

$$\lim_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau : \varphi(s+i\tau,\omega) \in A\}}\, d\tau = P_{1,\varphi}(A)$$

for almost all $\omega \in \Omega$. Thus in view of (9), $P_w(A) = P_{1,\varphi}(A)$ for any continuity set of the measure P_w. This implies that $P_w(A) = P_{1,\varphi}(A)$ for all $A \in \mathcal{B}(H(D_1))$, since the continuity sets constitute a determining class. The Theorem is proved.

Now let P denote the limit measure in the theorem of Matsumoto [5], i.e., for $\sigma_0 > \alpha + \beta + 1$, $\lim_{T\to\infty}(1/T)$ meas $\{t \in [0,T] : \log \varphi(\sigma_0 + it) \in A\} = P(A)$ for all continuity sets A of P. Our Theorem allows us to identify the measure P.

Corollary. *We have*, *for* $\sigma_0 > \alpha + \beta + 1$,

$$P(A) = m(\log \varphi(\sigma_0, \omega) \in A), \qquad A \in \mathcal{B}(\mathbb{C}).$$

Proof. Let $h : H(D_1) \to \mathbb{C}$ be given by $h(f) = \log f(\sigma_0)$ for $f \in H(D_1)$. Then the weak convergence of the measure $P_{T,w}$ to $P_{1,\varphi}$ (in this case $w(\tau) \equiv 1$) together with Lemma 4 implies the weak convergence of $P_{T,1}h^{-1}$ to $P_{1,\varphi}h^{-1}$. This proves the Corollary.

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