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On the greatest prime factor of $(ab+1)(ac+1)(bc+1)$

by

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Dedicated to Professor J. W. S. Cassels
on the occasion of his seventy-fifth birthday

1. Introduction. For any integer n larger than one let $P(n)$ denote the greatest prime factor of n. In [3], Győry, Sárközy and Stewart conjectured that if a, b and c denote distinct positive integers then

$$P((ab+1)(ac+1)(bc+1)) \to \infty \tag{1}$$

as the maximum of a, b and c tends to infinity. We shall show that (1) holds provided that

$$\frac{\log a}{\log(c+1)} \to \infty.$$

This is a consequence of the following result.

Theorem 1. *Let a, b and c be positive integers with $a \ge b > c$. There exists an effectively computable positive number C_0 such that*

$$P((ab+1)(ac+1)(bc+1)) > C_0 \log(\log a/\log(c+1)). \tag{2}$$

Recently, Győry [2] has proved that (1) holds provided that at least one of $P(a)$, $P(b)$, $P(c)$, $P(a/b)$, $P(a/c)$ and $P(b/c)$ is bounded. While we have not been able to prove (1) we have been able to prove that if a, b, c and d are positive integers with $a \ne d$ and $b \ne c$ then

$$P((ab+1)(ac+1)(bd+1)(cd+1)) \to \infty$$

as the maximum of a, b, c and d tends to infinity. Notice, by symmetry, that there is no loss of generality in assuming that $a \ge b > c$ and that $a > d$.

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In fact, we are able to give an effective lower bound for the greatest prime factor of $(ab+1)(ac+1)(bd+1)(cd+1)$ in terms of a.

THEOREM 2. *Let a, b, c and d denote positive integers with $a \ge b > c$ and $a > d$. There exists an effectively computable positive number C_1 such that*

$$P((ab+1)(ac+1)(bd+1)(cd+1)) > C_1 \log\log a. \tag{3}$$

The proofs of Theorems 1 and 2 depend upon estimates for linear forms in the logarithms of algebraic numbers. We are able to estimate the greatest prime factor of more general polynomials than those considered in Theorems 1 and 2. To this end we make the following definition.

DEFINITION. Let n and t be positive integers with $t \ge 2$. $\{L, M\}$ is said to be a *balanced pair of t-sets of a set* $\{h_1, \ldots, h_n\}$ if L and M are disjoint sets of t-element subsets of $\{h_1, \ldots, h_n\}$ and each element h_i, with $1 \le i \le n$, occurs in some element of L and, further, occurs in elements of L the same number of times it occurs in elements of M.

Thus, for example, if $L = \{\{1,2\},\{3,4\}\}$ and $M = \{\{1,3\},\{2,4\}\}$ then $\{L, M\}$ is a balanced pair of 2-sets of $\{1,2,3,4\}$.

THEOREM 3. *Let n and t be integers with $2 \le t < n$. Suppose that $\{L, M\}$ is a balanced pair of t-sets of $\{1, \ldots, n\}$. Let $a_1, \ldots, a_n$ denote positive integers for which*

$$\prod_{\{i_1,\ldots,i_t\}\in L} (a_{i_1}\ldots a_{i_t}+1) \ne \prod_{\{i_1,\ldots,i_t\}\in M} (a_{i_1}\ldots a_{i_t}+1). \tag{4}$$

Put

$$a^+ = \max\{3, a_1, \ldots, a_n\} \quad \textit{and} \quad a^- = \min_{\{i_1,\ldots,i_t\}\in L\cup M}\{a_{i_1}\ldots a_{i_t}\}.$$

Then

$$P\Big(\prod_{\{i_1,\ldots,i_t\}\in L\cup M} (a_{i_1}\ldots a_{i_t}+1)\Big) \to \infty \tag{5}$$

as a^- tends to infinity. Further, there exists a positive number C_2, which is effectively computable in terms of t and the cardinality of L, such that

$$P\Big(\prod_{\{i_1,\ldots,i_t\}\in L\cup M} (a_{i_1}\ldots a_{i_t}+1)\Big) > C_2 \log\left(\frac{\log a^-}{\log\log a^+}\right). \tag{6}$$

To prove (5) we shall appeal to a theorem on S-unit equations due to van der Poorten and Schlickewei [4, 5] and independently to Evertse [1]. This result in turn depends upon a p-adic version of Schmidt's Subspace Theorem due to Schlickewei [6]. As a consequence we are not able to give an effective lower bound for the quantity on the left hand side of (5). To prove (6) we shall appeal to a version of Baker's estimates for linear forms in logarithms due to Waldschmidt [7].

Let n be an even integer with $n \geq 4$. Let $L = \{(2i, 2i-1) | i = 1, \ldots, n/2\}$ and $M = \{(1, n)\} \cup \{(2i, 2i+1) \mid i = 1, \ldots, n/2-1\}$. Notice that $\{L, M\}$ is a balanced pair of 2-sets of $\{1, \ldots, n\}$ and so the following result is a direct consequence of Theorem 3.

Corollary 1. *Let n be an even integer with $n \geq 4$. Let $a_1, \ldots, a_n$ be positive integers for which*

$$\prod_{i=1}^{n/2}(a_{2i}a_{2i-1}+1) \neq \prod_{i=1}^{n/2}(a_{2i}a_{2i+1}+1)$$

with the convention that $a_{n+1} = a_1$. Then

$$P\Big(\prod_{i=1}^{n}(a_i a_{i+1}+1)\Big) \to \infty \quad as \quad \min_i(a_i a_{i+1}) \to \infty.$$

Another consequence of Theorem 3 is the following.

Corollary 2. *Let a, b, c, d and e be positive integers with*

$$(ab+1)(ac+1)(de+1) \neq (ad+1)(ae+1)(bc+1).$$

Then

$$P((ab+1)(ac+1)(ad+1)(ae+1)(bc+1)(de+1)) \to \infty$$

as $\min(b, c, d, e) \to \infty$.

Finally we mention a result which comes from applying Theorem 3 with a certain balanced pair of 3-sets of $\{1, \ldots, 6\}$.

Corollary 3. *Let a, b, c, d, e and f be positive integers with*

$$(abc+1)(cde+1)(aef+1) \neq (adf+1)(ace+1)(bce+1).$$

Then

$$P((abc+1)(ace+1)(adf+1)(aef+1)(bce+1)(cde+1)) \to \infty$$

as $\min(a, e) \to \infty$.

2. Preliminary lemmas. For any rational number x we may write $x = p/q$ with p and q coprime integers. We define the height of x to be the maximum of $|p|$ and $|q|$. Let $a_1, \ldots, a_n$ be rational numbers with heights at most $A_1, \ldots, A_n$ respectively. We shall suppose that $A_i \geq 4$ for $i = 1, \ldots, n$. Next let $b_1, \ldots, b_n$ be rational integers. Suppose that B and B_n are positive real numbers with

$$B \geq \max_{1 \leq j \leq n-1} |b_j| \quad \text{and} \quad B_n \geq \max(3, |b_n|).$$

Put

$$\Lambda = b_1 \log a_1 + \ldots + b_n \log a_n,$$

where log denotes the principal branch of the logarithm.

LEMMA 1. *There exists an effectively computable positive number C_3 such that if $\Lambda \neq 0$ then*

$$|\Lambda| > \exp\left(-C_3 n^{4n} \log A_1 \ldots \log A_n \log\left(B_n + \frac{B}{\log A_n}\right)\right).$$

Proof. This follows from Corollaire 10.1 of Waldschmidt [7]. Waldschmidt proved this result under the assumption that $b_n \neq 0$. If $b_n = 0$ then we apply the same theorem with b_n replaced by b_j where j is the largest integer for which $b_j \neq 0$. Notice that $j \geq 1$ since $\Lambda \neq 0$. Since $\log A_n \log(3 + B/(\log A_n))$ is larger than $\frac{1}{2}\log B$ the result follows.

We shall employ Lemma 1 in the following manner. Let r be a positive integer and let $p_1, \ldots, p_r$ be distinct prime numbers with p_r the largest. Let $h_1, \ldots, h_r$ be integers of absolute value at most H. Let α be a rational number with height at most A (≥ 4) and let h_0 be an integer of absolute value at most H_0 (≥ 2). We consider

$$\log T = h_1 \log p_1 + \ldots + h_r \log p_r + h_0 \log \alpha.$$

LEMMA 2. *Let U be a positive real number and suppose that*

$$0 < |\log T| < U^{-1}. \tag{7}$$

Then there exists an effectively computable number C_4 such that

$$p_r > C_4 \log\left(\frac{\log U}{\log A \log(H_0 + H/(\log A))}\right).$$

Proof. Let $C_5, C_6, \ldots$ denote effectively computable positive numbers. By Lemma 1,

$$|\log T| > \exp\left(-C_5(r+1)^{4(r+1)} \log p_1 \ldots \log p_r \log A \log\left(H_0 + \frac{H}{\log A}\right)\right). \tag{8}$$

Observe that

$$(r+1)^{4(r+1)} \log p_1 \ldots \log p_r < e^{4(r+1)\log(r+1) + r\log\log p_r} < e^{C_6 p_r}, \tag{9}$$

by the prime number theorem. Therefore by (7)–(9),

$$C_5 e^{C_6 p_r} \log A \log\left(H_0 + \frac{H}{\log A}\right) > \log U,$$

hence

$$p_r > C_7 \log\left(\frac{\log U}{\log A \log(H_0 + H/(\log A))}\right).$$

We shall also require the following theorem on S-unit equations.

LEMMA 3. *Let* $S = \{p_1, \ldots, p_s\}$ *be a set of prime numbers and let* n *be a positive integer. There are only finitely many* n*-tuples* $(x_1, \ldots, x_n)$ *of integers, all whose prime factors are from* S, *satisfying*:

(i) $\gcd(x_1, \ldots, x_n) = 1$,

(ii) $x_1 + \ldots + x_n = 0$, *and*

(iii) $x_{i_1} + \ldots + x_{i_k} \neq 0$ *for each proper, non-empty subset* $\{i_1, \ldots, i_k\}$ *of* $\{1, \ldots, n\}$.

Proof. See van der Poorten and Schlickewei [4, 5] and Evertse [1].

3. Proof of Theorem 1. Let $C_8, C_9, \ldots$ denote effectively computable positive numbers. The proof proceeds by a comparison of estimates for T_1 and T_2 where

$$T_1 = \frac{b}{c} \cdot \frac{ac+1}{ab+1} \tag{10}$$

and

$$T_2 = \frac{(ac+1)(bc+1)}{(ab+1)c^2}.$$

Let $p_1, \ldots, p_r$ be the distinct prime factors of $(ab+1)(ac+1)(bc+1)$ and suppose that p_r is the largest of them.

We may assume $a \geq 16$. Then

$$\log T_1 = \log\left(1 + \frac{b-c}{abc+c}\right) < \log\left(1 + \frac{1}{ac}\right) \leq \log\left(1 + \frac{1}{a}\right) < a^{-1/2}.$$

Further,

$$\log T_1 = h_1 \log p_1 + \ldots + h_r \log p_r + \log(b/c),$$

where $h_1, \ldots, h_r$ are integers of absolute value at most $6 \log a$. Since $b > c$, we find that $\log T_1 > 0$ and thus, by Lemma 2,

$$p_r > C_8 \log\left(\frac{\log a}{\log b \log\left(\frac{2\log a}{\log b}\right)}\right). \tag{11}$$

Observe that we may assume $b \geq 16$ since otherwise our result follows from (11). Next notice that

$$\begin{aligned}\log T_2 &= \log\left(1 + \frac{ac+bc+1-c^2}{abc^2+c^2}\right) < \log\left(1 + \frac{ac+bc}{abc^2}\right) \\ &= \log\left(1 + \frac{1}{bc} + \frac{1}{ac}\right) < \log\left(1 + \frac{2}{b}\right) < \frac{4}{b} < b^{-1/2}.\end{aligned} \tag{12}$$

We have

$$\log T_2 = l_1 \log p_1 + \ldots + l_r \log p_r - 2 \log c,$$

where $l_1, \ldots, l_r$ are integers of absolute value at most $6\log a$. Since $\log T_2 > 0$ it follows from Lemma 2 with $U = b^{1/2}$ that

$$p_r > C_9 \log\left(\frac{\log b}{\log(c+1)\log\left(\frac{2\log a}{\log(c+1)}\right)}\right). \tag{13}$$

Our result now follows from (11) and (13) on noting that if x, y and z are positive real numbers then

$$\tfrac{1}{2}\log xy \le \max(\log x, \log y)$$

and, for $z > 9$, $\log(z/(\log z)^2) > \frac{1}{5}\log z$.

4. Proof of Theorem 2. Let C_{10} and C_{11} denote effectively computable positive numbers. The proof depends on a comparison of estimates for T_1, T_3 and T_4 where T_1 is given by (10),

$$T_3 = \frac{(ac+1)(bd+1)}{(ab+1)cd} \quad \text{and} \quad T_4 = \frac{(ab+1)(cd+1)}{(ac+1)(bd+1)}.$$

We suppose that $p_1, \ldots, p_r$ are the distinct prime factors of $(ab+1)(ac+1)(bd+1)(cd+1)$ and that p_r is the largest of them.

We have (11), just as in the proof of Theorem 1. Since (11) holds we may assume $b \ge 16$. Then

$$\log T_3 = \log\left(1 + \frac{ac+bd-cd+1}{abcd+cd}\right) < \log\left(1+\frac{2}{b}\right) < b^{-1/2}. \tag{14}$$

We have

$$\log T_3 = l_1\log p_1 + \ldots + l_r \log p_r - \log cd,$$

where $l_1, \ldots, l_r$ are integers of absolute value at most $6\log a$. Since $\log T_3 > 0$ it follows from (14) and Lemma 2 that

$$p_r > C_{10}\log\left(\frac{\log b}{\log(2cd)\log\log a}\right). \tag{15}$$

It follows from (11) and (15) that we may assume that $cd \ge 16$ since otherwise the theorem holds. Note that

$$\log T_4 = \log\left(1 + \frac{(a-d)(b-c)}{abcd+ac+bd+1}\right) < \log\left(1+\frac{2}{cd}\right) < (cd)^{-1/2}. \tag{16}$$

Since $a > d$ and $b > c$, we find that $\log T_4 > 0$. Further,

$$\log T_4 = m_1 \log p_1 + \ldots + m_r \log p_r,$$

where $m_1, \ldots, m_r$ are integers of absolute value at most $6\log a$. We may apply Lemma 2 with $h_0 = 1$, $\alpha = 1$ and $U = (cd)^{1/2}$ to obtain

$$p_r > C_{11}\log\left(\frac{\log 2cd}{\log\log a}\right). \tag{17}$$

Our result now follows from (11), (15) and (17).

5. Proof of Theorem 3. For each integer i with $1 \le i \le n$ let $k(i)$ denote the number of subsets of L containing i. The polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$ given by

$$\prod_{(i_1,\ldots,i_t)\in L} (x_{i_1}\ldots x_{i_t}+1) - \prod_{i=1}^{n} x_i^{k(i)}$$

can be expressed as a finite sum of terms of the form

$$\prod_{(i_1,\ldots,i_t)\in L'} (x_{i_1}\ldots x_{i_t}+1)$$

where L' is a proper subset of L. Here the empty set is permitted and in that case the product is 1. This may be proved by induction on the cardinality of L. The corresponding assertion holds with M in place of L. It then follows that

$$\prod_{(i_1,\ldots,i_t)\in L} (x_{i_1}\ldots x_{i_t}+1) - \prod_{(i_1,\ldots,i_t)\in M} (x_{i_1}\ldots x_{i_t}+1) = \sum_R c_R \prod_{(i_1,\ldots,i_t)\in R} (x_{i_1}\ldots x_{i_t}+1), \tag{18}$$

where the sum on the right hand side of (18) is over all proper subsets R of L and of M and where c_R is an integer for each such R.

Let s be a positive integer and let $S = \{p_1, \ldots, p_s\}$ be the set of the first s prime numbers. We choose s sufficiently large that the prime factors of c_R lie in S for all proper subsets R of L and of M. Suppose that $a_1, \ldots, a_n$ are positive integers for which (4) holds and for which

$$P\Big(\prod_{(i_1,\ldots,i_t)\in L\cup M} (a_{i_1}\ldots a_{i_t}+1)\Big) \le p_s. \tag{19}$$

Then, by (18),

$$\prod_{(i_1,\ldots,i_t)\in L} (a_{i_1}\ldots a_{i_t}+1) - \prod_{(i_1,\ldots,i_t)\in M} (a_{i_1}\ldots a_{i_t}+1) - \sum_R c_R \prod_{(i_1,\ldots,i_t)\in R} (a_{i_1}\ldots a_{i_t}+1) = 0 \tag{20}$$

is an S-unit equation. By (4) there is a subsum of the sum on the left hand side of equality (20) which is zero and has no vanishing subsum and which involves $\prod_{(i_1,\ldots,i_t)\in L}(a_{i_1}\ldots a_{i_t}+1)$ and at least one term of the form $-c_R\prod_{(i_1,\ldots,i_t)\in R}(a_{i_1}\ldots a_{i_t}+1)$ with $c_R \ne 0$, where R is a proper subset of L or of M. Let g be the greatest common divisor of the terms in this subsum. It follows from Lemma 3 that $\big(\prod_{(i_1,\ldots,i_t)\in L}(a_{i_1}\ldots a_{i_t}+1)\big)/g$ is bounded in terms of p_s. Plainly

$$g \le |c_R| \prod_{(i_1,\ldots,i_t)\in R} (a_{i_1}\ldots a_{i_t}+1) \le 2^{|R|}|c_R| \prod_{(i_1,\ldots,i_t)\in R} (a_{i_1}\ldots a_{i_t}),$$

where $|R|$ denotes the cardinality of R. Since

$$\prod_{(i_1,\ldots,i_t)\in M} (a_{i_1}\ldots a_{i_t}) = \prod_{(i_1,\ldots,i_t)\in L} (a_{i_1}\ldots a_{i_t}), \tag{21}$$

we find that

$$\Big(\prod_{(i_1,\ldots,i_t)\in L} (a_{i_1}\ldots a_{i_t}+1)\Big)/g \ge \frac{\min_{(i_1,\ldots,i_t)\in L\cup M}(a_{i_1}\ldots a_{i_t})}{2^{|R|}|c_R|} = \frac{a^-}{2^{|R|}|c_R|}$$

and so a^- is bounded in terms of p_s as required.

We shall now prove (6). Let $C_{12}, C_{13},\ldots$ denote positive numbers which are effectively computable in terms of t and the cardinality of L. Let $p_1,\ldots$ $\ldots, p_r$ be the distinct prime factors of

$$\prod_{(i_1,\ldots,i_t)\in L\cup M} (a_{i_1}\ldots a_{i_t}+1)$$

and suppose that p_r is the largest of them. We may assume without loss of generality, by (4), that

$$\prod_{(i_1,\ldots,i_t)\in L} (a_{i_1}\ldots a_{i_t}+1) > \prod_{(i_1,\ldots,i_t)\in M} (a_{i_1}\ldots a_{i_t}+1).$$

Put

$$T = \Big(\prod_{(i_1,\ldots,i_t)\in L} (a_{i_1}\ldots a_{i_t}+1)\Big)/\prod_{(i_1,\ldots,i_t)\in M} (a_{i_1}\ldots a_{i_t}+1). \tag{22}$$

Then

$$\log T = l_1\log p_1 + \ldots + l_r \log p_r,$$

where $l_1,\ldots,l_r$ are integers of absolute value at most $C_{12}\log a^+$. By (22),

$$0 < \log T < \log(1 + C_{13}Z), \tag{23}$$

where

$$Z = \max_R \Big(\prod_{(i_1,\ldots,i_t)\in R} (a_{i_1}\ldots a_{i_t})\Big)/\prod_{(i_1,\ldots,i_t)\in M} (a_{i_1}\ldots a_{i_t})$$

and where the maximum is taken over all proper subsets R of L. Further, by (21),

$$Z = (\min_{(i_1,\ldots,i_t)\in L} a_{i_1}\ldots a_{i_t})^{-1} \le 1/a^-. \tag{24}$$

Therefore, provided that a^- exceeds C_{14}, which we may assume, we find from (23) and (24) that

$$0 < \log T < 1/(a^-)^{1/2}.$$

Our result now follows from Lemma 2 on taking $\alpha = h_0 = 1, U = (a^-)^{1/2}$ and $H = C_{12} \log a^+$.

6. Proof of Corollary 2. Denote a, b, c, d and e by a_1, a_2, a_3, a_4 and a_5 respectively. We apply Theorem 3 with the balanced pair of sets of 2-element subsets of $\{1, \ldots, 5\}$ given by $\{L, M\}$ where $L = \{(1, 2), (1, 3), (4, 5)\}$ and $M = \{(1, 4), (1, 5), (2, 3)\}$. Condition (4) becomes

$$(ab + 1)(ac + 1)(de + 1) \neq (ad + 1)(ae + 1)(bc + 1)$$

and our result now follows since

$$\min\{ab, ac, ad, ae, bc, de\} \geq \min\{b, c, d, e\}.$$

7. Proof of Corollary 3. Denote a, b, c, d, e and f by a_1, a_2, a_3, a_4, a_5 and a_6 respectively. We now apply Theorem 3 with the balanced pair of 3-sets of $\{1, 2, 3, 4, 5, 6\}$ given by $\{L, M\}$ where $L = \{(1, 2, 3), (3, 4, 5), (1, 5, 6)\}$ and $M = \{(1, 4, 6), (1, 3, 5), (2, 3, 5)\}$. The result follows on noting that

$$\min\{abc, aef, adf, ace\} \geq a \quad \text{and} \quad \min\{cde, bce\} \geq e.$$

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