ACTA ARITHMETICA
LXXVIII.2 (1996)

The equation $x + y = 1$ in finitely generated groups

by

F. Beukers (Utrecht) and H. P. Schlickewei (Ulm)

1. Introduction. Let H be a finitely generated subgroup of rank r in $(\mathbb{C}^*)^2$. Denote by G the $\mathbb{Q}$-closure of H, i.e. the subgroup of $(\mathbb{C}^*)^2$ consisting of all pairs $\mathbf{a} = (a_1, a_2) \in (\mathbb{C}^*)^2$ such that $\mathbf{a}^N = (a_1^N, a_2^N) \in H$ for some $N \in \mathbb{N}$. We are interested in an upper bound for the number of solutions $(x, y) \in G$ of the equation

$$x + y = 1. \tag{1}$$

A special case of (1) is obtained if we restrict x and y to the group of so-called S-units in an algebraic number field K. Here S is assumed to be a finite set of places of K including all infinite ones. Supposing that $d = [K : \mathbb{Q}]$, $s = \#S$ and letting $a, b \in K^*$ be fixed, J. H. Evertse [3, Theorem 1] showed that

$$ax + by = 1 \tag{2}$$

has not more than $3 \cdot 7^{d+2s}$ solutions. Since $s \geq d/2$ this implies that (2) has at most $3 \cdot 7^{4s}$ solutions. We can apply this result to equation (1). However, the estimate will depend on the degree of the field containing H, and on s, the number of places for which the elements of H have non-trivial valuation. Note that for fixed r the number s may have arbitrarily large values.

We shall be interested in bounds which depend only on r. The first such uniform result for a general subgroup G of $(\mathbb{C}^*)^2$ was given in [5]. There the bound $2^{2^{26}+36r^2}$ was derived for the number of solutions of equation (1). This was improved in [6] to $2^{13r+63}r^r$.

In this paper we obtain

Theorem 1.1. *Let G be the $\mathbb{Q}$-closure of a finitely generated subgroup of $(\mathbb{C}^*)^2$ of rank r. Then the equation*

$$x + y = 1, \qquad (x, y) \in G,$$

has not more than 2^{8r+8} solutions.

Note that this bound, apart from the numerical constants, has the same shape as Evertse's upper bound.

It is well known that a particular application of Theorem 1.1 deals with the multiplicity of binary recurrences. Let $\{u_m\}_{m\in\mathbb{Z}}$ be a sequence of complex numbers satisfying the recurrence relation

$$u_{m+2} = \nu_1 u_{m+1} + \nu_0 u_m$$

with $\nu_0, \nu_1 \in \mathbb{C}$, $\nu_0 \neq 0$. Suppose that we have initial values $(u_0, u_1) \neq (0,0)$. Write $f(z) = z^2 - \nu_1 z - \nu_0$. Let α, β be its zeros. Note that $\nu_0 \neq 0$ implies $\alpha, \beta \neq 0$. Let us assume that $\alpha \neq \beta$. Then there exist $a, b \in \mathbb{C}$ such that

$$u_m = a\alpha^m + b\beta^m.$$

Given $c \in \mathbb{C}$ we are interested in the number of solutions $m \in \mathbb{Z}$ of $u_m = c$. Note that the cases a, b or c equal to zero are uninteresting since they have either at most one solution or infinitely many trivial ones. So we assume they are non-zero. Divide on both sides by c, and from now on we shall be interested in the equation

$$\lambda\alpha^x + \mu\beta^x = 1 \quad \text{in } x \in \mathbb{Z}, \tag{3}$$

where $\lambda\mu\alpha\beta \neq 0$. We shall also assume that α, β are not both roots of unity.

As a fine point we add that if α, β are roots of unity, then the set

$$\{(\alpha^x, \beta^x) : x \text{ solution of (3)}\}$$

consists of at most two elements. This is a consequence of the fact that there exist precisely two triangles in the complex plane two of whose sides have lengths $|\lambda|, |\mu|$, whose third side is the segment $[0,1]$ and such that the side of length $|\lambda|$ ends in 0.

Straightforward application of Theorem 1.1 with the group H generated by (λ, μ) and (α, β) shows that (3) has not more than 2^{24} solutions. However, one can do much better:

Theorem 1.2. *Under the assumptions just mentioned the equation*

$$\lambda\alpha^x + \mu\beta^x = 1 \quad \text{in } x \in \mathbb{Z}$$

has at most 61 *solutions.*

As a curiosity we mention that the equation with the largest number of solutions known is

$$\frac{\theta_2 - \theta_3}{\theta_2 - \theta_1}\left(\frac{\theta_1}{\theta_3}\right)^x + \frac{\theta_1 - \theta_3}{\theta_1 - \theta_2}\left(\frac{\theta_2}{\theta_3}\right)^x = 1$$

where the θ_i are the zeros of $X^3 - 2X^2 + 4X - 4$. The solutions are $x = 0, 1, 4, 6, 13, 52$. It would be interesting to have examples with more than 6 solutions, if they exist.

The first result in the situation of Theorem 1.2 with a universal bound was derived in [4] with the bound $2^{2^{23}}$.

The improvements we give in the current paper in comparison with [4]–[6] depend upon two ingredients. First we use an explicit version of Thue's method via hypergeometric polynomials as given in [1], whereas the previous papers are based on a quantitative version of Roth's Theorem. To get bounds that do not depend upon degrees of number fields involved, previously a result from [7] was used on lower bounds for heights of solutions of equations. Here we apply the strongly improved bound given in Corollary 2.4 of [2].

2. Lemmas on algebraic numbers. First we fix our notations concerning heights. Let K be an algebraic number field of degree d over $\mathbb{Q}$. For any valuation v we write $d_v = [K_v : \mathbb{Q}_v]$, where $K_v, \mathbb{Q}_v$ are the completions of $K, \mathbb{Q}$ with respect to v. For archimedean v we normalise the valuation by $|x|_v = |x|^{d_v/d}$ where $|\cdot|$ is the ordinary complex absolute value. When v is non-archimedean we take $|p|_v = p^{-d_v/d}$ where p is the unique rational prime such that $|p|_v < 1$. The height of an algebraic number $\alpha \in K^*$ is defined by

$$H(\alpha) = \prod_v \max(1, |x|_v).$$

Because of our normalisation $H(\alpha)$ does not depend on the choice of the field K in which α is contained. More generally, for any $(n+1)$-tuple $(x_0, x_1, \ldots, x_n) \in K^n$ with not all x_i zero we define

$$H(x_0, \ldots, x_n) = \prod_v \max(|x_0|_v, \ldots, |x_n|_v).$$

Note that by the product formula we have $H(\lambda x_0, \ldots, \lambda x_n) = H(x_0, \ldots, x_n)$ for any $\lambda \in K^*$, so we can view this height as a height on the K-rational points of the projective space $\mathbb{P}^n$. In particular, we have $H(\alpha) = H(1, \alpha)$.

We start with an easy lemma.

LEMMA 2.1. *Let $a, a', b, b', A, B \in \overline{\mathbb{Q}}^*$ and $c, c' \in \overline{\mathbb{Q}}$ be such that $ab' \neq a'b$ and*

$$aA + bB = c, \qquad a'A + b'B = c'.$$

Then $H(A, B, 1) \leq 2H(a, b, c)H(a', b', c')$.

Proof. Fix a number field K in which all numbers involved are contained. For each infinite valuation v let $r_v = 2^{d_v/d}$ and let $r_v = 1$ if v is finite. Notice that $\prod_v r_v = 2$.

One easily finds that

$$A = \frac{bc' - b'c}{\Delta}, \qquad B = \frac{a'c - ac'}{\Delta}$$

where $\Delta = a'b - ab'$. Hence

$$\begin{aligned} H(A,B,1) &= H(bc' - b'c, a'c - ac', ba' - ab') \\ &= \prod_v \max(|bc' - b'c|_v, |a'c - ac'|_v, |ba' - ab'|_v) \\ &\le \prod_v r_v \max(|a|_v, |b|_v, |c|_v) \max(|a'|_v, |b'|_v, |c'|_v) \\ &= 2H(a,b,c)H(a',b',c'). \ \blacksquare \end{aligned}$$

As a corollary we get

Corollary 2.2. *Let $a, b, A, B \in \overline{\mathbb{Q}}^*$ be such that $a \ne b$ and*

$$A + B = 1, \qquad aA + bB = 1.$$

Then $H(A,B,1) \le 2H(a,b,1)$.

The next lemma follows from an explicit version of Thue's method using hypergeometric polynomials.

Lemma 2.3. *Let $a, b, A, B \in \overline{\mathbb{Q}}^*$ and $\varrho \in \mathbb{N}$ be such that*

$$A + B = 1, \qquad aA^{2\varrho} + bB^{2\varrho} = 1.$$

Then $H(A,B,1) \le 2^{1/\varrho} c H(a,b,1)^{1/\varrho}$, where $c = 6\sqrt{3}$.

Proof. We infer from Lemma 6 of [1] that there exist three polynomials $P_\varrho, Q_\varrho, R_\varrho$ of degree $\le \varrho$ such that

$$\begin{gathered} z^{2\varrho} P_\varrho(z) + (1-z)^{2\varrho} Q_\varrho(z) = R_\varrho(z), \qquad \forall z \in \mathbb{C}, \\ bP_\varrho(A) \ne aQ_\varrho(A), \\ H(P_\varrho(A), Q_\varrho(A), R_\varrho(A)) \le (6\sqrt{3})^\varrho H(A)^\varrho. \end{gathered}$$

Substitute $z = A$ in the polynomial identity. Application of the previous lemma with $A^{2\varrho}, B^{2\varrho}$ instead of A, B and $c = 1$, $a' = P_\varrho(A)$, $b' = Q_\varrho(A)$, $c' = R_\varrho(A)$ yields

$$\begin{aligned} H(A,B,1)^{2\varrho} &\le 2H(a,b,1)H(P_\varrho(A), Q_\varrho(A), R_\varrho(A)) \\ &\le 2c^\varrho H(a,b,1)H(A)^\varrho \le 2c^\varrho H(a,b,1)H(A,B,1)^\varrho. \end{aligned}$$

Divide on both sides by $H(A,B,1)^\varrho$ and take ϱth roots to obtain our lemma. ■

The following lemma is due to an improvement of [7] by Corollary 2.4 in [2].

Lemma 2.4. *Let $\lambda, \mu \in \overline{\mathbb{Q}}^*$ and suppose that $\lambda + \mu = 1$. Let (p_i, q_i), $i = 1, 2$, be two solutions in $\overline{\mathbb{Q}}$ of $\lambda p + \mu q = 1$ such that the pairs $(p_1, q_1), (p_2, q_2)$ and $(1, 1)$ are all distinct. Then*

$$H(p_1, q_1, 1)H(p_2, q_2, 1) \ge 1.0942711\ldots$$

By application of this lemma with $\lambda = x_0, \mu = y_0$ and $p_i = x_i/x_0,\ q_i = y_i/y_0$ we obtain

COROLLARY 2.5. *Let $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ be three distinct solutions of $x+y=1$ in $x, y \in \overline{\mathbb{Q}}^*$. Then*

$$\max_{i=1,2}(\max(H(x_i/x_0), H(y_i/y_0))) \geq 1.022777\ldots$$

3. Normed vector spaces. Let $m \in \mathbb{N}$. For any subgroup $H \subset (\overline{\mathbb{Q}}^*)^m$ we let the $\mathbb{Q}$-closure of H be the set of all $\mathbf{a} \in (\overline{\mathbb{Q}}^*)^m$ such that $\mathbf{a}^N \in H$ for some $N \in \mathbb{N}$. Let G be the $\mathbb{Q}$-closure of a finitely generated subgroup of $(\overline{\mathbb{Q}}^*)^m$ of rank r. Let T be the torsion subgroup of G. Then $G/T = G \otimes_{\mathbb{Z}} \mathbb{Q}$ has the natural structure of a $\mathbb{Q}$-vector space of dimension r. Consider the logarithmic height function $h(x) = \log H(x)$. The function

$$\|(x_1, \ldots, x_m)\| = \max_{i=1,\ldots,m} h(x_i)$$

provides a natural norm on $G \otimes_{\mathbb{Z}} \mathbb{Q}$ as $\mathbb{Q}$-vector space. By continuity we can extend this norm to the real vector space $V_G = G \otimes_{\mathbb{Z}} \mathbb{R}$.

LEMMA 3.1. *The* (*semi*)*-norm $\|\cdot\|$ is positive definite on V_G.*

Proof. Let us write down the semi-norm $\|\cdot\|$ in an explicit way. Suppose the $\mathbb{Q}$-generators of G are given by

$$\mathbf{a}_i = (a_{i1}, \ldots, a_{im}), \quad i = 1, \ldots, r.$$

Any element of G can be written, modulo roots of unity, in the form $\mathbf{x} = (x_1, \ldots, x_m) = \prod_{i=1}^r (a_{i1}, \ldots, a_{im})^{e_i}$ for some $e_i \in \mathbb{Q}$. Hence, using $h(a) = (1/2)\sum_v |\log(|a|_v)|$,

$$\|\mathbf{x}\| = \max_{j=1,\ldots,m} h\Big(\prod_{i=1}^r a_{ij}^{e_i}\Big) = \max_{j=1,\ldots,m} (1/2) \sum_v \Big| \sum_{i=1}^r e_i \log(|a_{ij}|_v)\Big|.$$

Extending $\|\cdot\|$ to the reals is now straightforward, simply extend e_i to $\mathbb{R}$. We also remark that if we take the e_i integral, the components of $\mathbf{x}$ all lie in the same number field, hence the non-trivial elements of the group generated (over $\mathbb{Z}$) by the $\mathbf{a}_i$ have a norm uniformly bounded below by a positive constant, γ, say.

We now prove positive definiteness of $\|\cdot\|$. Suppose there exists $\mathbf{y} \in V_G$, non-zero, such that $\|\mathbf{y}\| = 0$. This implies that there exist $e_i \in \mathbb{R}$, not all zero, such that $|\sum_{i=1}^r e_i \log(|a_{ij}|_v)| = 0$ for all valuations v and all j. Using Dirichlet's box principle we can then show that for any $\varepsilon > 0$ there exist integers m_i, not all zero, such that $|\sum_{i=1}^r m_i \log(|a_{ij}|_v)| < \varepsilon$ for all v and j. This contradicts the existence of the uniform lower bound γ. Hence $\|\mathbf{y}\| = 0$ implies that $e_i = 0$ for all i, as desired. ∎

From now on we suppose that $G \subset (\overline{\mathbb{Q}}^*)^2$. We want to bound the number of solutions of the equation

$$\text{(M)} \qquad x + y = 1, \quad (x, y) \in G.$$

Consider the natural projection $p : G \to V_G$.

LEMMA 3.2. *Let $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ be three distinct solutions of* (M). *Then their images under p cannot be all equal.*

Proof. If all three images were the same then x_i/x_0 and y_i/y_0 would be roots of unity for $i = 1, 2$. But this is impossible in view of Corollary 2.5. ∎

Let $\mathcal{M}$ be the image under p of the solution set of (M). Then the number of solutions to (M) is bounded by $2(\#\mathcal{M})$.

We now restate the lemmas of the previous section in terms of the set $\mathcal{M} \subset V_G$. In the derivations we use the fact that $\max(H(a), H(b)) \le H(a, b, 1) \le \max(H(a), H(b))^2$.

Corollary 2.2 becomes

LEMMA 3.3. *Let $\mathbf{w}_1, \mathbf{w}_2$ be distinct points of $\mathcal{M}$. Then*

$$\|\mathbf{w}_1\| \le \log 2 + 2\|\mathbf{w}_2 - \mathbf{w}_1\|.$$

Lemma 2.3 becomes

LEMMA 3.4. *Let $\mathbf{w}_1, \mathbf{w}_2$ be distinct points of $\mathcal{M}$ and $\varrho \in \mathbb{N}$. Then*

$$\|\mathbf{w}_1\| \le \log c + \frac{1}{\varrho}(\log 2 + 2\|\mathbf{w}_2 - 2\varrho \mathbf{w}_1\|).$$

Corollary 2.5 becomes

LEMMA 3.5. *Let $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$ be distinct points of $\mathcal{M}$. Then*

$$\max(\|\mathbf{w}_1 - \mathbf{w}_0\|, \|\mathbf{w}_2 - \mathbf{w}_0\|) \ge 0.022522\ldots$$

It will turn out that the cardinality of any set satisfying the inequalities in the above three lemmas can be bounded in terms of the dimension of V_G.

We need some additional lemmas on coverings of convex bodies. The first is straightforward.

LEMMA 3.6. *Let V be an m-dimensional normed real vector space with norm $\|\cdot\|$. Let $R > \delta > 0$. Consider the ball B of radius R around the origin and suppose it contains a set U such that $\|\mathbf{u}_1 - \mathbf{u}_2\| \ge \delta$ for any two distinct $\mathbf{u}_1, \mathbf{u}_2 \in U$. Then $\#U \le (1 + 2R/\delta)^m$.*

Proof. Let V_0 be the volume of the unit ball $\{\mathbf{x} : \|\mathbf{x}\| < 1\}$. Around any point $\mathbf{u} \in U$ we consider the open ball $B_u = \{\mathbf{x} : \|\mathbf{x} - \mathbf{u}\| < \delta/2\}$. Since these balls are disjoint their union fills up a region of volume $(\#U)(\delta/2)^m V_0$ in the ball of radius $R + \delta/2$. The latter ball has volume $(R + \delta/2)^m V_0$. Hence $(\#U)(\delta/2)^m \le (R + \delta/2)^m$ and our lemma follows. ∎

LEMMA 3.7. *Let Ψ be a convex symmetric body in $\mathbb{R}^r$. By $\lambda\Psi$ we denote the convex body obtained by multiplying the points of Ψ by λ. Then, for any $\lambda > 1$, the set $\lambda\Psi$ can be covered by $(4 + 2\lambda)^r$ translated copies of Ψ.*

The proof of this lemma can be found in [6, Lemma 7.2]. However, we really need the following corollary.

COROLLARY 3.8. *Let V be an r-dimensional normed real vector space with norm $\|\cdot\|$. Let $\varepsilon > 0$. Then there is a finite set $E \subset V$ of unit vectors such that every $\mathbf{v} \in V$ can be written as $\mathbf{v} = \|\mathbf{v}\|\mathbf{e} + \mathbf{v}'$ with $\mathbf{e} \in E$ and $\|\mathbf{v}'\| \le \varepsilon\|\mathbf{v}\|$. Moreover, E can be chosen such that $\#E < (4 + 4/\varepsilon)^r$.*

Proof. Let B be the unit ball with respect to $\|\cdot\|$. According to Lemma 3.7 the ball B can be covered by $(4 + 4/\varepsilon)^r$ translates of $(\varepsilon/2)B$. Consider such a covering and let Δ be the subset of $(\varepsilon/2)$-balls which have non-trivial intersection with the boundary of B. Clearly the balls in Δ give a covering of the boundary of B. For the set E we take the unit vectors $\mathbf{c}/\|\mathbf{c}\|$ where $\mathbf{c}$ runs over the centers of the $(\varepsilon/2)$-balls in Δ.

Now let $\mathbf{v} \in \mathbb{R}^r$ be arbitrary. Let $\mathbf{c}$ be the center of the $(\varepsilon/2)$-ball in Δ which contains $\mathbf{v}/\|\mathbf{v}\|$ and let $\mathbf{e} = \mathbf{c}/\|\mathbf{c}\|$. Notice that $\|\mathbf{c} - \mathbf{e}\| = |1 - \|\mathbf{c}\|\,| \le \varepsilon/2$. Hence,

$$\left\|\frac{\mathbf{v}}{\|\mathbf{v}\|} - \mathbf{e}\right\| \le \left\|\frac{\mathbf{v}}{\|\mathbf{v}\|} - \mathbf{c}\right\| + \|\mathbf{c} - \mathbf{e}\| \le \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus we find $\|\mathbf{v} - \|\mathbf{v}\|\mathbf{e}\| \le \varepsilon\|\mathbf{v}\|$, $\mathbf{e} \in E$ and our corollary follows. ∎

4. Proof of Theorem 1.1. Let Σ be a subset of a normed vector space V satisfying

1. $\|\mathbf{w}_1\| \le \log 2 + 2\|\mathbf{w}_2 - \mathbf{w}_1\|$ for any two distinct $\mathbf{w}_1, \mathbf{w}_2 \in \Sigma$.
2. There exists c_1 such that $\|\mathbf{w}_1\| \le c_1 + (1/\varrho)(\log 2 + 2\|\mathbf{w}_2 - 2\varrho\mathbf{w}_1\|)$ for any two distinct $\mathbf{w}_1, \mathbf{w}_2 \in \Sigma$ and any $\varrho \in \mathbb{N}$.
3. There exists $c_0 > 0$ such that $\max(\|\mathbf{w}_1 - \mathbf{w}_0\|, \|\mathbf{w}_2 - \mathbf{w}_0\|) \ge c_0$ for any three distinct $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2 \in \Sigma$.

PROPOSITION 4.1. *Let* $c_2 = \max(2\log 2, c_1 + (\log 2)/20)$. *Then*

$$\#\Sigma \le \frac{1}{2}\left(44 + 2\frac{c_2}{c_0}\right)^{r+1}$$

where r is the dimension of V.

Proof. Let ε be a real number such that $0 < \varepsilon < 0.1$. Let $\mathbf{e}$ be a unit vector in V and consider the cone

$$C_{\mathbf{e}} = \{\mathbf{v} \in V : \mathbf{v} = \|\mathbf{v}\|\mathbf{e} + \mathbf{v}',\ \|\mathbf{v}'\| \le \varepsilon\|\mathbf{v}\|\}.$$

Let

$$c_3(\varepsilon) = \frac{c_2}{1-10\varepsilon}.$$

We will show that for any two $\mathbf{w}_1, \mathbf{w}_2 \in \Sigma \cap C_{\mathbf{e}}$ with $c_3(\varepsilon) < \|\mathbf{w}_1\| \le \|\mathbf{w}_2\|$ we have

$$(4) \qquad (5/4)\|\mathbf{w}_1\| \le \|\mathbf{w}_2\| \le (1+4/\varepsilon)\|\mathbf{w}_1\|.$$

Suppose first $\mathbf{w}_1, \mathbf{w}_2 \in \Sigma \cap C_{\mathbf{e}}$ and $\|\mathbf{w}_1\| \le \|\mathbf{w}_2\| < (5/4)\|\mathbf{w}_1\|$. Write $\mathbf{w}_i = \|\mathbf{w}_i\|\mathbf{e} + \mathbf{w}_i'$. Then, from the first inequality on Σ, we infer

$$\begin{aligned}\|\mathbf{w}_1\| &\le \log 2 + 2\|(\|\mathbf{w}_2\| - \|\mathbf{w}_1\|)\mathbf{e} + \mathbf{w}_2' - \mathbf{w}_1'\| \\ &\le \log 2 + 2(\|\mathbf{w}_2\| - \|\mathbf{w}_1\|) + 2\varepsilon(\|\mathbf{w}_2\| + \|\mathbf{w}_1\|) \\ &\le \log 2 + 2(1/4)\|\mathbf{w}_1\| + 2\varepsilon(9/4)\|\mathbf{w}_1\|.\end{aligned}$$

We obtain

$$\|\mathbf{w}_1\| \le \frac{2\log 2}{1-9\varepsilon} \le c_3(\varepsilon).$$

Suppose next that $\mathbf{w}_1, \mathbf{w}_2 \in \Sigma \cap C_{\mathbf{e}}$ and $\|\mathbf{w}_2\| > (1+4/\varepsilon)\|\mathbf{w}_1\|$. Choose $\varrho \in \mathbb{N}$ such that $\|\mathbf{w}_2\| = (2\varrho + \delta)\|\mathbf{w}_1\|$ with $|\delta| \le 1$. Notice that $\varrho \ge 2/\varepsilon$. From the second inequality on Σ it follows that

$$\begin{aligned}\|\mathbf{w}_1\| &\le c_1 + \frac{1}{\varrho}(\log 2 + 2\|\delta\|\mathbf{w}_1\|\mathbf{e} + \mathbf{w}_2' - 2\varrho\mathbf{w}_1'\|) \\ &\le c_1 + (\log 2)/20 + \frac{2}{\varrho}(\|\mathbf{w}_1\| + \varepsilon(\|\mathbf{w}_2\| + 2\varrho\|\mathbf{w}_1\|)) \\ &\le c_2 + \frac{2}{\varrho}\|\mathbf{w}_1\| + \varepsilon(8+4/\varrho)\|\mathbf{w}_1\| \\ &\le c_2 + \varepsilon\|\mathbf{w}_1\| + 9\varepsilon\|\mathbf{w}_1\|.\end{aligned}$$

We get

$$\|\mathbf{w}_1\| \le \frac{c_2}{1-10\varepsilon} \le c_3(\varepsilon).$$

We now put the above considerations together. Let N be the smallest integer such that $(5/4)^{N-1} > 1 + 4/\varepsilon$. Suppose $C_{\mathbf{e}}$ contains N points $\mathbf{w}_1, \ldots, \mathbf{w}_N$ larger than $c_3(\varepsilon)$. Suppose they are ordered by size. Then, for each i, $\|\mathbf{w}_{i+1}\|/\|\mathbf{w}_i\| \ge 5/4$. This implies $\|\mathbf{w}_N\|/\|\mathbf{w}_1\| > (5/4)^{N-1} > 1+4/\varepsilon$, which is impossible by inequality (4). Hence any cone $C_{\mathbf{e}}$ contains at most $N-1$ elements from Σ of norm $\ge c_3(\varepsilon)$. According to Lemma 3.8 the space V can be covered by $(4+4/\varepsilon)^r$ such cones and so the total number of points of Σ larger than $c_3(\varepsilon)$ can be estimated by $(N-1)(4+4/\varepsilon)^r$. Since $\varepsilon < 0.1$ it is not hard to see that $N - 1 < 2/\varepsilon$. Hence the number of large points is bounded by $(2/\varepsilon)(4+4/\varepsilon)^r$.

It remains to count the elements of Σ with norm at most $c_3(\varepsilon)$. By the third inequality on Σ a ball of radius c_0 around a point of Σ contains at most one other element from Σ. Consider a subset Σ' of Σ such that a ball of radius c_0 around any point of Σ' contains no other point of Σ'. We can do this in such a way that $|\Sigma| \le 2|\Sigma'|$. According to Lemma 3.6 the number of points in Σ' can be bounded from above by $(1+2c_3(\varepsilon)/c_0)^r$. Thus we conclude

$$|\Sigma| \le \frac{2}{\varepsilon}\left(4+\frac{4}{\varepsilon}\right)^r + 2\left(\frac{2c_3(\varepsilon)}{c_0}+1\right)^r.$$

Now we choose ε such that $4/\varepsilon = 2c_3(\varepsilon)/c_0$, i.e. $\varepsilon = (10+0.5c_2/c_0)^{-1}$. Our proposition then follows immediately. ■

Proof of Theorem 1.1. By a specialisation argument as in [5] we may assume that $G \subset (\overline{\mathbb{Q}}^*)^2$. We now complete the line of argument started in Section 3. There we had the set $\mathcal{M}$. This set satisfies the conditions of Proposition 4.1 for the values $c_0 = 0.022522\ldots$, $c_1 = \log(6\sqrt{3}) = 2.3410\ldots$ Hence the cardinality of $\mathcal{M}$ is bounded by $\frac{1}{2}\cdot 256^{r+1}$. Since the number of solutions of (M) is bounded by $2\#\mathcal{M}$ our theorem follows. ■

5. Proof of Theorem 1.2. We first need a lemma

LEMMA 5.1. *Consider the equation $\lambda\alpha^x + \mu\beta^x = 1$ in $x \in \mathbb{Z}$ where $\lambda, \mu, \alpha, \beta$ are as in the introduction and assumed to be algebraic numbers. Suppose we have the solutions $x = 0, r, s, t$. Suppose that $t \ge 14s$. Then*

$$s - 8.4r \le \frac{9.1}{\log H(\alpha,\beta,1)}.$$

Proof. Application of Corollary 2.2 with $A=\lambda$, $B=\mu$ yields

$$H(\lambda,\mu,1) \le 2H(\alpha,\beta,1)^r.$$

Apply Lemma 2.3 with $A=\lambda\alpha^s$, $B=\mu\beta^s$ and ϱ such that $t = 2s\varrho+\delta$, with $0\le\delta<2s$. Note that $\varrho \ge 7$. We obtain

$$\begin{aligned} H(\lambda\alpha^s,\mu\beta^s,1) &\le 2^{1/\varrho} c H(\alpha^\delta\lambda^{1-2\varrho}, \beta^\delta\mu^{1-2\varrho})^{1/\varrho} \\ &\le 2^{1/\varrho} c H(\alpha,\beta,1)^{\delta/\varrho} H(\lambda^{-1},\mu^{-1},1)^{2-1/\varrho}. \end{aligned}$$

Notice that

$$\begin{aligned} H(\alpha,\beta,1)^s &\le H(\lambda^{-1},\mu^{-1},1) H(\lambda\alpha^s,\mu\beta^s,1) \\ &\le 2^{1/\varrho} c H(\alpha,\beta,1)^{\delta/\varrho} H(\lambda^{-1},\mu^{-1},1)^{3-1/\varrho} \end{aligned}$$

and use $H(\lambda^{-1},\mu^{-1},1) \le H(\lambda,\mu,1)^2 \le 4H(\alpha,\beta,1)^{2r}$ to obtain

$$H(\alpha,\beta,1)^{s-\delta/\varrho} \le 2^{1/\varrho} c 2^{6-2/\varrho} H(\alpha,\beta,1)^{6r} < 64cH(\alpha,\beta,1)^{6r}.$$

Taking log's and using $\log(64c) \le 6.5$ yields

$$s - \delta/\varrho - 6r \le 6.5/\log H(\alpha,\beta,1)$$

from which our lemma is immediate via $\delta/\varrho \le 2s/7$. ■

Proof of Theorem 1.2. By Theorem 2 of [1] we may assume that $\alpha, \beta, \lambda, \nu \in \overline{\mathbb{Q}}$. Without loss of generality we can also assume that

$$H(\alpha, \beta, 1) \le H(\alpha^{-1}, \beta^{-1}, 1).$$

Let q be the length of the shortest closed interval containing three solutions. Let $n, n+p, n+q$ be such three solutions. Application of Lemma 2.4 to the equation $\lambda\alpha^{n+p}X + \mu\beta^{n+p}Y = 1$ yields

$$H(\alpha, \beta, 1)^{q-p} H(\alpha^{-1}, \beta^{-1}, 1)^p \ge c_4,$$

where $c_4 = 1.0942711\ldots$ Hence $H(\alpha^{-1}, \beta^{-1}, 1)^q \ge c_4$.

Define $\gamma = \log 8/\log c_4$ and note that $\gamma < 23.1$.

Now let $k < l < m < n$ be any four solutions. First of all application of Corollary 2.2 with $A = \lambda\alpha^k,\ B = \mu\beta^k$ yields

$$H(\lambda\alpha^k, \mu\beta^k, 1) \le 2H(\alpha, \beta, 1)^{l-k}. \tag{5}$$

In a similar way application of Corollary 2.2 with $A = \lambda\alpha^n,\ B = \mu\beta^n$ yields

$$H(\lambda\alpha^n, \mu\beta^n, 1) \le 2H(\alpha^{-1}, \beta^{-1}, 1)^{n-m}. \tag{6}$$

Application of Lemma 2.1 with $A = \alpha^{k-n},\ B = \beta^{k-n}$ yields

$$\begin{aligned} H(\alpha^{k-n}, \beta^{k-n}, 1) &\le 2H(\lambda\alpha^n, \mu\beta^n, 1) H(\lambda\alpha^{n-k+l}, \mu\beta^{n-k+l}, 1) \\ &\le 2H(\lambda\alpha^n, \mu\beta^n, 1)^2 H(\alpha^{l-k}, \beta^{l-k}, 1). \end{aligned}$$

With (6) and $H(\alpha, \beta, 1) \le H(\alpha^{-1}, \beta^{-1}, 1)$ we get

$$H(\alpha^{-1}, \beta^{-1}, 1)^{n-k} \le 8H(\alpha^{-1}, \beta^{-1}, 1)^{2(n-m)+l-k}.$$

Using our lower bound $H(\alpha^{-1}, \beta^{-1}, 1) \ge c_4^{1/q}$ we find that

$$n - 2m + l \ge -\gamma q \quad \text{hence} \quad n - l - \gamma q \ge 2(m - l - \gamma q).$$

Denote the smallest solution by n_0 and the second smallest by n_1. Application of the inequality with $k = n_0, l = n_1$ yields

$$n - n_1 - \gamma q \ge 2(m - n_1 - \gamma q) \tag{7}$$

for any two solutions m, n with $n_1 < m < n$. We divide our solutions into three intervals,

- $I_1 = [n_0, n_1 + (0.9 + \gamma)q[$,
- $I_2 = [n_1 + (0.9 + \gamma)q, n_1 + (230 + \gamma)q[$,
- $I_3 = [n_1 + (230 + \gamma)q, \infty[$.

Since any interval of length $< q$ contains at most two solutions, the interval I_1 contains at most $1 + 2([\gamma + 0.9] + 1) \le 49$ solutions. Because of (7) the interval I_2 contains at most 8 solutions.

We finally show that I_3 contains at most 4 solutions. Suppose I_3 contains 5 solutions, the largest being denoted by N, the smallest by M. Furthermore, we let k be a solution such that there exists another solution l such that

$k < l < k + q$. Because of (7) we find $k < n_1 + (1 + \gamma)q$. Since there exists at least one closed interval of length q containing three solutions such a k exists and we may moreover assume that $k \geq n_1$. From (7) it follows that $(N - n_1 - \gamma q) \geq 16(M - n_1 - \gamma q)$. Since $k \geq n_1$ this implies $(N - k - \gamma q) \geq 16(M - k - \gamma q)$ and since $N - k > M - k > 229q$ we get $N - k \geq (16 - 15\gamma/229)(M - k) > 14(M - k)$. Application of Lemma 5.1 to the equation $\lambda\alpha^k\alpha^x + \mu\beta^k\beta^x = 1$ with $r = l - k, s = M - k, t = N - k$ yields

$$M - k - 8.4(l - k) \leq \frac{9.1}{\log H(\alpha, \beta, 1)}.$$

Using the lower bound $H(\alpha, \beta, 1) \geq c_4^{1/(2q)}$ and $l - k < q$ we get $M - k < 211q$, contradicting $M - k > 229q$.

So we conclude that I_3 contains at most 4 solutions, which leaves us with a total of at most $49 + 8 + 4 = 61$ solutions. ∎

References

[1] F. Beukers and R. Tijdeman, *On the multiplicities of binary complex recurrences*, Compositio Math. 51 (1984), 193–213.

[2] F. Beukers and D. Zagier, *Lower bounds for heights of points on hypersurfaces*, Math. Z., to appear.

[3] J. H. Evertse, *On equations in S-units and the Thue–Mahler equation*, Invent. Math. 75 (1984), 561–584.

[4] H. P. Schlickewei, *The multiplicity of binary recurrences*, ibid., to appear.

[5] —, *Equations $ax + by = 1$*, Ann. of Math., to appear.

[6] H. P. Schlickewei and W. M. Schmidt, *Linear equations in variables which lie in a multiplicative group*, to appear.

[7] H. P. Schlickewei and E. Wirsing, *Lower bounds for the heights of solutions of linear equations*, Invent. Math., to appear.

Department of Mathematics
University of Utrecht
Budapestlaan 6
P.O. Box 80010
3508 TA Utrecht, The Netherlands
E-mail: beukers@math.ruu.nl

Abteilung für Mathematik II
Universität Ulm
Helmholtzstr. 18
89069 Ulm, Germany
E-mail: hps@mathematik.uni-ulm.de

Received on 18.3.1996 (2946)