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On zero-free subset sums

by

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1. Introduction. An old problem of Erdős and Heilbronn (see [5]) has been to prove the existence of a constant c such that every subset S of an abelian group G with $|S| \geq c|G|^{1/2}$ contains a nonempty subset summing to 0. They also conjectured that the above statement holds for $c = 2$. It was later stipulated by Erdős [3] that the result holds with $c = \sqrt{2}$. A slightly more precise conjecture when G is of prime order is attributed to Selfridge [4, p. 95].

The existence of c was first proved by Szemerédi [10] . The validity of the above conjecture with $c = 2$ in the case when G is a group of prime order follows using a more general result by Olson [7]. The validity of the above conjecture with $c = 3$ in the case of an arbitrary finite group was obtained later [8].

In this paper we further reduce the constant c and get arbitrarily close to $c = \sqrt{2}$ in the following sense. We prove that, when G is of prime order, any subset S of G such that $|S| \geq \sqrt{2}|G|^{1/2} + 5\ln|G|$ contains a nonempty subset summing to zero (Theorem 3.3). When G is an arbitrary abelian group, we prove that any subset S of G such that $|S| \geq \sqrt{2}|G|^{1/2} + \varepsilon(|G|)$ contains a nonempty subset summing to zero, where $\varepsilon(n) = O(n^{1/3}\ln n)$ (Theorem 4.5).

2. Notation and preliminaries. Let G be a finite abelian group. For $S \subset G$, denote by $\Sigma(S)$ the set of sums of distinct elements of S,

$$\Sigma(S) = \Big\{\sum_{t\in T} t : T \subset S\Big\},$$

and

$$\Sigma^\star(S) = \Big\{\sum_{t\in T} t : T \subset S,\ T \neq \emptyset\Big\}.$$

Denote by $\langle X\rangle$ the subgroup of G generated by X. For $X \subset G$ such that

$0 \notin X$ and $X \cup \{0\} \neq \langle X\rangle$, define

$$\kappa(X) = \min_M |(X + M) \setminus M|$$

where M ranges over all nonempty subsets of $\langle X\rangle$ such that $M \cup (X+M) \neq \langle X\rangle$.

We shall make use of the following three classical addition theorems.

THEOREM 2.1 (Cauchy–Davenport [1, 2]). *Let A and B be nonempty subsets of $\mathbb{Z}/p\mathbb{Z}$. Then*

$$|A + B| \geq \min(p, |A| + |B| - 1).$$

THEOREM 2.2 (Kneser [6]). *Let M and N be finite nonempty subsets of an abelian group G. There exists a subgroup H of G such that*

$$M + N + H = M + N \quad \textit{and} \quad |M + N| \geq |M + H| + |N + H| - |H|.$$

THEOREM 2.3 (Scherk [9]). *Let A and B be nonempty subsets of a finite abelian group G such that $A \cap (-B) = \{0\}$. Then*

$$|A + B| \geq |A| + |B| - 1.$$

We shall use the following three theorems of Olson:

THEOREM 2.4 [7]. *Let $S \subset \mathbb{Z}/p\mathbb{Z}$. Suppose $S \cap (-S) = \emptyset$. Then*

$$|\Sigma(S)| \geq \min\left(\frac{p+3}{2}, \frac{|S|(|S|+1)}{2}\right).$$

THEOREM 2.5 [8]. *Let G be an abelian group and let $S \subset G$. If $|S| \geq 3\sqrt{|G|}$, then $0 \in \Sigma^\star(S)$.*

THEOREM 2.6 [7]. *Let G be a finite group of prime order p. Let $S \subset G$. If $|S| > \sqrt{4p-3}$, then $0 \in \Sigma^\star(S)$.*

We shall make heavy use of an averaging technique introduced by Erdős and Heilbronn and developed by Olson. For $B \subset G$ and $x \in G$, define

$$\lambda_B(x) = |(B + x) \cap \overline{B}|.$$

For any B, x, y we have (see [7])

$$\lambda_B(x) = \lambda_B(-x) \tag{1}$$

and

$$\lambda_B(x + y) \leq \lambda_B(x) + \lambda_B(y). \tag{2}$$

LEMMA 2.7 (Olson). *Let A and B be subsets of G such that $0 \notin A$. Let $a = |A|$ and $b = |B|$. Suppose that $a \geq 2b - 1$. Then*

$$\sum_{x \in A} \lambda_B(x) \geq b^2 \tag{3}$$

and

$$\exists x \in A \quad \lambda_B(x) \ge (b+1)/2. \tag{4}$$

The following lemma is essentially in [7]. We give here a slightly modified version.

LEMMA 2.8. *Let A and B be subsets of G such that $0 \notin A$. Let $S \subset A$ be such that $S \cup (-S) = A \cup (-A)$. Put $a = |A|$ and $b = |B|$. Suppose that $a < 2b - 1$, and let $\kappa = \kappa(A)$. Suppose that $|\langle A\rangle| \ge 2b$. Then*

$$\exists x \in S \quad \lambda_B(x) \ge (\kappa/2)\left(\frac{4b}{4b+\kappa}\right)^2. \tag{5}$$

Proof. Define $A_m = (A \cup (A+A) \cup \ldots \cup mA) \setminus \{0\}$. Let m and r be such that $2b - 1 = m\kappa + r$, with $0 \le r < \kappa$. Note that, since $\kappa \le a$, A_{m+1} must have at least $2b$ elements. Throw away elements of mA, and if need be of $(m-1)A$, and so on to obtain a set $A' \subset A_{m+1}$ with exactly $|A'| = 2b-1$ elements. Put $\alpha = \max_{x\in S} \lambda_B(x)$. Notice that, because of (1), $\lambda_B(x) \le \alpha$, for every $x \in A$.

Using (2), we obtain

$$\sum_{x\in A'} \lambda_B(x) \le \alpha\kappa + 2\alpha\kappa + \ldots + m\alpha\kappa + (m+1)r\alpha.$$

Hence, applying (3), we have

$$b^2 \le \alpha\left(\frac{m(m+1)}{2}\kappa + (m+1)r\right),$$
$$2b^2 \le \alpha(m+1)(2b-1+r),$$

hence, writing $(m+1)\kappa = m\kappa + r + \kappa - r = 2b - 1 + \kappa - r$, we get

$$2b^2\kappa \le \alpha(2b-1+\kappa-r)(2b-1+r)$$

and minimizing

$$\alpha \ge \kappa\frac{2b^2}{(2b-1+u)(2b-1+v)}$$

for $u \ge 0, v \ge 0, u+v = \kappa$, we obtain

$$\alpha \ge \kappa\frac{2b^2}{(2b+\kappa/2)(2b-1+\kappa/2)},$$

hence the result. ∎

The following lemma is an application of Lemma 2.7.

LEMMA 2.9. *Let G be a finite abelian group. Let $S \subset G$ be such that $S \cap (-S) = \emptyset$ and let $k \ge 2$ be an integer such that $|S| \ge k + \log_{3/2} k$. There exists $S_0 \subset S$ such that $|S \setminus S_0| \ge k$ and $|\Sigma^\star(S_0)| \ge k$.*

Proof. Prove by induction on v the existence of a set $V \subset S$ such that $|V| = v$ and $|\Sigma^\star(V)| \geq \min((3/2)^v, k)$. This is clear for $v = 2$. Suppose V exists for $|V| = v$. Let $B = \Sigma^\star(V)$, and suppose $|B| \leq k$, otherwise there is nothing to prove. Let $A = (S \setminus V) \cup -(S \setminus V)$, and note that $2|B| \leq |A|$. Apply Lemma 2.7(4) to A and B to deduce that there exists $x \in A$ such that $\lambda(x) \geq |B|/2$. Note that (1) implies that we may suppose $x \in S \setminus V$. Now $V \cup \{x\}$ gives the result for $v+1$. The lemma is proved by chosing for S_0 a set V with $v = \lceil \log_{3/2} k \rceil$. ∎

3. The prime order case. In this section, we suppose G is a cyclic group of prime order p. Let $S \subset G$, $0 \notin S$. The following lemma will be crucial in the proof of Theorem 3.3.

LEMMA 3.1. *Let $S \subset G$ be such that $S \cap (-S) = \emptyset$. Let $k \geq 2$ be an integer such that $|S| \geq k + \log_{3/2} k$. Let $m \leq k$. There exists $T \subset S$ such that $|S \setminus T| = k - m$, and*

$$|\Sigma^\star(T)| \geq \min\left(\frac{p+1}{2}, \sum_{i=0}^{m}(k-i) - \frac{9}{4}\sum_{i=1}^{m}\frac{k}{i}\right). \tag{6}$$

Proof. Choose S_0 as in Lemma 2.9 and let $K = S \setminus S_0$. We have $k = |K|$. Let us prove by induction on m the existence of a set $M_m \subset K$ such that $|M_m| = m$ and such that $T = S_0 \cup M_m$ satisfies

$$|\Sigma^\star(T)| \geq \min\left(\frac{p+1}{2}, \frac{4}{9}\sum_{i=0}^{m}(k-i)\right) \tag{7}$$

together with (6). The result holds clearly for $m = 0$, since $|\Sigma^\star(S_0)| \geq k$. Assume it is proved for m.

Suppose $m + 1 \leq k$. Let $K' = K \setminus M_m$. Set $A = K' \cup (-K')$. We have $|A| = 2|K'|$. Set $B = \Sigma^\star(M_m \cup S_0)$. Note that $|A| < 2|B| - 1$, so that we can apply Lemma 2.8 whenever $|B| \leq (p+1)/2$. Choose $x \in K'$ which maximizes $\lambda_B(x)$ among $x \in K'$, and set $M_{m+1} = M_m \cup \{x\}$. Let $\delta = \lambda_B(x)$.

By the Cauchy–Davenport theorem we have

$$\kappa(A) = |A| = 2|K'| = 2(k - m),$$

so that, by Lemma 2.8,

$$\delta \geq |K'|\left(\frac{2|B|}{2|B| + |K'|}\right)^2. \tag{8}$$

Now (7) and (6) for $m + 1$ will follow from

$$\Sigma^\star(S_0 \cup M_{m+1}) \geq \Sigma^\star(S_0 \cup M_m) + \delta \tag{9}$$

with the induction hypothesis to evaluate $\Sigma^\star(S_0 \cup M_m)$ and (8) to evaluate δ.

Since $|B| > |K'|$, we deduce from (8) that $\delta > 4|K'|/9$. Hence, applying (7) to $\Sigma^\star(S_0 \cup M_m)$, (9) gives

$$\Sigma^\star(S_0 \cup M_m \cup \{x\}) \geq \frac{4}{9}\sum_{i=0}^{m}(k-i) + \frac{4}{9}(k-m) > \frac{4}{9}\sum_{i=0}^{m+1}(k-i),$$

which gives (7) for $m+1$.

Now (8) gives

$$\delta \geq |K'|\left(1 - \frac{|K'|}{2|B| + |K'|}\right)^2 \geq |K'|\left(1 - \frac{|K'|}{|B|}\right).$$

By applying (7) to $|B|$ we get

$$\delta \geq k - m - \frac{9}{4}\cdot\frac{(k-m)^2}{\sum_{i=0}^{m}(k-i)} \geq k - m - \frac{9}{4}\cdot\frac{(k-m)^2}{m(k-m)} \geq k - m - \frac{9}{4}\cdot\frac{k}{m}.$$

Hence, (9) gives (6) for $m+1$. ∎

COROLLARY 3.2. *Let $S \subset G$ be such that $S \cap (-S) = \emptyset$. Let k be a positive integer such that $|S| \geq k + \log_{3/2} k$. Then*

$$|\Sigma^\star(S)| \geq \min\left(p, \frac{k(k+1)}{2} - \frac{9}{4}k(1 + \ln k)\right).$$

Proof. 1. If

$$\sum_{i=0}^{k}(k-i) - \frac{9}{4}\sum_{i=1}^{k}\frac{k}{i} \leq \frac{p+1}{2}$$

then apply Lemma 3.1 with $m = k$, and bound $\sum_{i=1}^{k} 1/i$ from above by $1 + \ln k$ to obtain the result.

2. If not, then let $m \leq k$ be the smallest integer satisfying

$$\sum_{i=0}^{m}(k-i) - \frac{9}{4}\sum_{i=1}^{m}\frac{k}{i} > \frac{p+1}{2}.$$

Choose T as in Lemma 3.1, and note that we have

$$|\Sigma^\star(T)| \geq \sum_{i=0}^{m-1}(k-i) - \frac{9}{4}\sum_{i=1}^{m-1}\frac{k}{i}.$$

Note that $\Sigma^\star(S) \supset \Sigma^\star(T) + \Sigma(S \setminus T)$ and $|S \setminus T| = k - m$. Apply Theorem 2.4 to $|\Sigma(S \setminus T)|$ to obtain

$$|\Sigma(S \setminus T)| \geq \min\left(\frac{p+3}{2}, \sum_{i=m}^{k}(k-i)\right).$$

The Cauchy–Davenport Theorem 2.1 applied to $|\Sigma^\star(T)+\Sigma(S\setminus T)|$ gives

$$|\Sigma^\star(S)| \ge \sum_{i=0}^{k}(k-i) - \frac{9}{4}\sum_{i=1}^{k}\frac{k}{i},$$

hence the result. ∎

THEOREM 3.3. *For any group G of prime order p, and for $S\subset G$,*

$$|S| \ge \sqrt{2p}+5\ln p \quad \textit{implies} \quad 0\in \Sigma^\star(S).$$

P r o o f. Suppose the result does not hold. Then we must have $S\cap(-S) = \emptyset$, if not, $0\in\Sigma^\star(S)$ trivially. Let $s=|S|$. Let $k=s-\lceil \log_{3/2} s\rceil$. Corollary 3.2 yields a contradiction provided we check that

$$\frac{k(k+1)}{2} - \frac{9}{4}k(1+\ln k) \ge p. \tag{10}$$

To check (10), note that one may suppose $p\ge 1000$, because otherwise Theorem 2.6 implies the result. Similarly, we may suppose $s<2\sqrt{p}$. Hence $k\ge s-\log_{3/2}(2\sqrt{p})-1\ge\sqrt{2p}+3\ln p$. Using this lower bound on k and the upper bound $\ln k\le \ln p$ it is a straightforward computation to obtain (10). ∎

4. The case of an arbitrary abelian group *G*. Let G be an arbitrary abelian group with n elements. This section is devoted to proving Theorem 4.5.

The proof shall follow similar lines to the case when G is of prime order, but we need to accomodate several points. When switching to an arbitrary abelian group, the proof of Theorem 3.3 fails essentially because of two differences with the prime order case.

1. It is not necessarily true that $0\notin\Sigma^\star(S)$ implies $S\cap(-S)=\emptyset$. And more importantly, it is not necessarily true that $\kappa(A)=|A|$ as in Lemma 3.1, because the Cauchy–Davenport theorem does not hold any more.

2. The proof of Lemma 3.1 may fail because $\langle A\rangle\neq G$, so that Lemma 2.8 may not apply any more.

To deal with the first problem we shall use the following lemma.

LEMMA 4.1. *Let $S\subset G$ be such that $0\notin\Sigma^\star(S)$ and $|S|\ge k+\log_2 n$.*

(i) *There is a subset $K\subset S$ such that $|K|=k$ and $K\cap(-K)=\emptyset$.*

(ii) *Let $A=S\cup(-S)$. Then*

$$\kappa(A)\ge 2k-6\sqrt{k}.$$

P r o o f. To prove (i), notice that the set of elements x of G such that $x+x=0$ is a vector space over the field on two elements, and that its dimension is at most $\log_2 n$. Hence the number of selfinverse elements of

S must be smaller than $\log_2 n$, otherwise some nonempty subset of those elements sums to zero.

(ii) Set $Y = K \cup (-K)$ and suppose first that $Y \cup \{0\} \neq \langle Y \rangle$; we intend to show that $\kappa(Y) \geq 2|K| - 6\sqrt{|K|}$.

By definition of $\kappa = \kappa(Y)$, there exists $M \subset \langle Y \rangle$ such that $M \cup (M+Y) \neq \langle Y \rangle$ and $|(M+Y) \setminus M| = \kappa$. Set $N = Y \cup \{0\}$.

By Kneser's Theorem 2.2 there is a subgroup H such that $H+M+N = M+N$ and $\kappa = |(M+N) \setminus M| \geq |(N+H)| - |H|$ (observe that $0 \in N$). Clearly, we may suppose $|H| \geq 2$, otherwise we have $\kappa \geq |Y|$ and the result holds. We now claim $|(H+N) \setminus H| \geq 2|H|$. Otherwise $N \subset H \cup b+H$, for some b where $2b \in H$. But this implies $\langle Y \rangle = H \cup b+H = H+N+M = M \cup M+N$, which contradicts the definition of M. It follows that

$$|H| \leq \kappa/2 \leq |Y|/2 = |K|.$$

Now we have, by Olson's Theorem 2.5, $|K \cap H| \leq 3\sqrt{|H|} \leq 3\sqrt{|K|}$. Therefore $|Y \cap H| \leq 6\sqrt{|K|}$. Now since $\kappa = |(M+Y) \setminus M| = |M+N| - |M| \geq |N+H| - |H| \geq |N \setminus H|$, it follows that

$$\kappa \geq |N| - |N \cap H| \geq |Y| + 1 - (|Y \cap H| + 1) \geq 2|K| - 6\sqrt{|K|}.$$

If $Y \cup \{0\} = \langle Y \rangle$, then argue as above after replacing K by $K \cup \{x\}$ for some $x \in S \setminus K$. ∎

The following lemma replaces Lemma 3.1 in the prime order case.

LEMMA 4.2. *Let $S \subset G$ be such that $0 \notin \Sigma^\star(G)$. Let k, d, r and $\Delta \geq \log_2 n$ be nonnegative integers such that $|S| \geq \Delta + k + r\log_{3/2} k$. Let $m \leq k$ and suppose that any subset $X \subset S$ with $|X| \geq \Delta$ is such that $|\langle X \rangle| \geq n/d$. Then there exists $T \subset S$ such that $|S \setminus T| \geq \Delta + k - m + (r-1)\log_{3/2} k$ and*

$$|\Sigma^\star(T)| \geq \min\left(\frac{n}{2d}, \sum_{i=0}^{m}(k - i - 4\sqrt{k-i}) - 3\sum_{i=1}^{m}\frac{k}{i}\right). \tag{11}$$

Proof. The proof is basically the same as that of Lemma 3.1. Apply Lemma 2.9 to obtain S_0 such that $|S \setminus S_0| \geq \Delta + (r-1)\log_{3/2} k + k$ and $|\Sigma^\star(S_0)| > k$. Let $K = S \setminus S_0$ and, as in Lemma 3.1, prove by induction on m the existence of $M_m \subset K$ of size m such that $T = S_0 \cup M_m$ satisfies (11) together with

$$|\Sigma^\star(T)| \geq \min\left(\frac{n}{2d}, \frac{1}{3}\sum_{i=0}^{m}(k - i - 4\sqrt{i})\right). \tag{12}$$

Mimic the proof of Lemma 3.1. (12) is obtained by writing

$$\delta \geq \frac{\kappa(A)}{2}\left(\frac{2|B|}{2|B| + |K'|}\right)^2 \geq \frac{4}{9} \cdot \frac{\kappa(A)}{2} \geq \frac{1}{3}(k - m - 4\sqrt{m}),$$

the last inequality being given by Lemma 4.1.

Then (11) is obtained by writing

$$\delta \geq \frac{\kappa(A)}{2}\left(1 - \frac{\kappa(A)/2}{|B|}\right)$$

and applying (12) to $|B|$, so that we get

$$\begin{aligned}\delta &\geq k - m - 4\sqrt{k-m} - 3\frac{(k-m-4\sqrt{k-m})^2}{\sum_{i=0}^{m}(k-i-4\sqrt{k-i})}\\ &\geq k - m - 4\sqrt{k-m} - 3\frac{(k-m-4\sqrt{k-m})^2}{m(k-m-4\sqrt{k-m})}\\ &\geq k - m - 4\sqrt{k-m} - 3\frac{k}{m}. \ \blacksquare\end{aligned}$$

Lemma 4.3. *Let $S \subset G$ be such that $0 \notin \Sigma^\star(G)$. Let k, d and $\Delta \geq \log_2 n$ be nonnegative integers such that $|S| \geq \Delta + k + 2d\log_{3/2} k$. Let $m \leq k$ and $j \leq 2d$. Suppose that any subset $X \subset S$ with $|X| \geq \Delta$ is such that $|\langle X\rangle| \geq n/d$. Then there exists $T \subset S$ such that $|S \setminus T| \geq \Delta + k - m + (2d-j)\log_{3/2} k$ and*

$$|\Sigma^\star(T)| \geq \min\left(\frac{jn}{2d}, \sum_{i=0}^{m}(k-i-4\sqrt{k-i}) - 3jk(1+\ln k)\right).$$

Proof. Use induction on j. The result holds for $j = 1$ by a direct application of Lemma 4.2. Suppose the lemma holds for $j < 2d$, and let m be given. Let m_1 be the smallest integer such that

$$\sum_{i=0}^{m_1}(k-i-4\sqrt{k-i}) - jk(1+\ln k) > \frac{jn}{2d}.$$

If $m_1 > m$, there is nothing to prove. If not, then let T_1 be such that $\Sigma^\star(T_1) \geq jn/(2d)$, and $|S \setminus T_1| \geq \Delta + k - m_1 + (2d-j)\log_{3/2} k$. Apply Lemma 4.2 to obtain $T_2 \subset S \setminus T_1$ such that

$$|S \setminus (T_1 \cup T_2)| \geq \Delta + k - m + (2d - j - 1)\log_{3/2} k$$

and, using $\sum_{i=1}^{s} 1/i \leq 1 + \ln k$,

$$|\Sigma^\star(T_2)| \geq \min\left(\frac{n}{2d}, \sum_{i=m_1}^{m}(k-i-4\sqrt{k-i}) - 3k(1+\ln k)\right).$$

Let now $A = \{0\} \cup \Sigma^\star(T_1)$ and $B = \{0\} \cup \Sigma^\star(T_2)$. Since $0 \notin \Sigma^\star(S)$, A and B satisfy the hypothesis of Theorem 2.3, so that we obtain

$$|\Sigma^\star(T_1 \cup T_2)| \geq |(A+B) \setminus \{0\}| \geq |\Sigma^\star(T_1)| + |\Sigma^\star(T_2)|$$

and $T = T_1 \cup T_2$ satisfies the conclusion of the lemma for $j + 1$. $\blacksquare$

COROLLARY 4.4. *Let $S \subset G$ be such that $0 \notin \Sigma^\star(G)$, and let k, d, $\Delta \geq \log_2 n$ be nonnegative integers such that $|S| \geq \Delta + k + 2d\log_{3/2} k$. Suppose that any subset $X \subset S$ with $|X| \geq \Delta$ is such that $|\langle X\rangle| \geq n/d$. Then there exists $T \subset S$ such that $|S \setminus T| = \Delta$ and*

$$|\Sigma^\star(T)| \geq \min\left(n, \frac{1}{2}k(k+1) - 3(k+1)^{3/2} - 3dk(1+\ln k)\right). \tag{13}$$

Proof. Apply Lemma 4.3 with $j = 2d$, and bound $\sum_{i=1}^k \sqrt{i}$ from above by $\frac{2}{3}(k+1)^{3/2}$. ∎

THEOREM 4.5. *There is a function $\varepsilon(n) = O(n^{1/3}\ln n)$ such that for any subset S of any finite abelian group G of order n,*

$$|S| > \sqrt{2n} + \varepsilon(n) \quad \textit{implies} \quad 0 \in \Sigma^\star(S).$$

Proof. Suppose $S \subset G$ is such that $0 \notin \Sigma^\star(G)$. Apply Corollary 4.4 to obtain $|\Sigma^\star(S)| \geq n$ and hence a contradiction.

By Theorem 2.5, the hypothesis of Corollary 4.4 holds if

$$|S| > 3\sqrt{\frac{n}{d}} + k + 2d\log_{3/2} n. \tag{14}$$

We must choose k such that $\frac{1}{2}k(k+1) - 3(k+1)^{3/2} - 3dk(1+\ln k) > n$.

Therefore, the desired contradiction is obtained when $|S|$ satisfies (14), together with

$$k > \sqrt{2n}\left(1 + \frac{3}{(2n)^{1/4}} + \frac{3d\ln n}{\sqrt{2n}}\right).$$

Choose $d \sim n^{1/3}$ to optimize. ∎

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