ACTA ARITHMETICA
LXXVIII.1 (1996)

# On the number-theoretic functions $\nu(n)$ and $\Omega(n)$

by

Jiahai Kan (Nanjing)

**1. Introduction.** Let $d(n)$ denote the divisor function, $\nu(n)$ the number of distinct prime factors, and $\Omega(n)$ the total number of prime factors of $n$, respectively. In 1984 Heath-Brown [4] proved the well-known Erdős–Mirsky conjecture [1] (which seemed at one time as hard as the twin prime conjecture, cf. [4, p. 141]):

(A) "There exist infinitely many positive integers $n$ for which
$$d(n+1) = d(n)."$$

The method of Heath-Brown [4] can also be used to prove the conjecture:

(B) "There exist infinitely many positive integers $n$ for which
$$\Omega(n+1) = \Omega(n)."$$

Another conjecture of Erdős for $d(n)$ is (cf. e.g. [5, p. 308]):

(C) "Every positive real number is a limit point of the sequence $\{d(n+1)/d(n)\}$,"

and the similar conjecture for $\Omega(n)$ is

(D) "Every positive real number is a limit point of the sequence $\{\Omega(n+1)/\Omega(n)\}$."

It follows from the results of Heath-Brown that 1 is a limit point of the sequence $\{d(n+1)/d(n)\}$, and also a limit point of the sequence $\{\Omega(n+1)/\Omega(n)\}$.

As for $\nu(n)$, Erdős has similar conjectures:

(E) "There exist infinitely many positive integers $n$ for which
$$\nu(n+1) = \nu(n)."$$

(F) "Every positive real number is a limit point of the sequence $\{\nu(n+1)/\nu(n)\}$."

Compared with the status of conjectures (A), (B), (C), (D), much less is known about conjectures (E) and (F). The best result up to date for conjecture (E) is the following

Theorem (Erdős–Pomerance–Sárközy) (cf. [2, p. 251, Theorem 1]). *There exist infinitely many positive integers $n$ for which*

$$|\nu(n+1) - \nu(n)| \le c$$

*where $c$ denotes a positive constant.*

And for conjecture (F), no limit point of the sequence $\{\nu(n+1)/\nu(n)\}$ is known yet.

The purpose of this paper is (i) to improve the result of Erdős–Pomerance–Sárközy about conjecture (E), and (ii) to prove conjectures (F) and (D). In fact, the following more general results will be proved here. Let $b$ denote any given nonzero integer, and $k$ denote any fixed integer greater than one. We have

Theorem 1. *There exist infinitely many positive integers $n$ for which*

$$|\nu(n+b) - \nu(n)| \le 1 \quad \text{and} \quad \nu(n) = k.$$

Theorem 2. *Every positive real number is a limit point of the sequence $\{\nu(n+b)/\nu(n)\}$.*

Theorem 3. *Every positive real number is a limit point of the sequence $\{\Omega(n+b)/\Omega(n)\}$.*

**2. Lemmas.** We deduce in this section some lemmas by the sieve method. Terminology and notations here have their customary meaning and coincide with those of [3] and [6].

Let $\mathcal{A}$ denote a finite set of integers, $|\mathcal{A}| \sim X$. Let

$$\mathcal{A}_d = \{a : a \in \mathcal{A},\ d \mid a\},$$

and assume that, for squarefree $d$,

$$|\mathcal{A}_d| = \frac{\omega(d)}{d} X + r_d, \qquad \text{and } \omega(d) \text{ is multiplicative.}$$

Define $\mathcal{P} = \{p : p \mid a,\ a \in \mathcal{A}\}$ (i.e., $\mathcal{P}$ is the set of all primes dividing at least one $a$ in $\mathcal{A}$), and $\overline{\mathcal{P}}$ the complement of $\mathcal{P}$ with respect to the set of all primes.

In the following conditions the $A_i$'s denote positive constants.

$(\Omega_1)$ $$0 \le \omega(p)/p \le 1 - 1/A_1.$$

$(\Omega_2^*(1))$ $$-A_2 \ln\ln 3X \le \sum_{w \le p < z} \frac{\omega(p)}{p} \ln p - \ln \frac{z}{w} \le A_2 \quad \text{if } 2 \le w \le z.$$

$$(\Omega_3) \qquad \sum_{z\le p<y,\, p\in\mathcal{P}} |\mathcal{A}_{p^2}| \le A_3\left(\frac{X\ln X}{z}+y\right) \quad \text{if } 2\le z\le y.$$

($R^*(1,\alpha)$) There exists $\alpha$ ($0<\alpha\le 1$) such that, for any given $A>0$, there is $B=B(A)>0$ such that

$$\sum_{d<X^\alpha \ln^{-B} X,\, (d,\overline{\mathcal{P}})=1} \mu^2(d)3^{\nu(d)}|r_d| \le A_4 X\ln^{-A} X.$$

As a kind of exponential measure for the magnitude of the $a$'s of $\mathcal{A}$ we introduce, for each positive integer $r$, the function

$$\Lambda_r = r+1-\frac{\ln(4/(1+3^{-r}))}{\ln 3}. \tag{1}$$

Clearly $\Lambda_r$ is increasing, $\Lambda_1=1$ and

$$r+1-\frac{\ln 4}{\ln 3}\le \Lambda_r \le r+1-\frac{\ln 3.6}{\ln 3} \quad \text{for } r\ge 2. \tag{2}$$

Lemma 1. *Let* ($\Omega_1$), ($\Omega_2^*(1)$), ($\Omega_3$) *and* ($R^*(1,\alpha)$) *hold. Suppose that*

$$(a,\overline{\mathcal{P}})=1 \quad \textit{for all } a\in\mathcal{A}. \tag{3}$$

*Let $\delta$ be a real number satisfying*

$$0<\delta<\Lambda_2, \tag{4}$$

*and let $r_0$ be the least integer of all $r$'s ($r\ge 2$) satisfying*

$$|a|\le X^{\alpha(\Lambda_r-\delta)} \quad \textit{for all } a\in\mathcal{A}. \tag{5}$$

*Then we have*, *for $X\ge X_0$*,

$$\begin{aligned}(6)\quad &\#\{n: n\in\mathcal{A},\ n=p_1\ldots p_{t+1} \text{ or } p_1\ldots p_{t+2} \text{ or } \ldots \text{or } p_1\ldots p_r,\\ &\quad p_1<\ldots<p_t<X^{1/\ln\ln X},\ X^{\alpha/4}\le p_{t+1}<p_{t+2}<\ldots<p_r\}\\ &\quad\quad > \frac{c(r_0,\delta)}{t!\alpha}c(\omega)X\ln^{-1}X(\ln\ln X)^t\left(1-O\left(\frac{\ln\ln\ln X}{\ln\ln X}\right)\right),\end{aligned}$$

*where $p_i$'s denote primes*, $t=r-r_0$,

$$c(r_0,\delta)=2(r_0+1-(1+3^{-r_0})(\Lambda_{r_0}-\delta))^{-1}\delta(1+3^{-r_0})\ln 3, \tag{7}$$

*and*

$$c(\omega)=\prod_p(1-\omega(p)/p)(1-1/p)^{-1}. \tag{8}$$

Proof. This lemma follows from [6, Theorem 1 and p. 281, (39) of Remark 3].

Lemma 2. *Let $F(n)$ ($\ne \pm n$) be an irreducible polynomial of degree $g$ ($\ge 1$) with integer coefficients. Let $\varrho(p)$ denote the number of solutions of the congruence*

$$F(m)\equiv 0 \bmod p.$$

*Suppose that*

$$\varrho(p) < p \quad \text{for all } p, \tag{9}$$

*and also that*

$$\varrho(p) < p - 1 \quad \text{if } p \nmid F(0) \text{ and } p \le g + 1. \tag{10}$$

*Then we have, for any fixed* $r \ge r_0 = 2g + 1$ *and for* $x > x_0 = x_0(F)$,

$$\begin{aligned} \#\{p : p < x,\ F(p) &= p_1 \ldots p_{r-r_0+1} \text{ or } p_1 \ldots p_{r-r_0+2} \text{ or } \ldots \text{ or } \\ &\qquad p_1 \ldots p_r,\ p_1 < p_2 < \ldots < p_r\} \\ > \frac{3/2}{(r-r_0)!} \prod_{p \nmid F(0)} &\frac{1 - \varrho(p)/(p-1)}{1 - 1/p} \prod_{p \mid F(0)} \frac{1 - (\varrho(p) - 1)/(p-1)}{1 - 1/p} \\ &\times x \ln^{-2} x (\ln\ln x)^{r-r_0}. \end{aligned} \tag{11}$$

Proof. We consider the sequence

$$\mathcal{A} = \{F(p) : p < x\},$$

and we take $\mathcal{P}$ to be the set of all primes.

In [3, pp. 22–24, Example 6] (with $k = 1$), in accordance with [3, p. 23 (3.48), p. 28 (4.15), p. 24 (3.51)], we choose

$$X = \operatorname{li} x, \quad \omega(p) = \frac{\varrho_1(p)}{p-1} p \quad \text{for all } p, \tag{12}$$

where (cf. [3, p. 24 (3.53)])

$$\varrho_1(p) = \begin{cases} \varrho(p) & \text{if } p \nmid F(0), \\ \varrho(p) - 1 & \text{if } p \mid F(0). \end{cases} \tag{13}$$

From [3, p. 28 (4.15), p. 24 (3.52) and p. 24 (3.55)] we have

$$|R_d| \le g^{\nu(d)}(E(x,d) + 1) \quad \text{if } \mu(d) \ne 0, \tag{14}$$

where (cf. [3, p. 22 (3.41)])

$$E(x,d) = \max_{2 \le y \le x} \max_{\substack{1 \le a \le d \\ (a,d)=1}} |\pi(y; d, a) - \operatorname{li} y / \varphi(d)|.$$

It is now a matter of confirming the conditions under which Lemma 1 is valid.

First consider $(\Omega_1)$. Here we see that, for $p \le g + 1$, (13), (10) and (9) imply that

$$\varrho_1(p) \le p - 2,$$

and hence that

$$\omega(p) \le \frac{p-2}{p-1} p \le \left(1 - \frac{1}{g}\right) p \quad \text{if } p \le g + 1;$$

if, on the other hand, $p \geq g+2$, then, by [3, p. 24 (3.54)],

$$\varrho_1(p) \leq \varrho(p) \leq g,$$

and we find that

$$\omega(p) \leq \frac{g}{p-1}p \leq \frac{g}{g+1}p = \left(1 - \frac{1}{g+1}\right)p,$$

thus verifying $(\Omega_1)$ with $A_1 = g+1$.

Condition $(\Omega_2^*(1))$ is a consequence of Nagel's result (cf. [3, p. 18 (3.17)] with $k=1$)

$$\sum_{p<w} \frac{\varrho(p)}{p} \ln p = \ln w + O_F(1).$$

Moreover, since

$$\#\{p' : p' < x,\ F(p') \equiv 0 \bmod p^2\} \leq \#\{n : n < x,\ F(n) \equiv 0 \bmod p^2\} \ll \frac{x}{p^2} + 1 \ll \frac{X \ln X}{p^2} + 1,$$

it is easy to see that $(\Omega_3)$ is satisfied.

As for $(R^*(1,\alpha))$, we see from (14) and Bombieri's theorem (cf. [3, p. 111, Lemma 3.3, p. 115, Lemmas 3.4 and 3.5]) that, for any given $A > 0$, there is $B = B(A) > 0$ such that

$$\sum_{d < X^{1/2} \ln^{-B} X} \mu^2(d) 3^{\nu(d)} |r_d| \ll \frac{x}{\ln^{A+1} x} \ll \frac{X}{\ln^A X}.$$

Thus $(R^*(1,\alpha))$ holds with

$$\alpha = 1/2. \tag{15}$$

Finally, because of our choice of $\mathcal{P}$, (3) is trivially true (cf. [6, p. 285 (40)]).

We may now apply Lemma 1. We take

$$\delta = 2/3 \quad \text{and} \quad r_0 = 2g+1$$

and find that, by (15) and (2), for $r \geq r_0$,

$$\alpha(\Lambda_r - \delta) > \frac{1}{2}\left(2g + 1 - \frac{2}{7} - \frac{2}{3}\right) = g + \frac{5}{14} - \frac{1}{3},$$

so that (5) is satisfied if $x > x_1 = x_1(F)$. Hence, by Lemma 1, (12) and (15), we have

$$\begin{aligned} &\#\{p : p < x,\ F(p) = p_1 \ldots p_{r-r_0+1} \text{ or } p_1 \ldots p_{r-r_0+2} \text{ or } \ldots \text{or} \\ &\qquad\qquad p_1 \ldots p_r,\ p_1 < p_2 < \ldots < p_r\} \\ &\geq \frac{2}{(r-r_0)!} c(r_0,\delta) \prod_p \frac{1 - \varrho_1(p)/(p-1)}{1 - 1/p} \cdot \frac{x}{\ln^2 x} (\ln\ln x)^{r-r_0}. \end{aligned} \tag{16}$$

It follows from (7), (2) and $\delta = 2/3$ that

$$c(r_0,\delta) > 2(r_0+1-\Lambda_{r_0}+\delta)^{-1}\delta\ln 3 \geq 2\left(\frac{\ln 4}{\ln 3}+\frac{2}{3}\right)^{-1}\frac{2}{3}\ln 3 > 0.7595. \tag{17}$$

Combining (16), (17) and (13) we obtain (11), and the proof of Lemma 2 is complete.

Lemma 3. *Let $a$ and $b$ be integers satisfying*

$$ab \neq 0, \quad (a,b)=1 \quad \text{and} \quad 2\,|\,ab. \tag{18}$$

*Then, for any fixed integer $r \geq 3$ and for $x \geq x_0 = x_0(a,b)$, we have*

$$\#\{p : p < x,\ ap+b = p_1\ldots p_{r-2} \text{ or } p_1\ldots p_{r-1} \text{ or } p_1\ldots p_r,\ p_1<p_2<\ldots<p_r\} > \frac{3}{(r-3)!}\prod_{p>2}(1-(p-1)^{-2})\prod_{2<p\mid ab}\frac{p-1}{p-2}\cdot\frac{x}{\ln^2 x}(\ln\ln x)^{r-3}. \tag{19}$$

Proof. In Lemma 2 let $F(n) = an+b$. Since (18) implies (9), (10) and $b\neq 0$, by Lemma 2 we have the assertion.

**3. Proof of the Theorems.** Let $q_i$ denote a prime. In Lemma 3 we take $a = q_1q_2\ldots q_{r-2}$ with $q_1<q_2<\ldots<q_{r-2}$, and let $n = ap$. Then from (19) it is easy to see that there are infinitely many $n$ for which

$$\nu(n) = \nu(ap) = \nu(q_1q_2\ldots q_{r-2}p) = r-1$$

and

$$\nu(n+b) = \nu(ap+b) = t,$$

where

$$t = r-2 \text{ or } r-1 \text{ or } r;$$

so for such $n$,

$$|\nu(n+b)-\nu(n)| \leq 1 \quad \text{and} \quad \nu(n) = r-1.$$

This completes the proof of Theorem 1.

If in Lemma 3 we take $a = q_1q_2\ldots q_{s-1}$, $q_1<q_2<\ldots<q_{s-1}$, and let $n = ap$, then from (19) again we see that there are infinitely many $n$ for which

$$\nu(n) = \Omega(n) = s \quad \text{and} \quad \nu(n+b) = \Omega(n+b) = t,$$

where

$$t = r-2 \text{ or } r-1 \text{ or } r;$$

so for such $n$,

$$\frac{\nu(n+b)}{\nu(n)} = \frac{\Omega(n+b)}{\Omega(n)} = \frac{t}{s}.$$

Moreover, for any given positive real number $\alpha$ and for any small $\varepsilon > 0$, the fraction $t/s$ (where $t$, $s$ are both variable) may be chosen to approximate $\alpha$ arbitrarily closely, i.e.

$$|\alpha - t/s| < \varepsilon.$$

Thus $\alpha$ is a limit point of the sequence $\{\nu(n+b)/\nu(n)\}$, as well as a limit point of the sequence $\{\Omega(n+b)/\Omega(n)\}$. We have thus completed the proof of Theorems 2 and 3.

Remark. The method here gives for the number of solutions of

$$|\nu(n+b) - \nu(n)| \le 1, \qquad n \le x,$$

a lower bound $\gg x \ln^{-2} x (\ln\ln x)^t$ for $t$ arbitrarily large but fixed. In view of the works of Heath-Brown, Hildebrand, and Erdős–Pomerance–Sárközy, it seems reasonable to conjecture that this lower bound is $\gg x/\sqrt{\ln\ln x}$.

## References


[1] P. Erdős and L. Mirsky, *The distribution of the values of the divisor function $d(n)$*, Proc. London Math. Soc. (3) 2 (1952), 257–271.

[2] P. Erdős, C. Pomerance and A. Sárközy, *On locally repeated values of certain arithmetic functions II*, Acta Math. Hungar. 49 (1987), 251–259.

[3] H. Halberstam and H.-E. Richert, *Sieve Methods*, Academic Press, London, 1974.

[4] D. R. Heath-Brown, *The divisor function at consecutive integers*, Mathematika 31 (1984), 141–149.

[5] A. Hildebrand, *The divisor function at consecutive integers*, Pacific J. Math. 129 (1987), 307–319.

[6] J. Kan, *On the lower bound sieve*, Mathematika 37 (1990), 273–286.



Nanjing Institute of Post and Telecommunications
210003 Nanjing
Nanjing, China



*Received on 12.2.1996*
*and in revised form on 5.6.1996* (2928)