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Gauss sums for $O^+(2n, q)$

by

Dae San Kim and In-Sok Lee (Seoul)

1. Introduction. Let λ be a nontrivial additive character of the finite field $\mathbb{F}_q$, and let χ be a multiplicative character of $\mathbb{F}_q$. Throughout this paper, we assume that q is a power of an odd prime. Then we consider the exponential sum

$$\sum_{g \in \mathrm{SO}^+(2n,q)} \lambda(\operatorname{tr} g), \tag{1.1}$$

where $\mathrm{SO}^+(2n, q)$ is a special orthogonal group over $\mathbb{F}_q$ (cf. (2.6)) and $\operatorname{tr} g$ is the trace of g. Also, we consider

$$\sum_{g \in O^+(2n,q)} \chi(\det g)\lambda(\operatorname{tr} g), \tag{1.2}$$

where $O^+(2n, q)$ is an orthogonal group over $\mathbb{F}_q$ (cf. (2.2)) and $\det g$ is the determinant of g.

The purpose of this paper is to find explicit expressions for the sums (1.1) and (1.2). It turns out that both of them are polynomials in q with coefficients involving powers of ordinary Kloosterman sums.

In [5], Hodges expressed certain exponential sums in terms of what we call the "generalized Kloosterman sum over nonsingular symmetric matrices" $K_{\mathrm{sym},t}(A, B)$ (for m even in the main theorem of [5]) and the "signed generalized Kloosterman sum over nonsingular symmetric matrices" $L_{\mathrm{sym},t}(A, B)$ (for m odd in the main theorem of [5]), where A, B are $t \times t$ symmetric matrices over $\mathbb{F}_q$ (cf. (6.1) and [10], (7.1)).

Some of their general properties were investigated in [5], and, for A or B zero, they were evaluated in [4] (see also [5], Theorem 10). However, they have never been explicitly computed for both A and B nonzero.

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From a corollary to the main theorem in [5] and using an explicit expression of the similar sum to (1.2) but over $O(2n+1,q)$, we were able to find, in [10], an expression for $L_{\mathrm{sym},2n+1}\big(\frac{a^2}{4}C^{-1},C\big)$, where C is a nonsingular symmetric matrix of size $2n+1$ and $0\neq a\in\mathbb{F}_q$.

In this paper, from the corollary mentioned above and Theorem 5.1, we will be able to find an explicit expression for $K_{\mathrm{sym},2n}\big(\frac{a^2}{4}C^{-1},C\big)$, where C is now a nonsingular symmetric matrix of size $2n$ with $C\sim J^+$ (cf. (2.3) and (2.13)) and $0\neq a\in\mathbb{F}_q$ as before. On the other hand, $K_{\mathrm{sym},2n}\big(\frac{a^2}{4}C^{-1},C\big)$ for $C\sim J^-$ (cf. (2.14)) was determined in [9].

Similar sums for other classical groups over a finite field have been considered and the results for these sums will appear in various places.

Finally, we would like to state the main results of this paper. For some symbols, one is referred to the next section.

THEOREM A. *The sum* $\sum_{g\in\mathrm{SO}^+(2n,q)}\lambda(\mathrm{tr}\,g)$ *in* (1.1) *equals*

$$q^{n^2-n-1}\sum_{r=0}^{[n/2]}q^{r(r+1)}\begin{bmatrix}n\\2r\end{bmatrix}_q\prod_{j=1}^{r}(q^{2j-1}-1)\\ \times\sum_{l=1}^{[(n-2r+2)/2]}q^lK(\lambda;1,1)^{n-2r+2-2l}\sum(q^{j_1}-1)\dots(q^{j_{l-1}}-1),$$

where $K(\lambda;1,1)$ *is the usual Kloosterman sum as in* (2.7) *and the innermost sum is over all integers* $j_1,\dots,j_{l-1}$ *satisfying* $2l-3\le j_1\le n-2r-1$, $2l-5\le j_2\le j_1-2,\dots,1\le j_{l-1}\le j_{l-2}-2$.

THEOREM B. *The sum* $\sum_{g\in O^+(2n,q)}\chi(\det g)\lambda(\mathrm{tr}\,g)$ *in* (1.2) *equals*

$$q^{n^2-n-1}\Bigg\{\sum_{r=0}^{[n/2]}q^{r(r+1)}\begin{bmatrix}n\\2r\end{bmatrix}_q\prod_{j=1}^{r}(q^{2j-1}-1)\\ \times\sum_{l=1}^{[(n-2r+2)/2]}q^lK(\lambda;1,1)^{n-2r+2-2l}\sum(q^{j_1}-1)\dots(q^{j_{l-1}}-1)\\ +\chi(-1)\sum_{r=0}^{[(n-1)/2]}q^{r(r+1)}\begin{bmatrix}n\\2r+1\end{bmatrix}_q\prod_{j=1}^{r+1}(q^{2j-1}-1)\\ \times\sum_{l=1}^{[(n-2r+1)/2]}q^lK(\lambda;1,1)^{n-2r+1-2l}\sum(q^{j_1}-1)\dots(j^{j_{l-1}}-1)\Bigg\},$$

where the first unspecified sum runs over the same set of integers as in Theorem A *above and the second unspecified sum runs over all integers*

$j_1,\dots,j_{l-1}$ satisfying $2l-3 \le j_1 \le n-2r-2$, $2l-5 \le j_2 \le j_1-2,\dots,$ $1 \le j_{l-1} \le j_{l-2}-2$.

THEOREM C. *Let $0 \ne a \in \mathbb{F}_q$. Then, for any nonsingular symmetric matrix over $\mathbb{F}_q$ of size $2n$ with $C \sim J^+$, the Kloosterman sum below over nonsingular symmetric matrices* (cf. (6.1)) *is independent of C, and*

$$K_{\mathrm{sym},2n}\left(\frac{a^2}{4}C^{-1}, C\right) = q^n \sum_{g \in O^+(2n,q)} \lambda_a(\operatorname{tr} g),$$

so that it equals q^n times the expression in Theorem B *above with χ trivial, $\lambda = \lambda_a$* (cf. (2.1)).

The above Theorems A, B, and C are respectively stated as Theorem 4.3, Theorem 5.1, and Theorem 6.2.

2. Preliminaries. In this section, we will fix some notations that will be used throughout this paper, describe some basic groups, recall the usual Kloosterman sum and mention the q-binomial theorem. One may refer to [1] and [12] for some elementary facts of the following.

Let $\mathbb{F}_q$ denote the finite field with q elements, $q = p^d$ ($p > 2$ an odd prime, d a positive integer).

Let λ be an additive character of $\mathbb{F}_q$. Then $\lambda = \lambda_a$ for a unique $a \in \mathbb{F}_q$, where, for $\alpha \in \mathbb{F}_q$,

$$\lambda_a(\alpha) = \exp\left\{\frac{2\pi i}{p}(a\alpha + (a\alpha)^p + \ldots + (a\alpha)^{p^{d-1}})\right\}. \tag{2.1}$$

It is nontrivial if $a \ne 0$.

$\operatorname{tr} A$ and $\det A$ denote respectively the trace of A and the determinant of A for a square matrix A, and ${}^tB$ denotes the transpose of B for any matrix B.

$\mathrm{GL}(n,q)$ is the group of all nonsingular $n \times n$ matrices with entries in $\mathbb{F}_q$. Then

$$O^+(2n,q) = \{g \in \mathrm{GL}(2n,q) \mid {}^tgJ^+g = J^+\}, \tag{2.2}$$

where

$$J^+ = \begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix}. \tag{2.3}$$

We write $g \in O^+(2n,q)$ as

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A, B, C, D are of size n. Then (2.2) is given by

$$O^+(2n,q) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GL}(2n,q) \,\middle|\, {}^tAC + {}^tCA = 0, \ {}^tAD + {}^tCB = 1_n, \ {}^tBD + {}^tDB = 0 \right\} \tag{2.4}$$

$$= \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GL}(2n,q) \,\middle|\, A{}^tB + B{}^tA = 0, \ A{}^tD + B{}^tC = 1_n, \ C{}^tD + D{}^tC = 0 \right\}.$$

$P(2n,q)$ is the maximal parabolic subgroup of $O^+(2n,q)$ defined by

$$P(2n,q) = \left\{ \begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix} \,\middle|\, A \in \mathrm{GL}(n,q), \ {}^tB = -B \right\}. \tag{2.5}$$

Moreover,

$$\mathrm{SO}^+(2n,q) = \{ g \in O^+(2n,q) \mid \det g = 1 \}, \tag{2.6}$$

which is a subgroup of index 2 in $O^+(2n,q)$.

For a nontrivial additive character λ of $\mathbb{F}_q$, $a, b \in \mathbb{F}_q$, $K(\lambda; a, b)$ is the Kloosterman sum defined by

$$K(\lambda; a, b) = \sum_{\alpha \in \mathbb{F}_q^\times} \lambda(a\alpha + b\alpha^{-1}). \tag{2.7}$$

For integers n, r with $0 \le r \le n$, we define the q-binomial coefficients as

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \prod_{j=0}^{r-1} (q^{n-j} - 1)/(q^{r-j} - 1). \tag{2.8}$$

The order of the group $\mathrm{GL}(n,q)$ is denoted by

$$g_n = \prod_{j=0}^{n-1} (q^n - q^j) = q^{\binom{n}{2}} \prod_{j=1}^{n} (q^j - 1). \tag{2.9}$$

Then we have

$$\frac{g_n}{g_{n-r} g_r} = q^{r(n-r)} \begin{bmatrix} n \\ r \end{bmatrix}_q, \tag{2.10}$$

for integers n, r with $0 \le r \le n$.

For x an indeterminate, n a nonnegative integer,

$$(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1}). \tag{2.11}$$

Then the q-binomial theorem says

$$\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q (-1)^r q^{\binom{r}{2}} x^r = (x;q)_n. \tag{2.12}$$

$[y]$ denotes the greatest integer $\leq y$, for a real number y.

For $n \times n$ matrices A, B over $\mathbb{F}_q$, we will say A is equivalent to B and write

$$A \sim B \quad \text{if and only if} \quad B = {}^t g A g \text{ for some } g \in \mathrm{GL}(n,q). \tag{2.13}$$

Finally, for a fixed element ε in $\mathbb{F}_q^\times - \mathbb{F}_q^{\times 2}$,

$$J^- = \begin{bmatrix} 0 & & 1_{n-1} & 0 & 0 \\ & & & \vdots & \\ 1_{n-1} & & 0 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & -\varepsilon \end{bmatrix}. \tag{2.14}$$

3. Bruhat decomposition. In this section, we will discuss the Bruhat decomposition of $O^+(2n,q)$ with respect to the maximal parabolic subgroup $P(2n,q)$ of $O^+(2n,q)$ (cf. (2.5)).

This decomposition will play a key role in deriving the main theorems in Sections 4 and 5, and an elementary proof of that will be provided.

As a simple application, we will demonstrate that this decomposition, when combined with the q-binomial theorem, can be used to derive the order of the group $O^+(2n,q)$.

THEOREM 3.1. (a) *There is a one-to-one correspondence*

$$P(2n,q)\backslash O^+(2n,q) \to \mathrm{GL}(n,q)\backslash \Lambda$$

given by

$$P(2n,q) \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \mathrm{GL}(n,q)[C\ D],$$

where

$$\Lambda = \{[C\ D] \mid C, D\ n \times n \text{ matrices over } \mathbb{F}_q,\ \mathrm{rank}[C\ D] = n,\ C^t D + D^t C = 0\}.$$

(b) *For given $[C\ D] \in \Lambda$, there exists a unique r $(0 \leq r \leq n)$, $g \in \mathrm{GL}(n,q)$, $p \in P(2n,q)$ such that*

$$g[C\ D]p = \begin{bmatrix} 1_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{bmatrix}.$$

(c) *We have*

$$O^+(2n,q) = \coprod_{r=0}^{n} P \sigma_r P,$$

where $P = P(2n, q)$ *and*

$$\sigma_r = \begin{bmatrix} 0 & 0 & 1_r & 0 \\ 0 & 1_{n-r} & 0 & 0 \\ 1_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{bmatrix} \in O^+(2n, q). \tag{3.1}$$

P r o o f. The map in (a) is clearly well-defined and it is easy to see that it is injective. For the surjectivity, it suffices to show that, for a given $[C\ D] \in \Lambda$, there exists $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in O^+(2n, q)$ (cf. (2.4)) whose lower half is the given $[C\ D]$.

Choose $g' \in \mathrm{GL}(n, q)$ so that $g'[C\ D]$ is a row echelon matrix. Let r $(0 \le r \le n)$ be the number of pivots in $g'C$. Then, for some $h \in \mathrm{GL}(n, q)$,

$$g'[C\ D] \begin{bmatrix} h & 0 \\ 0 & {}^t h^{-1} \end{bmatrix} = \begin{bmatrix} 1_r & 0 & \\ & & D' \\ 0 & 0 & \end{bmatrix}. \tag{3.2}$$

Write

$$D' = \begin{bmatrix} D'_1 & D'_2 \\ D'_3 & D'_4 \end{bmatrix},$$

where D'_1 is of size r, and D'_4 is of size $(n - r)$, etc.

One can check directly that $\widetilde{g}[C\ D]\widetilde{p} \in \Lambda$, for any $\widetilde{g} \in \mathrm{GL}(n, q)$, $\widetilde{p} \in P(2n, q)$. Thus, in (3.2),

$$\begin{bmatrix} 1_r & 0 \\ 0 & 0 \end{bmatrix} {}^t D' = -D'^t \begin{bmatrix} 1_r & 0 \\ 0 & 0 \end{bmatrix}$$

must be satisfied, i.e., ${}^t D'_1 = -D'_1$, $D'_3 = 0$.

Write

$$p' = \begin{bmatrix} & -D'_1 & & -D'_2 \\ 1_n & & & \\ & {}^t D'_2 & & 0 \\ & & & \\ 0 & & 1_n & \end{bmatrix} \in P(2n, q).$$

Then (3.2) right multiplied by p' is

$$\begin{bmatrix} 1_r & 0 & 0 & 0 \\ 0 & 0 & 0 & D'_4 \end{bmatrix}. \tag{3.3}$$

Since (3.3) is of full rank, D'_4 must be invertible. Thus, with

$$g = \begin{bmatrix} 1_r & 0 \\ 0 & {D'_4}^{-1} \end{bmatrix} g' \in \mathrm{GL}(n, q), \qquad p = \begin{bmatrix} h & 0 \\ 0 & {}^t h^{-1} \end{bmatrix} p' \in P(2n, q),$$

we have

$$g[C\ D]p = \begin{bmatrix} 1_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{bmatrix}.$$

So (b) is proved. Moreover,

$$\begin{bmatrix} {}^t g & 0 \\ 0 & g^{-1} \end{bmatrix} \sigma_r p^{-1}$$

is a matrix in $O^+(2n,q)$ whose lower half is the given $[C\ D]$. Thus the proof of (a) is complete.

In view of (a), the Bruhat decomposition in (c) is equivalent to

$$\Lambda = \coprod_{r=0}^{n} G \begin{bmatrix} 1_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{bmatrix} P, \tag{3.4}$$

where $G = \mathrm{GL}(n,q)$, $P = P(2n,q)$. Λ is such a union of double cosets as in (3.4) by (b). The disjointness in (3.4) is easy to check. ∎

Write, for each r $(0 \le r \le n)$,

$$A_r = A_r(q) = \{p \in P(2n,q) \mid \sigma_r p \sigma_r^{-1} \in P(2n,q)\}. \tag{3.5}$$

Expressing $O^+(2n,q)$ as a disjoint union of right cosets of $P = P(2n,q)$, the Bruhat decomposition in (c) of Theorem 3.1 can be rewritten as follows.

Corollary 3.2.

$$O^+(2n,q) = \coprod_{r=0}^{n} P\sigma_r(A_r \backslash P), \tag{3.6}$$

where $P = P(2n,q)$, σ_r is as in (3.1) *and A_r is as in* (3.5).

Observing that $\det g = 1$ for $g \in P(2n,q)$ and $\det \sigma_r = (-1)^r$, we get the following.

Corollary 3.3.

$$\mathrm{SO}^+(2n,q) = \coprod_{\substack{0 \le r \le n \\ r\ even}} P\sigma_r(A_r \backslash P), \tag{3.7}$$

$$O^+(2n,q) = \coprod_{\substack{0 \le r \le n \\ r\ even}} P\sigma_r(A_r \backslash P) \amalg \coprod_{\substack{0 \le r \le n \\ r\ odd}} P\sigma_r(A_r \backslash P). \tag{3.8}$$

Write $p \in P(2n,q)$ as

$$p = \begin{bmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix}, \tag{3.9}$$

with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad {}^tA^{-1} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ -{}^tB_{12} & B_{22} \end{bmatrix}, \tag{3.10}$$
$$ {}^tB_{11} = -B_{11}, \quad {}^tB_{22} = -B_{22}.$$

Here A_{11}, A_{12}, A_{21}, and A_{22} are respectively of sizes $r \times r$, $r \times (n-r)$, $(n-r) \times r$, and $(n-r) \times (n-r)$, and similarly for ${}^tA^{-1}$ and B.

Then, by multiplying out, we see that

$$\sigma_r p \sigma_r^{-1} \in P(2n, q)$$

if and only if $A_{11}B_{11} - A_{12}{}^tB_{12} = 0$, $A_{12} = 0$, $E_{21} = 0$ if and only if $A_{12} = 0$, $B_{11} = 0$. Hence

$$|A_r(q)| = g_r g_{n-r} q^{\binom{n}{2}} q^{r(2n-3r+1)/2} \tag{3.11}$$

where g_n is as in (2.9). Also,

$$|P(2n, q)| = q^{\binom{n}{2}} g_n. \tag{3.12}$$

From (2.10), (3.11) and (3.12), we get

$$|A_r(q) \backslash P(2n, q)| = q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_q. \tag{3.13}$$

This will be used later in Sections 4 and 5. Also, from (3.12) and (3.13),

$$|P(2n, q)|^2 |A_r(q)|^{-1} = q^{\binom{n}{2}} g_n \begin{bmatrix} n \\ r \end{bmatrix}_q q^{\binom{r}{2}}. \tag{3.14}$$

From (3.6),

$$|O^+(2n, q)| = \sum_{r=0}^{n} |P(2n, q)|^2 |A_r(q)|^{-1}. \tag{3.15}$$

Applying the binomial theorem (2.12) with $x = -1$ and from (3.14) and (3.15), we get the following theorem. We note here that this result was already shown in [3]. See also Theorem 6.21 in [14].

Theorem 3.4.

$$|O^+(2n, q)| = 2q^{n^2-n}(q^n - 1) \prod_{j=1}^{n-1} (q^{2j} - 1). \tag{3.16}$$

P r o o f.

$$|O^+(2n,q)| = q^{\binom{n}{2}} g_n \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q q^{\binom{r}{2}} = q^{\binom{n}{2}} g_n(-1;q)_n$$
$$= 2q^{n^2-n} \prod_{j=1}^{n} (q^j - 1) \prod_{j=1}^{n-1} (q^j + 1)$$
$$= 2q^{n^2-n}(q^n - 1) \prod_{j=1}^{n-1} (q^{2j} - 1). \ \blacksquare$$

4. $\mathrm{SO}^+(2n,q)$ **case.** In this section, we will consider the sum in (1.1)

$$\sum_{g \in \mathrm{SO}^+(2n,q)} \lambda(\operatorname{tr} g)$$

for any nontrivial additive character λ of $\mathbb{F}_q$ and find an explicit expression for this by using the decomposition in (3.7).

The sum in (1.1) can be written, using (3.7), as

$$\sum_{\substack{0 \le r \le n \\ r \text{ even}}} |A_r \backslash P| \sum_{g \in P} \lambda(\operatorname{tr} g\sigma_r), \tag{4.1}$$

where $P = P(2n,q)$, $A_r = A_r(q)$ is as in (3.5), and σ_r is as in (3.1).

Here one has to note that, for each $h \in P$,

$$\sum_{g \in P} \lambda(\operatorname{tr} g\sigma_r h) = \sum_{g \in P} \lambda(\operatorname{tr} hg\sigma_r) = \sum_{g \in P} \lambda(\operatorname{tr} g\sigma_r).$$

Write $g \in P$ as in (3.9) with A, ${}^tA^{-1}$, B as in (3.10). Then $g\sigma_r$ is

$$\begin{bmatrix} M & & A_{12} & & \\ & & & * & \\ N & & A_{22} & & \\ & & & 0 & E_{12} \\ & * & & & \\ & & & 0 & E_{22} \end{bmatrix},$$

where $M = A_{11}B_{11} - A_{12}{}^tB_{12}$, $N = A_{21}B_{11} - A_{22}{}^tB_{12}$. So, for any r $(0 \le r \le n)$,

$$\sum_{g \in P} \lambda(\operatorname{tr} g\sigma_r) = \sum_{A,B} \lambda(\operatorname{tr} A_{11}B_{11} - \operatorname{tr} A_{12}{}^tB_{12} + \operatorname{tr} A_{22} + \operatorname{tr} E_{22}). \tag{4.2}$$

For each fixed A, the subsum over B in (4.2) is

$$\sum_{B} \lambda(\operatorname{tr} A_{11}B_{11} - \operatorname{tr} A_{12}{}^tB_{12}), \tag{4.3}$$

where the sum is over all B_{11}, B_{12}, B_{22} satisfying ${}^tB_{11} = -B_{11}$, ${}^tB_{22} = -B_{22}$. Since the summand is independent of B_{22}, it equals

$$q^{\binom{n-r}{2}} \sum_{B_{11}} \lambda(\operatorname{tr} A_{11}B_{11}) \sum_{B_{12}} \lambda(-\operatorname{tr} A_{12}{}^tB_{12}). \tag{4.4}$$

The sum over B_{12} in (4.4) is nonzero if and only if $A_{12} = 0$, in which case it is $q^{r(n-r)}$. On the other hand, we claim that the sum over B_{11} in (4.4) is nonzero if and only if A_{11} is symmetric, in which case it is $q^{\binom{r}{2}}$. To see this, we let $A_{11} = (\alpha_{ij})$, $B_{11} = (\beta_{ij})$. Then, since ${}^tB_{11} = -B_{11}$,

$$\operatorname{tr} A_{11}B_{11} = \sum_{i,j=1}^{r} \alpha_{ij}\beta_{ji} = \sum_{1\le i<j\le r} (\alpha_{ji} - \alpha_{ij})\beta_{ij}.$$

Thus the sum over B_{11} in (4.4) is nonzero if and only if $\alpha_{ij} = \alpha_{ji}$ for $1 \le i < j \le r$, i.e., A_{11} is symmetric. Further, it is $q^{\binom{r}{2}}$ in that case.

In summary, we have shown that the sum in (4.3) is nonzero if and only if

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

with A_{11} nonsingular symmetric, in which case it equals

$$q^{\binom{n-r}{2}+\binom{r}{2}+r(n-r)} = q^{\binom{n}{2}}.$$

For such an A,

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \begin{bmatrix} {}^tA_{11}^{-1} & * \\ 0 & {}^tA_{22}^{-1} \end{bmatrix},$$

and hence the sum in (4.2) is

$$q^{\binom{n}{2}} \sum_{A_{11},A_{21}} \sum_{A_{22}} \lambda(\operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1}) = q^{\binom{n}{2}+r(n-r)} s_r K_{\mathrm{GL}(n-r,q)}(\lambda; 1, 1),$$

where s_r denotes the number of $r \times r$ nonsingular symmetric matrices for $r \ge 1$ (also we agree that $s_r = 1$ for $r = 0$) and in [11], for $a, b \in \mathbb{F}_q$, $K_{\mathrm{GL}(t,q)}(\lambda; a, b)$ is defined as

$$K_{\mathrm{GL}(t,q)}(\lambda; a, b) = \sum_{g\in \mathrm{GL}(t,q)} \lambda(a \operatorname{tr} g + b \operatorname{tr} g^{-1}). \tag{4.5}$$

Putting everything together, the sum in (4.1) can now be written as

$$q^{\binom{n}{2}} \sum_{\substack{0\le r\le n \\ r \text{ even}}} |A_r\backslash P| q^{r(n-r)} s_r K_{\mathrm{GL}(n-r,q)}(\lambda; 1, 1). \tag{4.6}$$

The next proposition was shown in [2]. See also the elegant proof in [13].

PROPOSITION 4.1. *For each positive integer r, let s_r denote the number of all $r\times r$ nonsingular symmetric matrices over $\mathbb{F}_q$. Then*

$$s_r=\begin{cases} q^{r(r+2)/4}\displaystyle\prod_{i=1}^{r/2}(q^{2i-1}-1) & \text{if } r \text{ is even},\\ q^{(r^2-1)/4}\displaystyle\prod_{i=1}^{(r+1)/2}(q^{2i-1}-1) & \text{if } r \text{ is odd}.\end{cases} \tag{4.7}$$

An explicit expression for (4.5) was obtained in [11].

THEOREM 4.2. *For integers $t\ge 1$ and nonzero elements a, b of $\mathbb{F}_q$, the Kloosterman sum $K_{\mathrm{GL}(t,q)}(\lambda;a,b)$ is given by*

$$\begin{aligned} &K_{\mathrm{GL}(t,q)}(\lambda;a,b)\\ &\quad=q^{(t-2)(t+1)/2}\sum_{l=1}^{[(t+2)/2]}q^lK(\lambda;a,b)^{t+2-2l}\sum(q^{j_1}-1)\dots(q^{j_{l-1}}-1), \end{aligned} \tag{4.8}$$

where $K(\lambda;a,b)$ is the usual Kloosterman sum in (2.7) and the inner sum is over all integers $j_1,\dots,j_{l-1}$ satisfying $2l-3\le j_1\le t-1$, $2l-5\le j_2\le j_1-2,\dots,1\le j_{l-1}\le j_{l-2}-2$. Here we adopt the convention that the inner sum in (4.8) is 1 for $l=1$, and that $j_0=t+1$ for $l=2$.

From (4.6), (3.13), (4.7), (4.8) and replacing r by $2r$, we get the following theorem.

THEOREM 4.3. *For any nontrivial additive character λ of $\mathbb{F}_q$, the Gauss sum over $\mathrm{SO}^+(2n,q)$*

$$\sum_{g\in\mathrm{SO}^+(2n,q)}\lambda(\operatorname{tr}g)$$

is given by

$$\begin{aligned} &q^{n^2-n-1}\sum_{r=0}^{[n/2]}q^{r(r+1)}\begin{bmatrix}n\\2r\end{bmatrix}_q\prod_{j=1}^{r}(q^{2j-1}-1)\\ &\quad\times\sum_{l=1}^{[(n-2r+2)/2]}q^lK(\lambda;1,1)^{n-2r+2-2l}\sum(q^{j_1}-1)\dots(q^{j_{l-1}}-1), \end{aligned} \tag{4.9}$$

where $K(\lambda;1,1)$ is the usual Kloosterman sum as in (2.7) and the innermost sum is over all integers $j_1,\dots,j_{l-1}$ satisfying $2l-3\le j_1\le n-2r-1$, $2l-5\le j_2\le j_1-2,\dots,1\le j_{l-1}\le j_{l-2}-2$.

R e m a r k. Comparing the expression of the Gauss sum over $\mathrm{SO}^+(2n,q)$ in the above theorem and that over $\mathrm{Sp}(2n,q)$ in [11], we see that

$$\sum_{g\in \mathrm{SO}^+(2n,q)} \lambda(\mathrm{tr}\, g) = q^{-n} \sum_{g\in \mathrm{Sp}(2n,q)} \lambda(\mathrm{tr}\, g).$$

5. $O^+(2n,q)$ case. Let χ be a multiplicative character of $\mathbb{F}_q$, and let λ be a nontrivial additive character of $\mathbb{F}_q$. We will consider the sum in (1.2)

$$\sum_{g\in O^+(2n,q)} \chi(\det g)\lambda(\mathrm{tr}\, g)$$

and find an explicit expression for it.

From the decompositions in (3.7) and (3.8), the sum in (1.2) is $\sum_{g\in \mathrm{SO}^+(2n,q)} \lambda(\mathrm{tr}\, g)$ plus

$$\chi(-1) \sum_{\substack{0\le r\le n\\ r\,\mathrm{odd}}} |A_r\backslash P| \sum_{g\in P} \lambda(\mathrm{tr}\, g\sigma_r). \tag{5.1}$$

Glancing through the argument in Section 4, we see that (5.1) equals

$$\chi(-1)q^{\binom{n}{2}} \sum_{\substack{0\le r\le n\\ r\,\mathrm{odd}}} |A_r\backslash P| q^{r(n-r)} s_r K_{\mathrm{GL}(n-r,q)}(\lambda;1,1). \tag{5.2}$$

Using (3.13), (4.7), (4.8) and replacing r by $2r+1$, one gets an explicit expression for (5.2). This expression combined with that in (4.9) yields:

Theorem 5.1. *For any multiplicative character χ of $\mathbb{F}_q$ and any nontrivial additive character λ of $\mathbb{F}_q$, the Gauss sum over $O^+(2n,q)$*

$$\sum_{g\in O^+(2n,q)} \chi(\det g)\lambda(\mathrm{tr}\, g)$$

is given by

$$\begin{aligned} q^{n^2-n-1}\Bigg\{ &\sum_{r=0}^{[n/2]} q^{r(r+1)} \begin{bmatrix} n \\ 2r \end{bmatrix}_q \prod_{j=1}^{r} (q^{2j-1}-1) \\ &\times \sum_{l=1}^{[(n-2r+2)/2]} q^l K(\lambda;1,1)^{n-2r+2-2l} \sum (q^{j_1}-1)\dots(q^{j_{l-1}}-1) \\ &+ \chi(-1) \sum_{r=0}^{[(n-1)/2]} q^{r(r+1)} \begin{bmatrix} n \\ 2r+1 \end{bmatrix}_q \prod_{j=1}^{r+1} (q^{2j-1}-1) \\ &\times \sum_{l=1}^{[(n-2r+1)/2]} q^l K(\lambda;1,1)^{n-2r+1-2l} \sum (q^{j_1}-1)\dots(q^{j_{l-1}}-1)\Bigg\}, \end{aligned}$$

where $K(\lambda;1,1)$ is the usual Kloosterman sum as in (2.7), *and the first and second unspecified sums are respectively over all integers $j_1,\dots,j_{l-1}$ satisfying $2l-3 \le j_1 \le n-2r-1$, $2l-5 \le j_2 \le j_1-2,\dots,1 \le j_{l-1} \le j_{l-2}-2$ and over the same set of integers satisfying $2l-3 \le j_1 \le n-2r-2$, $2l-5 \le j_2 \le j_1-2,\dots,1 \le j_{l-1} \le j_{l-2}-2$.*

6. Application to Hodges' Kloosterman sum. In [5], the generalized Kloosterman sum over nonsingular symmetric matrices is defined, for $t \times t$ symmetric matrices A, B over $\mathbb{F}_q$, as

$$K_{\mathrm{sym},t}(A,B) = \sum_g \lambda_1(\mathrm{tr}(Ag + Bg^{-1})), \tag{6.1}$$

where g runs over the set of all nonsingular symmetric matrices over $\mathbb{F}_q$ of size t.

Unlike his other papers [6]–[8], Hodges neglected to mention an important special case of the main theorem in [5]. Namely, if $m = t$ and U is a nonsingular matrix in the main theorem of [5], then $s_1 = s_2 = 0$.

Now, we take $m = t = 2n$, $A = B = J^+$ in (2.3), $U = \frac{a}{2}1_{2n}$ with $0 \ne a \in \mathbb{F}_q$, in the main theorem of [5]. Then we have the identity

$$\sum_{g \in O^+(2n,q)} \lambda_a(\mathrm{tr}\, g) = q^{-n} K_{\mathrm{sym},2n}\left(\frac{a^2}{4}(J^+)^{-1}, J^+\right), \tag{6.2}$$

where λ_a is as in (2.1).

We summarize this as the following theorem.

THEOREM 6.1. *For $0 \ne a \in \mathbb{F}_q$, we have the identity*

$$\begin{aligned}\sum_{g \in O^+(2n,q)} \lambda_a(\mathrm{tr}\, g) &= q^{-n} K_{\mathrm{sym},2n}\left(\frac{a^2}{4}(J^+)^{-1}, J^+\right) \\ &= q^{-n} K_{\mathrm{sym},2n}\left(\frac{a^2}{4}C^{-1}, C\right),\end{aligned} \tag{6.3}$$

where λ_a is as in (2.1) *and C is any nonsingular symmetric matrix over $\mathbb{F}_q$ of size $2n$ with $C \sim J^+$* (cf. (2.3)).

R e m a r k. The second identity in (6.3) is clear from the definition of the Kloosterman sum in (6.1).

Combining Theorem 5.1 and Theorem 6.1, we get the following.

THEOREM 6.2. *Let $0 \ne a \in \mathbb{F}_q$, and let C be any nonsingular symmetric matrix over $\mathbb{F}_q$ of size $2n$ with $C \sim J^+$* (cf. (2.3)). *Then the generalized Kloosterman sum below over nonsingular symmetric matrices is the same*

for any such C, and

$$K_{\mathrm{sym},2n}\left(\frac{a^2}{4}C^{-1}, C\right) \tag{6.4}$$

$$= q^{n^2-1}\Bigg\{ \sum_{r=0}^{[n/2]} q^{r(r+1)} \begin{bmatrix} n \\ 2r \end{bmatrix}_q \prod_{j=1}^{r}(q^{2j-1}-1)$$

$$\times \sum_{l=1}^{[(n-2r+2)/2]} q^l K(\lambda_a;1,1)^{n-2r+2-2l} \sum (q^{j_1}-1)\dots(q^{j_{l-1}}-1)$$

$$+ \sum_{r=0}^{[(n-1)/2]} q^{r(r+1)} \begin{bmatrix} n \\ 2r+1 \end{bmatrix}_q \prod_{j=1}^{r+1}(q^{2j-1}-1)$$

$$\times \sum_{l=1}^{[(n-2r+1)/2]} q^l K(\lambda_a;1,1)^{n-2r+1-2l} \sum (q^{j_1}-1)\dots(q^{j_{l-1}}-1)\Bigg\},$$

where $K(\lambda_a;1,1)$ is the Kloosterman sum as in (2.7)*, the first unspecified sum in* (6.4) *is over all integers $j_1,\dots,j_{l-1}$ satisfying $2l-3 \le j_1 \le n-2r-1$, $2l-5 \le j_2 \le j_1-2,\dots,1 \le j_{l-1} \le j_{l-2}-2$ and the second one in* (6.4) *is over all integers $j_1,\dots,j_{l-1}$ satisfying $2l-3 \le j_1 \le n-2r-2$, $2l-5 \le j_2 \le j_1-2,\dots,1 \le j_{l-1} \le j_{l-2}-2$.*

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Department of Mathematics
Seoul Women's University
Seoul 139-774, Korea

Department of Mathematics
Seoul National University
Seoul 151-742, Korea

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