ACTA ARITHMETICA
LXXVIII.1 (1996)

A note on the number of solutions of the generalized Ramanujan–Nagell equation $x^2 - D = k^n$

by

Maohua Le (Zhanjiang)

1. Introduction. Let $\mathbb{Z}$, $\mathbb{N}$ be the sets of integers and positive integers respectively. Let D be a nonzero integer, and let k be a positive integer such that $k > 1$ and $\gcd(D, k) = 1$. Further let $N(D, k)$ denote the number of solutions (x, n) of the generalized Ramanujan–Nagell equation

$$x^2 - D = k^n, \quad x, n \in \mathbb{N}. \tag{1}$$

There have been many papers concerned with upper bounds for $N(D, k)$. Let C_i $(i = 1, 2, \ldots)$ denote effectively computable absolute constants. The known results include the following:

1 (Apéry [1, 2]). If $D < 0$, k is a prime and $(D, k) \neq (-7, 2)$, then $N(D, k) \leq 2$.

2 (Beukers [3]). If $D < -7$, then $N(-23, 2) = N(-2^r + 1, 2) = 2$ for some $r \in \mathbb{N}$, otherwise $N(D, 2) \leq 1$.

3 (Le [10]). If $D < 0$, k is an odd prime and $|D| > C_1$, then $N(-3s^2 - 1, 4s^2 + 1) = 2$ for some $s \in \mathbb{N}$, otherwise $N(D, k) \leq 1$.

4 (Xu and Le [15]). If $D < 0$, $2 \nmid k$ and $|D| > C_2$, then

$$N(D,k) \leq \begin{cases} 2^{\omega(k)-1} + 1 & \text{if } D = -3s^2 \pm \text{ and } k^r = 4s^2 \mp 1 \\ & \qquad \text{for some } r, s \in \mathbb{N}, \\ 2^{\omega(k)-1} & \text{otherwise,} \end{cases}$$

where $\omega(k)$ is the number of distinct prime factors of k.

5 (Beukers [3, 4]). If $D > 0$ and k is a prime, then $N(D, k) \leq 4$.

6 (Le [9]). If $D > 0$, then $N(2^{2r} - 3 \cdot 2^{r+1} + 1, 2) = 4$ for some $r \in \mathbb{N}$, otherwise $N(D, 2) \leq 3$.

7 (Le [8]). If $D > 0$, k is an odd prime and $\max(D, k) > C_3$, then $N(D, k) \leq 3$.

1991 *Mathematics Subject Classification*: 11D61.

Supported by the National Natural Science Foundation of China and the Guangdong Provincial Natural Science Foundation.

8 (Chen and Le [6]). If $D > 0$, $2 \nmid k$ and $\max(D, k) > C_4$, then $N(D, k) \le 3 \cdot 2^{\omega(k)-1} + 1$.

So far we have not been able to find references to the case where $2 \mid k$ and k is not a power of 2. In this note we prove the following general result:

THEOREM. *Let $\omega(D)$ be the number of distinct prime factors of $|D|$. Then*

$$N(D,k) \le \begin{cases} 2^{\omega(D)+1} & \text{if } D < 0, \\ 2^{\omega(D)+1} + 1 & \text{if } D > 0. \end{cases}$$

2. Preliminaries

LEMMA 1. *If $D > 0$ and D is not a square, then* (1) *has at most one solution (x, n) with $k^n < \sqrt{D}$.*

Proof. By [7, Theorem $10 \cdot 8 \cdot 2$], if $k^n < \sqrt{D}$, then $x/1$ must be a convergent of $\sqrt{D}$ with $x/1 > \sqrt{D}$. Notice that $\sqrt{D}$ has at most one convergent p/q satisfying $q = 1$ and $p/q > \sqrt{D}$. The lemma is proved.

LEMMA 2. *If k is not a square and the equation*

$$X^2 - kY^2 = D, \quad X, Y \in \mathbb{Z}, \ \gcd(X, Y) = 1 \tag{2}$$

has solutions (X, Y), then all solutions of (2) *can be put into at most $2^{\omega(D)-1}$ classes. Moreover, every solution (X, Y) in the class T can be expressed as*

$$X + Y\sqrt{k} = (X_0 + \delta Y_0 \sqrt{k})(u + v\sqrt{k}), \quad \delta \in \{-1, 1\},$$

where (X_0, Y_0) is a fixed positive integer solution in T, (u, v) is a solution of the equation

$$u^2 - kv^2 = 1, \quad u, v \in \mathbb{Z}. \tag{3}$$

Proof. This is a special case of [11, Theorem 2] for $D_1 = 1$ and $z = 1$.

LEMMA 3. *For $1 \le D \le 5$, the equation*

$$X^2 + D = Y^n, \quad X, Y, n \in \mathbb{N}, \ \gcd(X, Y) = 1, \ n > 3$$

has no solutions (X, Y, n).

Proof. This follows immediately from the results of [5], [12] and [13].

LEMMA 4. *For $r, r' \in \mathbb{N}$ with $r < r'$, let S, S' be the sets of positive integer solutions (u, v) of* (3) *satisfying*

$$k^r \mid v, \quad \gcd(k, v/k^r) = 1, \tag{4}$$

and

$$k^{r'} \mid v, \quad \gcd(k, v/k^{r'}) = 1, \tag{5}$$

respectively. If $S \neq \emptyset$, $S' \neq \emptyset$, (U,V) *and* (U',V') *are least solutions of* S *and* S' *respectively*, *then*

$$U' + V'\sqrt{k} = (U + V\sqrt{k})^{k^{r'-r}}. \tag{6}$$

P r o o f. Since (U,V) is the least solution of S, $U + (V/k^r)\sqrt{k^{2r+1}}$ is the fundamental solution of the equation

$$u'^2 - k^{2r+1}v'^2 = 1, \quad u', v' \in \mathbb{Z}. \tag{7}$$

Further, since $(U', V'/k^r)$ is a positive integer solution of (7), there exists a suitable $t \in \mathbb{N}$ such that

$$U' + \frac{V'}{k^r}\sqrt{k^{2r+1}} = \left(U + \frac{V}{k^r}\sqrt{k^{2r+1}}\right)^t,$$

whence we get

$$U' + V'\sqrt{k} = (U + V\sqrt{k})^t. \tag{8}$$

Let $s = [(t-1)/2]$. From (8), we get

$$V' = V\sum_{i=0}^{s}\binom{t}{2i+1}U^{t-2i-1}(kV^2)^i. \tag{9}$$

Notice that $r < r'$, $k^r \mid V$, $k^{r'} \mid V'$ and $\gcd(k, V/k^r) = \gcd(k,U) = 1$. We see from (9) that $k \mid t$ and

$$\frac{V'}{V} = \sum_{i=0}^{s}\binom{t}{2i+1}U^{t-2i-1}(kV^2)^i \equiv 0 \pmod{k^{r'-r}}. \tag{10}$$

Let $k = p_1^{\alpha_1}\dots p_m^{\alpha_m}$ be the factorization of k, and let $p_j^{\beta_j} \parallel t$ for $j = 1,\dots,m$. Further, let $p_j^{\gamma_{ij}} \parallel 2i+1$ for any $i \in \mathbb{N}$ and $j = 1,\dots,m$. Then we have $\gamma_{ij} \le (\log(2i+1))/\log p_j < 2i$, and hence,

$$\begin{aligned}\binom{t}{2i+1}U^{t-2i-1}(kV^2)^i &= tU^{t-2i-1}\binom{t-1}{2i}\frac{(kV^2)^i}{2i+1} \\ &\equiv 0 \pmod{p_j^{\beta_j+1}}, \quad j = 1,\dots,m.\end{aligned} \tag{11}$$

By (10) and (11), we get $k^{r'-r} \mid t$ and $t = k^{r'-r}t_1$, where $t_1 \in \mathbb{N}$. Therefore, by (4), if (U',V') satisfies (6), then it is the least positive integer solution of (3) satisfying (5). The lemma is proved.

LEMMA 5 ([14, Theorem I·2]). *If* k *is not a square and* (x,n) *is a solution of* (1) *satisfying* $k^n \ge 4^{1+s/r}D^{2+s/r}$ *for some* $r, s \in \mathbb{N}$, *then*

$$\left|\frac{x'}{k^{n'/2}} - 1\right| > \frac{8}{2187k^{n(3+\nu/2)}}\left(\frac{81k^n}{4}\right)^{1/s}k^{-n'(1+\nu)/2}$$

for any $x', n' \in \mathbb{N}$ *with* $2 \nmid n'$, *where* ν *satisfies* $k^{n\nu} = 9(81k^n/4)^{r/s}$.

Lemma 6. *If k is not a square and* (1) *has a solution* (x,n) *such that* $k^n \ge \max(10^5, 4^3D^4)$, *then every solution* (x',n') *of* (1) *with* $2\nmid n'$ *satisfies* $n' < 39n$.

Proof. Let (x',n') be a solution of (1) with $2\nmid n'$. Then

$$\left|\frac{x'}{k^{n'/2}}-1\right| = \frac{D}{k^{n'/2}(k^{n'/2}+x')} < \frac{D}{k^{n'}}. \tag{12}$$

Since $k^n \ge \max(10^5, 4^3D^4)$, by Lemma 5, we get

$$\left|\frac{x'}{k^{n'/2}}-1\right| > \frac{8}{2187k^{n(3+\nu/2)}}\left(\frac{81k^n}{4}\right)^{1/2} k^{-n'(1+\nu)/2}, \tag{13}$$

where

$$\nu = \frac{\log 9}{\log k^n} + \frac{\log(81/4)}{2\log k^n} + \frac{1}{2} < 0.8215. \tag{14}$$

The combination of (12) and (13) yields

$$\frac{D}{k^{n'}} > \frac{8}{2187k^{n(3+\nu/2)}}\left(\frac{81k^n}{4}\right)^{1/2} k^{-n'(1+\nu)/2}. \tag{15}$$

Since $D \le (k^n/64)^{1/4}$ and $k^n \ge 10^5$, from (5) we get

$$k^{n(6+\nu)/2} > 60.75Dk^{n(5+\nu)/2} > k^{n'(1-\nu)/2}. \tag{16}$$

This implies that

$$n' < \left(\frac{6+\nu}{1-\nu}\right)n. \tag{17}$$

Substituting (14) into (17), we obtain $n' < 39n$. The lemma is proved.

3. Proof of Theorem. By the known results of [1]–[4], we may assume that k is not a prime power.

If k is a square, then from (1) we get $x + k^{n/2} = D_1$ and $x - k^{n/2} = D_2$, where D_1, D_2 are integers satisfying $D_1D_2 = D$, $\gcd(D_1,D_2) \le 2$, $D_1 > 0$ and $D_1 > D_2$. Notice that there exist at most $2^{\omega(D)-1}$ such pairs (D_1,D_2). So we have $N(D,k) \le 2^{\omega(D)-1}$ in this case. From the above, we may assume that k is not a square. Similarly, we see that (1) has at most $2^{\omega(D)-1}$ solutions (x,n) with $2\mid n$.

If (x,n) is a solution of (1) with $2\nmid n$, then the equation (2) has a solution $(X,Y) = (x, k^{(n-1)/2})$. By Lemma 2, all solutions (X,Y) of (2) can be put into at most $2^{\omega(D)-1}$ classes.

First we consider the case $D > 0$. We now suppose that (1) has five solutions (x_i,n_i) $(i = 1,\dots,5)$ such that $n_1 < \dots < n_5$, $k^{n_1} < \sqrt{D}$, $2\nmid n_i$ $(i = 1,\dots,5)$ and $(X,Y) = (x_i, k^{(n_i-1)/2})$ $(i = 1,\dots,n)$ belong to the same class

T of (2). By Lemma 2, there exists a fixed positive integer solution (X_0, Y_0) of (2) which satisfies

$$x_i + k^{(n_i-1)/2}\sqrt{k} = (X_0 + \delta_i Y_0\sqrt{k})(u_i + v_i\sqrt{k}), \quad \delta_i \in \{-1, 1\},\ i = 1, \ldots, 5, \tag{18}$$

where (u_i, v_i) $(i = 1, \ldots, 5)$ are solutions of (3). We find from (18) that

$$x_{j+1} + \delta_{j+1}k^{(n_{j+1}-1)/2}\sqrt{k} = (x_j + \delta_j k^{(n_j-1)/2}\sqrt{k})(u'_j + v'_j\sqrt{k}), \quad j = 1, \ldots, 4, \tag{19}$$

where (u'_j, v'_j) $(j = 1, \ldots, 4)$ are also solutions of (3). Since $x_1 < \ldots < x_5$, we see from (19) that

$$x_{j+1} + k^{(n_{j+1}-1)/2}\sqrt{k} = \begin{cases} (x_j + k^{(n_j-1)/2}\sqrt{k})(u''_j + v''_j\sqrt{k}) & \text{if } \delta_j = \delta_{j+1}, \\ (x_j - k^{(n_j-1)/2}\sqrt{k})(u''_j + v''_j\sqrt{k}) & \text{if } \delta_j \neq \delta_{j+1}, \end{cases} \tag{20}$$

$j = 1, \ldots, 4$, where (u''_j, v''_j) are positive integer solutions of (3). Notice that $x_{j+1} > x_j$ and

$$\frac{x_{j+1}}{x_j} > \frac{x_{j+1} + k^{(n_{j+1}-1)/2}\sqrt{k}}{x_j + k^{(n_j-1)/2}\sqrt{k}} > \frac{x_{j+1} + k^{(n_{j+1}-1)/2}\sqrt{k}}{x_j - k^{(n_j-1)/2}\sqrt{k}} > 0, \quad j = 1, \ldots, 4. \tag{21}$$

From (20) and (21), we obtain

$$\frac{x_{j+1}}{x_j} > u''_j + v''_j\sqrt{k}, \quad j = 1, \ldots, 4. \tag{22}$$

On the other hand, by (20), we get

$$k^{(n_{j+1}-1)/2} = x_j v''_j \pm k^{(n_j-1)/2}u''_j, \quad j = 1, \ldots, 4. \tag{23}$$

Since $\gcd(D, k) = \gcd(x_j, k) = 1$ $(j = 1, \ldots, 4)$, we see from (23) that

$$k^{(n_j-1)/2} \mid v''_j, \quad j = 1, \ldots, 4, \tag{24}$$

and $v''_j/k^{(n_j-1)/2}$ is a positive integer satisfying

$$k^{(n_{j+1}-n_j)/2} = x_j\frac{v''_j}{k^{(n_j-1)/2}} \pm u''_j, \quad j = 1, \ldots, 4. \tag{25}$$

Since $\gcd(u''_j, k) = 1$ $(j = 1, \ldots, 4)$, from (25) we get

$$\gcd(k, v''_j/k^{(n_j-1)/2}) = 1, \quad j = 1, \ldots, 4. \tag{26}$$

For $j = 1, \ldots, 4$, let (U_j, V_j) be the least positive integer solution of (3) such that $k^{(n_j-1)/2} \mid V_j$ and $\gcd(k, V_j/k^{(n_j-1)/2}) = 1$. By Lemma 4, we deduce from (22), (24) and (26) that

$$(27)\quad \frac{x_{j+2}}{x_{j+1}} > u''_{j+1} + v''_{j+1}\sqrt{k} \geq U_{j+1} + V_{j+1}\sqrt{k} = (U_j + V_j\sqrt{k})^{k^{(n_{j+1}-n_j)/2}}, \quad j = 1, 2, 3.$$

By Lemma 1, we have $k^{n_2} > \sqrt{D}$. Further, since $k^{(n_2-1)/2} \mid V_2$, we infer from (27) that

$$x_3^2 > x_2^2(U_2 + V_2\sqrt{k})^2 > 4x_2^2 k^{n_2} > 4x_2^2\sqrt{D}.$$

This implies that

$$(28)\quad k^{n_3} = x_3^2 - D > 4x_2^2\sqrt{D} - D = 4(D + k^{n_2})\sqrt{D} - D > 4D^{3/2} + 3D.$$

Since $k \geq 6$, by the same argument, we can prove that

$$(29)\quad \begin{aligned} k^{n_4} &= x_4^2 - D > x_3^2(u''_3 + v''_3\sqrt{k})^2 - D \geq x_3^2(U_3 + V_3\sqrt{k})^2 - D \\ &= x_3^2(U_2 + V_2\sqrt{k})^{2k^{(n_3-n_2)/2}} - D > x_3^2(4k^{n_2})^{k^{(n_3-n_2)/2}} - D \\ &> 4D^{3/2}(4D^{1/2})^k - D > 4^7 D^{9/2} - D > 4^3 D^4, \end{aligned}$$

and

$$(30)\quad \begin{aligned} k^{n_5} &= x_5^2 - D > x_4^2(U_4 + V_4\sqrt{k})^2 - D \\ &= x_4^2(U_3 + V_3\sqrt{k})^{2k^{(n_4-n_3)/2}} - D > k^{n_4 + n_3 k^{(n_4-n_3)/2}}. \end{aligned}$$

We see from (29) that (x_4, n_4) is a solution of (1) with $k^{n_4} > 4^3 D^4$. Moreover, if $D \geq 7$, then we have $k^{n_4} > 10^5$. Since k is not a prime power, k has at least two distinct prime factors p with $(D/p) = 1$, where (D/p) is Legendre's symbol. So we have $k \geq 7 \cdot 17$, $11 \cdot 13$ and $11 \cdot 19$ for $D = 2, 3$ and 5 respectively. Since $n_4 \geq 7$, this implies that $k^{n_4} > \max(10^5, 4^3 D^4)$. Therefore, by Lemma 6, we get

$$(31)\quad 39n_4 > n_5.$$

The combination of (30) and (31) yields

$$(32)\quad 38n_4 > n_3 k^{(n_4-n_3)/2}.$$

Since $n_3 \geq 5$, if $n_3 \leq n_4/4.6$ then $n_4 \geq 4.6n_3 \geq 23$ and

$$38n_4 > n_3 k^{9n_4/23} \geq 5 \cdot 6^{9n_4/23},$$

by (32). This is impossible for $n_4 \geq 23$. If $n_3 > n_4/4.6$, then from (22) and (32) we get

$$\begin{aligned} 174.8n_4 &> n_4 k^{(n_4-n_3)/2} = n_4\left(\frac{x_4^2 - D}{x_3^2 - D}\right)^{1/2} > n_4\frac{x_4}{x_3} > n_4(U_3 + V_3\sqrt{k}) \\ &> 2n_4 k^{n_3/2} > 2 \cdot 6^{5/2} n_4 > 176.3n_4, \end{aligned}$$

a contradiction. Thus, the equation (1) has at most four solutions (x_i, n_i) $(i = 1, \ldots, 4)$ such that $n_1 < \ldots < n_4$, $k^{n_1} < \sqrt{D}$, $2 \nmid n_i$ $(i = 1, \ldots, 4)$ and $(X, Y) = (x_i, k^{(n_i - 1)/2})$ $(i = 1, \ldots, 4)$ belong to the same class of (2). By the same argument, we can prove that (1) has at most three solutions (x_i, n_i) $(i = 1, \ldots, 3)$ such that $n_1 < \ldots < n_3$, $k^{n_1} > \sqrt{D}$, $2 \nmid n_i$ $(i = 1, \ldots, 3)$ and $(X, Y) = (x_i, k^{(n_i - 1)/2})$ $(i = 1, \ldots, 3)$ belong to the same class of (2). Further, by Lemma 1, (1) has at most one solution (x, n) that satisfies $k^n < \sqrt{D}$. This implies that if $D > 0$, then (1) has at most $3 \cdot 2^{\omega(D)-1} + 1$ solutions (x, n) with $2 \nmid n$. Recall that (1) has at most $2^{\omega(D)-1}$ solutions (x, n) with $2 \mid n$. So we have $N(D, k) \le 2^{\omega(D)+1} + 1$ for $D > 0$.

We next consider the case $D < 0$. By Lemma 3, if $-5 \le D \le -1$, then $N(D, k) \le 3$. We may therefore assume that $|D| \ge 6$. Notice that (1) has no solution (x, n) satisfying $k^n < |D|$. Therefore, by much the same argument as in the proof of the case $D > 0$, we can prove that (1) has at most three solutions (x, n) such that $2 \nmid n$ and $(X, Y) = (x, k^{(n-1)/2})$ belongs to the same class of (2). So we have $N(D, k) \le 2^{\omega(D)+1}$ for $D < 0$. The proof is complete.

References

[1] R. Apéry, *Sur une équation diophantienne*, C. R. Acad. Sci. Paris Sér. A 251 (1960), 1263–1264.

[2] —, *Sur une équation diophantienne*, ibid., 1451–1452.

[3] F. Beukers, *On the generalized Ramanujan–Nagell equation I*, Acta Arith. 38 (1981), 389–410.

[4] —, *On the generalized Ramanujan–Nagell equation II*, ibid. 39 (1981), 113–123.

[5] E. Brown, *The diophantine equation of the form* $x^2 + D = y^n$, J. Reine Angew. Math. 274/275 (1975), 385–389.

[6] X.-G. Chen and M.-H. Le, *On the number of solutions of the generalized Ramanujan–Nagell equation* $x^2 - D = k^n$, Publ. Math. Debrecen, to appear.

[7] L.-K. Hua, *Introduction to Number Theory*, Springer, Berlin, 1982.

[8] M.-H. Le, *On the generalized Ramanujan–Nagell equation* $x^2 - D = p^n$, Acta Arith. 58 (1991), 289–298.

[9] —, *On the number of solutions of the generalized Ramanujan–Nagell equation* $x^2 - D = 2^{n+2}$, ibid. 60 (1991), 149–167.

[10] —, *Sur le nombre de solutions de l'équation diophantienne* $x^2 + D = p^n$, C. R. Acad. Sci. Paris Sér. I Math. 317 (1993), 135–138.

[11] —, *Some exponential diophantine equations I: The equation* $D_1x^2 - D_2y^2 = \lambda k^z$, J. Number Theory 55 (1995), 209–221.

[12] V. A. Lebesgue, *Sur l'impossibilité, en nombres entiers, de l'équation* $x^m = y^2 + 1$, Nouv. Ann. Math. (1) 9 (1850), 178–181.

[13] T. Nagell, *Contributions to the theory of a category of diophantine equations of the second degree with two unknowns*, Nova Acta R. Soc. Sc. Uppsal. (4) 16 (1954), No. 2.

[14] N. Tzanakis and J. Wolfskill, *On the diophantine equation* $y^2 = 4q^n + 4q + 1$, J. Number Theory 23 (1986), 219–237.
[15] T.-J. Xu and M.-H. Le, *On the diophantine equation* $D_1x^2 + D_2 = k^n$, Publ. Math. Debrecen 47 (1995), 293–297.

Department of Mathematics
Zhanjiang Teachers College
524048 Zhanjiang, Guangdong
P.R. China

Received on 29.8.1995
and in revised form on 18.3.1996 (2849)