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On representations of a number as a sum of three triangles

by

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1. Introduction. Let $t(n)$ be the number of representations of n as the sum of three triangular numbers. Then

$$\sum_{n\ge 0} t(n)q^n = \left(\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\right)^3.$$

Gauss [2] showed that $t(n) > 0$, in other words, every number is the sum of three triangles. Recently, George E. Andrews [1] provided a proof of this fact via q-series.

We shall show that $t(n)$ satisfies infinitely many arithmetic identities. Thus, for $\lambda \ge 1$,

$$t(3^{2\lambda+1}n + (11\cdot 3^{2\lambda}-3)/8) = 3^\lambda \cdot t(3n+1),$$
$$t(3^{2\lambda+1}n + (19\cdot 3^{2\lambda}-3)/8) = (2\cdot 3^\lambda - 1)t(3n+2),$$
$$t(3^{2\lambda+2}n + (3^{2\lambda+1}-3)/8) = ((3^{\lambda+1}-1)/2)t(9n)$$

and

$$t(3^{2\lambda+2}n + (17\cdot 3^{2\lambda+1}-3)/8) = ((3^{\lambda+1}-1)/2)t(9n+6).$$

It should be noted that $t(n) = r(8n+3)$, where $r(n)$ is the number of representations of n in the form $n = k^2 + l^2 + m^2$ with k, l, m odd and positive. Indeed,

$$n = (r^2+r)/2 + (s^2+s)/2 + (t^2+t)/2$$

is equivalent to

$$8n+3 = (2r+1)^2 + (2s+1)^2 + (2t+1)^2.$$

This research was begun while the first author was enjoying leave.

Our results can be written in terms of $r(n)$. Thus, we have

$$r(9^\lambda n) = \begin{cases} 3^\lambda \cdot r(n) & \text{if } n \equiv 11 \pmod{24}, \\ (2 \cdot 3^\lambda - 1) r(n) & \text{if } n \equiv 19 \pmod{24}, \\ ((3^{\lambda+1} - 1)/2) r(n) & \text{if } n \equiv 3 \text{ or } 51 \pmod{72}. \end{cases}$$

We also obtain the generating function formulae

$$\sum_{n\ge 0} t(3n+1) q^n = 3\frac{(q^3;q^3)_\infty^3}{(q;q^2)_\infty^2}, \qquad \sum_{n\ge 0} t(3n+2) q^n = 3\frac{(q^6;q^6)_\infty^3}{(q;q^2)_\infty},$$

$$\sum_{n\ge 0} t(9n) q^n = \frac{(q^3;q^3)_\infty}{(q;q^6)_\infty (q^5;q^6)_\infty} \left\{ 1 + 6 \sum_{n\ge 1} \left(\frac{q^{3n-2}}{1-q^{3n-2}} - \frac{q^{3n-1}}{1-q^{3n-1}} \right) \right\},$$

and

$$\sum_{n\ge 0} t(9n+6) q^n = 6\frac{(q^3;q^6)_\infty^2 (q^6;q^6)_\infty^4}{(q;q)_\infty}.$$

2. The generating functions for $t(3n+1)$ and $t(3n+2)$. We have $t(3n+1) = r(24n+11)$, so we need to consider

$$24n + 11 = k^2 + l^2 + m^2$$

with k, l, m odd and positive. Modulo 6, this becomes

$$k^2 + l^2 + m^2 \equiv 5 \pmod 6.$$

This has the solutions, together with permutations,

$$(k, l, m) \equiv (\pm 1, \pm 1, 3) \pmod 6.$$

Conversely, if $(k, l, m) \equiv (\pm 1, \pm 1, 3) \pmod 6$ then $k^2 + l^2 + m^2 \equiv 5 \pmod{24}$.

It follows that

$$\sum_{n\ge 0} t(3n+1) q^{24n+11}$$
$$= \sum_{n\ge 0} r(24n+11) q^{24n+11} = 3 \sum_{m\ge 0} q^{(6k+1)^2 + (6l+1)^2 + (6m+3)^2}.$$

Here we have used the facts that

$$\sum_{\substack{k>0 \\ k\equiv \pm 1 \pmod 6}} q^{k^2} = \sum q^{(6k+1)^2} \quad \text{and} \quad \sum_{\substack{k>0 \\ k\equiv 3 \pmod 6}} q^{k^2} = \sum_{k\ge 0} q^{(6k+3)^2}.$$

Thus we have

$$\sum_{n\ge 0} t(3n+1) q^{24n+11} = 3q^{11} \sum_{m\ge 0} q^{36k^2 + 12k + 36l^2 + 12l + 36m^2 + 36m}$$

so

$$\sum_{n\ge 0} t(3n+1)q^n = 3\sum_{m\ge 0} q^{(3k^2+k)/2+(3l^2+l)/2+3((m^2+m)/2)}$$
$$= 3(-q;q^2)^2_\infty(-q^2;q^3)^2_\infty(q^3;q^3)^2_\infty\frac{(q^6;q^6)_\infty}{(q^3;q^6)_\infty}$$
$$= 3\frac{(q^3;q^3)^3_\infty}{(q;q^2)^2_\infty}.$$

Similarly we find that

$$\sum_{n\ge 0} t(3n+2)q^{24n+19} = \sum_{n\ge 0} r(24n+19)q^{24n+19}$$
$$= 3\sum_{l,m\ge 0} q^{(6k+1)^2+(6l+3)^2+(6m+3)^2}$$
$$= 3q^{19}\sum_{l,m\ge 0} q^{36k^2+12k+36l^2+36l+36m^2+36m}$$

so

$$\sum_{n\ge 0} t(3n+2)q^n = 3\sum_{l,m\ge 0} q^{(3k^2+k)/2+3((l^2+l)/2)+3((m^2+m)/2)}$$
$$= 3(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty\left(\frac{(q^6;q^6)_\infty}{(q^3;q^6)_\infty}\right)^2$$
$$= 3\frac{(q^6;q^6)^3_\infty}{(q;q^2)_\infty}.$$

3. The generating functions for $t(9n)$ and $t(9n+6)$. We have $t(9n) = r(72n+3)$, so we need to consider

$$72n+3 = k^2+l^2+m^2$$

with k, l, m odd and positive. Modulo 18, this becomes

$$k^2+l^2+m^2 \equiv 3 \pmod{18}.$$

The only solutions of this are, together with permutations,

$$(k,l,m) \equiv (\pm1,\pm1,\pm1), (\pm5,\pm5,\pm5), (\pm7,\pm7,\pm7) \text{ or } (\pm1,\pm5,\pm7) \pmod{18}.$$

Conversely, if

$$(k,l,m) \equiv (\pm1,\pm1,\pm1), (\pm5,\pm5,\pm5), (\pm7,\pm7,\pm7) \text{ or } (\pm1,\pm5,\pm7) \pmod{18}$$

then $k^2+l^2+m^2 \equiv 3 \pmod{72}$. Thus,

$$\begin{aligned}\sum_{n\ge 0} t(9n)q^{72n+3} &= \sum_{n\ge 0} r(72n+3)q^{72n+3}\\ &= \sum q^{(18k+1)^2+(18l+1)^2+(18m+1)^2}\\ &\quad+\sum q^{(18k-5)^2+(18l-5)^2+(18m-5)^2}\\ &\quad+\sum q^{(18k+7)^2+(18l+7)^2+(18m+7)^2}\\ &\quad+6\sum q^{(18k+1)^2+(18l-5)^2+(18m+7)^2}\\ &= \sum_{k+l+m\equiv 0 \pmod 3} q^{(6k+1)^2+(6l+1)^2+(6m+1)^2}.\end{aligned}$$

If we set $k+l+m=3u,\ 2k-l-m=3v,\ l-m=w$ then $v\equiv w \pmod 2$,

$$\begin{aligned}&(6k+1)^2+(6l+1)^2+(6m+1)^2\\ &\quad= 12(k+l+m)^2+6(2k-l-m)^2+18(l-m)^2+12(k+l+m)+3\\ &\quad= 12(3u)^2+6(3v)^2+18w^2+12(3u)+3\\ &\quad= 108u^2+54v^2+18w^2+36u+3\end{aligned}$$

and

$$\sum_{n\ge 0} t(9n)q^{72n+3} = \sum_{v\equiv w \pmod 2} q^{108u^2+54v^2+18w^2+36u+3}.$$

That is,

$$\begin{aligned}&\sum_{n\ge 0} t(9n)q^n\\ &\quad= \sum q^{(3u^2+u)/2} \sum_{v\equiv w \pmod 2} q^{(3v^2+w^2)/4}\\ &\quad= \sum q^{(3u^2+u)/2} \sum q^{(3(s+t)^2+(s-t)^2)/4} = \sum q^{(3u^2+u)/2} \sum q^{s^2+st+t^2}\\ &\quad= (-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty\left\{1+6\sum_{n\ge 1}\left(\frac{q^{3n-2}}{1-q^{3n-2}}-\frac{q^{3n-1}}{1-q^{3n-1}}\right)\right\}\\ &\quad= \frac{(q^3;q^3)_\infty}{(q;q^6)_\infty(q^5;q^6)_\infty}\left\{1+6\sum_{n\ge 1}\left(\frac{q^{3n-2}}{1-q^{3n-2}}-\frac{q^{3n-1}}{1-q^{3n-1}}\right)\right\}.\end{aligned}$$

(For a proof that

$$\sum q^{s^2+st+t^2} = 1+6\sum_{n\ge 1}\left(\frac{q^{3n-2}}{1-q^{3n-2}}-\frac{q^{3n-1}}{1-q^{3n-1}}\right)$$

see the Appendix.)

In similar fashion,

$$\begin{aligned}\sum_{n\ge 0} t(9n+6)q^{72n+51} &= \sum_{n\ge 0} r(72n+51)q^{72n+51}\\ &= 3\sum q^{(18k+1)^2+(18l-5)^2+(18m-5)^2}\\ &\quad + 3\sum q^{(18k-5)^2+(18l+7)^2+(18m+7)^2}\\ &\quad + 3\sum q^{(18k+7)^2+(18l+1)^2+(18m+1)^2}\\ &= \sum_{k+l+m\equiv 1 \pmod 3} q^{(6k+1)^2+(6l+1)^2+(6m+1)^2}.\end{aligned}$$

If we set $k+l+m=3u+1$, $2k-l-m=3v-1$, $l-m=w$ then $v\not\equiv w \pmod 2$,

$$\begin{aligned}&(6k+1)^2+(6l+1)^2+(6m+1)^2\\ &\quad = 12(k+l+m)^2+6(2k-l-m)^2+18(l-m)^2+12(k+l+m)+3\\ &\quad = 12(3u+1)^2+6(3v-1)^2+18w^2+12(3u+1)+3\\ &\quad = 108u^2+54v^2+18w^2+108u-36v+33\end{aligned}$$

and

$$\sum_{n\ge 0} t(9n+6)q^{72n+51} = \sum_{v\not\equiv w \pmod 2} q^{108u^2+54v^2+18w^2+108u-36v+33}.$$

That is,

$$\begin{aligned}\sum_{n\ge 0} t(9n+6)q^n &= \sum q^{3((u^2+u)/2)} \sum_{v\not\equiv w \pmod 2} q^{(3v^2+w^2-2v-1)/4}\\ &= \sum q^{3((u^2+u)/2)} \sum q^{(3(s+t+1)^2+(s-t)^2-2(s+t+1)-1)/4}\\ &= \sum q^{3((u^2+u)/2)} \sum q^{s^2+st+t^2+s+t}\\ &= 2\frac{(q^6;q^6)_\infty}{(q^3;q^6)_\infty}\cdot 3\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} = 6\frac{(q^3;q^6)_\infty^2(q^6;q^6)_\infty^4}{(q;q)_\infty}.\end{aligned}$$

(For a proof that

$$\sum q^{s^2+st+t^2+s+t} = 3\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}$$

see the Appendix.)

4. Proof of the main result in the case $\lambda = 1$. We want to show that

$$\begin{aligned}t(27n+12) &= 3t(3n+1),\\ t(27n+21) &= 5t(3n+2),\end{aligned}$$

$$t(81n+3) = 4t(9n),$$
$$t(81n+57) = 4t(9n+6).$$

As we shall see, it is crucial for us to prove

$$(*) \qquad \sum_{n\geq 0} t(27n+3)q^n = 4\sum_{n\geq 0} t(3n)q^n - 3\sum_{n\geq 0} t(n)q^{3n+1}.$$

The third and fourth of the above relations follow from $(*)$ on comparing coefficients of q^{3n} and q^{3n+2} respectively. Also, we shall require $(*)$ in Section 5 to prove the main result for $\lambda > 1$.

We shall prove $(*)$ in full detail, and outline the proofs of the first and second relations above.

First,

$$\begin{aligned}\sum_{n\geq 0} t(3n)q^{24n+3} &= \sum_{n\geq 0} r(24n+3)q^{24n+3} \\ &= \sum q^{(6k+1)^2+(6l+1)^2+(6m+1)^2} \\ &\quad + \sum_{k,l,m\geq 0} q^{(6k+3)^2+(6l+3)^2+(6m+3)^2} \\ &= q^3 \sum q^{36k^2+12k+36l^2+12l+36m^2+12m} \\ &\quad + q^{27} \sum_{k,l,m\geq 0} q^{36k^2+36k+36l^2+36l+36m^2+36m}\end{aligned}$$

so

$$\begin{aligned}\sum_{n\geq 0} t(3n)q^n &= \sum q^{(3k^2+k)/2+(3l^2+l)/2+(3m^2+m)/2} \\ &\quad + q \sum_{k,l,m\geq 0} q^{3((k^2+k)/2)+3((l^2+l)/2)+3((m^2+m)/2)} \\ &= \sum q^{(3k^2+k)/2+(3l^2+l)/2+(3m^2+m)/2} \\ &\quad + \sum_{n\geq 0} t(n)q^{3n+1}.\end{aligned}$$

Next, $t(27n+3) = r(216n+27)$, so we need to consider

$$216n + 27 = k^2 + l^2 + m^2$$

with k, l, m odd and positive. Modulo 54, this becomes

$$k^2 + l^2 + m^2 \equiv 27 \pmod{54}$$

and we find that

$$\sum_{n\geq 0} t(27n+3)q^{216n+27} = \sum_{n\geq 0} r(216n+27)q^{216n+27}$$

$$
\begin{aligned}
&= 3\Big\{ \sum q^{(54k-23)^2+(54l-17)^2+(54m-17)^2} + \sum q^{(54k-17)^2+(54l+25)^2+(54m+25)^2} \\
&\quad + \sum q^{(54k-11)^2+(54l+13)^2+(54m+13)^2} + \sum q^{(54k-5)^2+(54l+1)^2+(54m+1)^2} \\
&\quad + \sum q^{(54k+1)^2+(54l-11)^2+(54m-11)^2} + \sum q^{(54k+7)^2+(54l-23)^2+(54m-23)^2} \\
&\quad + \sum q^{(54k+13)^2+(54l+19)^2+(54m+19)^2} + \sum q^{(54k+19)^2+(54l+7)^2+(54m+7)^2} \\
&\quad + \sum q^{(54k+25)^2+(54l-5)^2+(54m-5)^2} \Big\} \\
&+6\Big\{ \sum q^{(54k-23)^2+(54l+19)^2+(54m+1)^2} + \sum q^{(54k-17)^2+(54l+7)^2+(54m-11)^2} \\
&\quad + \sum q^{(54k-11)^2+(54l-5)^2+(54m-23)^2} + \sum q^{(54k-5)^2+(54l-17)^2+(54m+19)^2} \\
&\quad + \sum q^{(54k+1)^2+(54l+25)^2+(54m+7)^2} + \sum q^{(54k+7)^2+(54l+13)^2+(54m-5)^2} \\
&\quad + \sum q^{(54k+13)^2+(54l+1)^2+(54m-17)^2} + \sum q^{(54k+19)^2+(54l-11)^2+(54m+25)^2} \\
&\quad + \sum q^{(54k+25)^2+(54l-23)^2+(54m+13)^2} \Big\} \\
&\quad + \sum q^{(18k+3)^2+(18l+3)^2+(18m+3)^2} \\
&\quad + \sum_{k,l,m\geq 0} q^{(18k+9)^2+(18l+9)^2+(18m+9)^2}.
\end{aligned}
$$

The last two terms constitute

$$\sum_{n\geq 0} t(3n)q^{216n+27}$$

so we have

$$
\begin{aligned}
&\sum_{n\geq 0} t(27n+3)q^{216n+27} - \sum_{n\geq 0} t(3n)q^{216n+27} \\
&\qquad\qquad = 3 \sum_{\substack{4k+l+m\equiv -4 \pmod 9 \\ l-m\equiv 0 \pmod 3}} q^{(6k+1)^2+(6l+1)^2+(6m+1)^2}.
\end{aligned}
$$

If we set $4k+l+m = 9u-4$, $-k+2l+2m = 9v+1$, $l-m = 3w$ then $u \equiv w \pmod 2$,

$$
\begin{aligned}
&(6k+1)^2 + (6l+1)^2 + (6m+1)^2 \\
&\qquad = 2(4k+l+m)^2 + 4(-k+2l+2m)^2 + 18(l-m)^2 \\
&\qquad\quad + 4(4k+l+m) + 4(-k+2l+2m) + 3
\end{aligned}
$$

$$= 2(9u-4)^2 + 4(9v+1)^2 + 18(3w)^2 + 4(9u-4) + 4(9v+1) + 3$$
$$= 162u^2 + 324v^2 + 162w^2 - 108u + 108v + 27$$

and

$$\sum_{n\geq 0} t(27n+3)q^{216n+27} - \sum_{n\geq 0} t(3n)q^{216n+27}$$
$$= 3 \sum_{u\equiv w \pmod 2} q^{162u^2+324v^2+162w^2-108u+108v+27}.$$

That is,

$$\sum_{n\geq 0} t(27n+3)q^n - \sum_{n\geq 0} t(3n)q^n$$
$$= 3\sum q^{(3v^2+v)/2} \sum_{u\equiv w\ (\mathrm{mod}\ 2} q^{(3u^2+3w^2-2u)/4}$$
$$= 3\sum q^{(3v^2+v)/2} \sum q^{(3(s+t)^2+3(s-t)^2-2(s+t))/4}$$
$$= 3\sum q^{(3v^2+v)/2+(3s^2-s)/2+(3t^2-t)/2}$$
$$= 3\Big(\sum_{n\geq 0} t(3n)q^n - \sum_{n\geq 0} t(n)q^{3n+1}\Big),$$

so

$$\sum_{n\geq 0} t(27n+3)q^n = 4\sum_{n\geq 0} t(3n)q^n - 3\sum_{n\geq 0} t(n)q^{3n+1}.$$

In the same way, we can show

$$\sum_{n\geq 0} t(27n+12)q^{216n+99} - \sum_{n\geq 0} t(3n+1)q^{216n+99}$$
$$= 3 \sum_{\substack{4k+l+m\equiv 2 \pmod 9 \\ l-m\equiv 0 \pmod 3}} q^{(6k+1)^2+(6l+1)^2+(6m+1)^2}$$

from which it follows that

$$\sum_{n\geq 0} t(27n+12)q^n - \sum_{n\geq 0} t(3n+1)q^n$$
$$= 3\sum q^{3((v^2+v)/2)} \sum_{u\equiv w \pmod 2} q^{(3u^2+3w^2+2u)/4}$$
$$= 6\sum_{v\geq 0} q^{3((v^2+v)/2)+(3s^2+s)/2+(3t^2+t)/2}$$
$$= 2\sum_{n\geq 0} t(3n+1)q^n$$

and

$$\sum_{n\ge 0} t(27n+21)q^{216n+171} - \sum_{n\ge 0} t(3n+2)q^{216n+171} = 3 \sum_{\substack{4k+l+m\equiv -1 \pmod 9 \\ l-m\equiv 0 \pmod 3}} q^{(6k+1)^2+(6l+1)^2+(6m+1)^2}$$

from which it follows that

$$\begin{aligned}\sum_{n\ge 0} t(27n+21)q^n - \sum_{n\ge 0} t(3n+2)q^n &= 3\sum q^{(3v^2+v)/2} \sum_{u\not\equiv w \pmod 2} q^{(3u^2+3w^2-3)/4} \\ &= 3\sum q^{(3v^2+v)/2+3((s^2+s)/2)+3((t^2+t)/2)} \\ &= 12\sum_{s,t\ge 0} q^{(3v^2+v)/2+3((s^2+s)/2)+3((t^2+t)/2)} \\ &= 4\sum_{n\ge 0} t(3n+2)q^n.\end{aligned}$$

Thus the first and second relations hold.

5. Proof of the main result for $\lambda > 1$. We have shown that

$$r(9^\lambda n) = \begin{cases} 3^\lambda r(n) & \text{if } n\equiv 11 \pmod{24}, \\ (2\cdot 3^\lambda - 1)r(n) & \text{if } n\equiv 19 \pmod{24}, \\ ((3^{\lambda+1}-1)/2)r(n) & \text{if } n\equiv 3 \text{ or } 51 \pmod{72} \end{cases}$$

for $\lambda = 1$, and it is trivially true for $\lambda = 0$.

Suppose now that $\lambda \ge 2$, and that the result is true for $\lambda - 1$ and $\lambda - 2$.

We have

$$\sum_{n\ge 0} t(27n+3)q^n = 4\sum_{n\ge 0} t(3n)q^n - 3\sum_{n\ge 0} t(n)q^{3n+1}.$$

If we consider the coefficient of q^{3n+1} we obtain

$$t(81n+30) = 4t(9n+3) - 3t(n)$$

or

$$r(648n+243) = 4r(72n+27) - 3r(8n+3),$$

or

$$r(81(8n+3)) = 4r(9(8n+3)) - 3r(8n+3).$$

It follows that for all n,

$$r(81n) = 4r(9n) - 3r(n),$$

for if $n \not\equiv 3 \pmod 8$ then $81n \equiv 9n \equiv n \not\equiv 3 \pmod 8$ so $r(81n) = r(9n) = r(n) = 0$ and the result is trivially true.

It follows that for every $\lambda \geq 2$,

$$r(9^\lambda n) = 4r(9^{\lambda-1}n) - 3r(9^{\lambda-2}n).$$

Thus, if $n \equiv 11 \pmod{24}$,

$$\begin{aligned} r(9^\lambda n) &= 4r(9^{\lambda-1}n) - 3r(9^{\lambda-2}n) \\ &= 4 \cdot 3^{\lambda-1} r(n) - 3 \cdot 3^{\lambda-2} r(n) \\ &= (4 \cdot 3^{\lambda-1} - 3 \cdot 3^{\lambda-2}) r(n) \\ &= 3^\lambda r(n), \end{aligned}$$

if $n \equiv 19 \pmod{24}$,

$$\begin{aligned} r(9^\lambda n) &= 4r(9^{\lambda-1}n) - 3r(9^{\lambda-2}n) \\ &= 4(2 \cdot 3^{\lambda-1} - 1) r(n) - 3(2 \cdot 3^{\lambda-2} - 1) r(n) \\ &= (4(2 \cdot 3^{\lambda-1} - 1) - 3(2 \cdot 3^{\lambda-2} - 1)) r(n) \\ &= (2 \cdot 3^\lambda - 1) r(n), \end{aligned}$$

while if $n \equiv 3$ or $51 \pmod{72}$,

$$\begin{aligned} r(9^\lambda n) &= 4r(9^{\lambda-1}n) - 3r(9^{\lambda-2}n) \\ &= 4((3^\lambda - 1)/2) r(n) - 3((3^{\lambda-1} - 1)/2) r(n) \\ &= (4((3^\lambda - 1)/2) - 3((3^{\lambda-1} - 1)/2)) r(n) \\ &= ((3^{\lambda+1} - 1)/2) r(n), \end{aligned}$$

and our result is proved.

Appendix. In [3], a generalisation of the following identity was established:

$$\begin{aligned} &\sum a^{s+t+u} q^{(s^2+t^2+u^2)/2} \\ &\qquad = (-aq^{1/2};q)_\infty^3 (-a^{-1}q^{1/2};q)_\infty^3 (q;q)_\infty^3 \\ &\qquad = c_0 (-a^3 q^{3/2};q^3)_\infty (-a^{-3} q^{3/2};q^3)_\infty (q^3;q^3)_\infty \\ &\qquad\quad + c_1 \{ a(-a^3 q^{5/2};q^3)_\infty (-a^{-3} q^{1/2};q^3)_\infty (q^3;q^3)_\infty \\ &\qquad\quad + a^{-1} (-a^3 q^{1/2};q^3)_\infty (-a^{-3} q^{5/2};q^3)_\infty (q^3;q^3)_\infty \}, \end{aligned}$$

where

$$c_0 = 1 + 6 \sum_{n \geq 1} \left(\frac{q^{3n-2}}{1-q^{3n-2}} - \frac{q^{3n-1}}{1-q^{3n-1}} \right) \quad \text{and} \quad c_1 = 3q^{1/2} \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}.$$

Thus we have

$$\begin{aligned}\sum q^{s^2+st+t^2} &= \sum q^{(s^2+t^2+(-s-t)^2)/2} = \sum_{s+t+u=0} q^{(s^2+t^2+u^2)/2} \\ &= \text{ the constant term in } \sum a^{s+t+u} q^{(s^2+t^2+u^2)/2} \\ &= c_0 \\ &= 1 + 6\sum_{n\ge 1}\left(\frac{q^{3n-2}}{1-q^{3n-2}} - \frac{q^{3n-1}}{1-q^{3n-1}}\right),\end{aligned}$$

and

$$\begin{aligned}&\sum q^{s^2+st+t^2+s+t} \\ &\quad= \sum q^{(s^2+t^2+(-s-t-1)^2-1)/2} \\ &\quad= q^{-1/2}\sum_{s+t+u=-1} q^{(s^2+t^2+u^2)/2} \\ &\quad= q^{-1/2}\cdot \text{the coefficient of } a^{-1} \text{ in } \sum a^{s+t+u} q^{(s^2+t^2+u^2)/2} \\ &\quad= q^{-1/2} c_1 = 3\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}.\end{aligned}$$

Addendum. We conjecture that for each odd prime p, each $\lambda \ge 1$, each $\mu \in \{0,1,\dots,p-1\}$ and all $n \ge 0$,

- if $p \equiv 1 \pmod 8$, then

$$t(p^{2\lambda+1}n + (3p^{2\lambda}-3)/8 + \mu p^{2\lambda}) = \begin{cases} p^\lambda t(pn+\mu) & \text{if } 8\mu+3 \text{ is a q.r. modulo } p, \\ \dfrac{(p+1)p^\lambda - 2}{p-1} t(pn+\mu) & \text{if } 8\mu+3 \text{ is a q.n.r. modulo } p, \end{cases}$$

$$t(p^{2\lambda+2}n + (3p^{2\lambda+1}-3)/8 + \mu p^{2\lambda+1}) = \frac{p^{\lambda+1}-1}{p-1} t(p^2 n + (3p-3)/8 + \mu p) \quad \text{for } \mu \ne (3p-3)/8,$$

- if $p \equiv 3 \pmod 8$, then

$$t(p^{2\lambda+1}n + (3p^{2\lambda}-3)/8 + \mu p^{2\lambda}) = \begin{cases} p^\lambda t(pn+\mu) & \text{if } 8\mu+3 \text{ is a q.n.r. modulo } p, \\ \dfrac{(p+1)p^\lambda - 2}{p-1} t(pn+\mu) & \text{if } 8\mu+3 \text{ is a q.r. modulo } p, \end{cases}$$

$$t(p^{2\lambda+2}n + (p^{2\lambda+1}-3)/8 + \mu p^{2\lambda+1}) = \frac{p^{\lambda+1}-1}{p-1} t(p^2 n + (p-3)/8 + \mu p) \quad \text{for } \mu \ne (3p-1)/8,$$

• if $p \equiv 5 \pmod 8$, then

$$t(p^{2\lambda+1}n + (3p^{2\lambda} - 3)/8 + \mu p^{2\lambda}) = \begin{cases} p^\lambda t(pn+\mu) & \text{if } 8\mu+3 \text{ is a q.r. modulo } p, \\ \dfrac{(p+1)p^\lambda - 2}{p-1} t(pn+\mu) & \text{if } 8\mu+3 \text{ is a q.n.r. modulo } p, \end{cases}$$

$$t(p^{2\lambda+2}n + (7p^{2\lambda+1} - 3)/8 + \mu p^{2\lambda+1}) = \frac{p^{\lambda+1}-1}{p-1} t(p^2 n + (7p-3)/8 + \mu p) \quad \text{for } \mu \neq (3p-7)/8,$$

• if $p \equiv 7 \pmod 8$, then

$$t(p^{2\lambda+1}n + (3p^{2\lambda} - 3)/8 + \mu p^{2\lambda}) = \begin{cases} p^\lambda t(pn+\mu) & \text{if } 8\mu+3 \text{ is a q.n.r. modulo } p, \\ \dfrac{(p+1)p^\lambda - 2}{p-1} t(pn+\mu) & \text{if } 8\mu+3 \text{ is a q.r. modulo } p, \end{cases}$$

$$t(p^{2\lambda+2}n + (5p^{2\lambda+1} - 3)/8 + \mu p^{2\lambda+1}) = \frac{p^{\lambda+1}-1}{p-1} t(p^2 n + (5p-3)/8 + \mu p) \quad \text{for } \mu \neq (3p-5)/8.$$

In terms of $r(n)$, these relations are:

• if $p \equiv 1 \pmod 8$, then

$$r(p^{2\lambda}n) = \begin{cases} p^\lambda r(n) \\ \quad \text{for } n \equiv 8\mu+3 \pmod{8p} \text{ if } 8\mu+3 \text{ is a q.r. modulo } p, \\ \dfrac{(p+1)p^\lambda - 2}{p-1} r(n) \\ \quad \text{for } n \equiv 8\mu+3 \pmod{8p} \text{ if } 8\mu+3 \text{ is a q.n.r. modulo } p, \\ \dfrac{p^{\lambda+1}-1}{p-1} r(n) \\ \quad \text{for } n \equiv 3p \pmod{8p},\ n \not\equiv 3p^2 \pmod{8p^2}, \end{cases}$$

• if $p \equiv 3 \pmod 8$, then

$$r(p^{2\lambda}n) = \begin{cases} p^\lambda r(n) \\ \quad \text{for } n \equiv 8\mu+3 \pmod{8p} \text{ if } 8\mu+3 \text{ is a q.n.r. modulo } p, \\ \dfrac{(p+1)p^\lambda - 2}{p-1} r(n) \\ \quad \text{for } n \equiv 8\mu+3 \pmod{8p} \text{ if } 8\mu+3 \text{ is a q.r. modulo } p, \\ \dfrac{p^{\lambda+1}-1}{p-1} r(n) \\ \quad \text{for } n \equiv p \pmod{8p},\ n \not\equiv 3p^2 \pmod{8p^2}, \end{cases}$$

- if $p \equiv 5 \pmod 8$, then

$$r(p^{2\lambda}n) = \begin{cases} p^{\lambda} r(n) \\ \quad \text{for } n \equiv 8\mu + 3 \pmod{8p} \text{ if } 8\mu+3 \text{ is a q.r. modulo } p, \\ \dfrac{(p+1)p^{\lambda} - 2}{p-1} r(n) \\ \quad \text{for } n \equiv 8\mu + 3 \pmod{8p} \text{ if } 8\mu+3 \text{ is a q.n.r. modulo } p, \\ \dfrac{p^{\lambda+1} - 1}{p-1} r(n) \\ \quad \text{for } n \equiv 7p \pmod{8p},\ n \not\equiv 3p^2 \pmod{8p^2}, \end{cases}$$

- if $p \equiv 7 \pmod 8$, then

$$r(p^{2\lambda}n) = \begin{cases} p^{\lambda} r(n) \\ \quad \text{for } n \equiv 8\mu + 3 \pmod{8p} \text{ if } 8\mu+3 \text{ is a q.n.r. modulo } p, \\ \dfrac{(p+1)p^{\lambda} - 2}{p-1} r(n) \\ \quad \text{for } n \equiv 8\mu + 3 \pmod{8p} \text{ if } 8\mu+3 \text{ is a q.r. modulo } p, \\ \dfrac{p^{\lambda+1} - 1}{p-1} r(n) \\ \quad \text{for } n \equiv 5p \pmod{8p},\ n \not\equiv 3p^2 \pmod{8p^2}. \end{cases}$$

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