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On the exceptional set of Goldbach numbers in a short interval

by

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1. Introduction. An even number which can be written as a sum of two primes is called a *Goldbach number*.

In 1937, I. M. Vinogradov proved the famous three primes theorem. Soon after that, employing Vinogradov's idea, Hua Loo Keng [3] proved that if B is a sufficiently large positive constant and N is sufficiently large, then the even numbers in $(2, N)$, except for $O(N\log^{-B} N)$ values, are all Goldbach numbers.

In 1973, Ramachandra [18] obtained the result in a short interval. He showed that if $A = N^{\varphi+\varepsilon}$ with $\varphi = \frac{3}{5}$, then the even numbers in $(N, N+A)$, except for $O(A\log^{-B} N)$ values, are all Goldbach numbers. If the estimation for the zero density of Huxley [4] is applied, $\varphi = \frac{7}{12}$ can be arrived at. In 1991, Jia Chao Hua [8] employed the sieve method combined with the circle method to get $\varphi = \frac{23}{42}$.

In 1993, Perelli and Pintz [17] developed a new technique in the circle method to make great progress in solving this problem. They showed $\varphi = \frac{7}{36}$. Mikawa [13] proved $\varphi = \frac{7}{48}$ by the sieve method.

In 1994, Jia Chao Hua [10] proved $\varphi = \frac{7}{78}$. Li Hongze [12] improved it to $\varphi = \frac{7}{81}$. Jia Chao Hua [11] got further $\varphi = \frac{1}{12}$. In [10], the new technique of [17], the sieve method and the mean value estimation for Dirichlet polynomials were applied.

In this paper, we shall develop further the technique of [10] to prove the following:

Theorem. *Suppose that B is a sufficiently large positive constant, ε is a sufficiently small positive constant, N is sufficiently large and $A = N^{\frac{7}{108}+12\varepsilon}$. Then the even numbers in $(N, N + A)$, except for $O(A\log^{-B} N)$ values, are all Goldbach numbers.*

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For $A = N^{\frac{7}{108}+12\varepsilon}$, we can also get the conclusion of Theorem 2 of [10].

Throughout this paper, we always suppose that B is a sufficiently large positive constant, ε is a sufficiently small positive constant and $E = B^2$, $\delta = \varepsilon^{\frac{1}{3}}$. We also suppose that N is sufficiently large and that $A = N^{\frac{7}{108}+12\varepsilon}$, $Y = N^{\frac{7}{12}+\varepsilon}$, $Q = \frac{1}{2}\sqrt{A}\log^{-64B} N$. Let c, c_1 and c_2 denote positive constants which have different values at different places. $m \sim M$ means that there are positive constants c_1 and c_2 such that $c_1 M < m \le c_2 M$. We often use $M(s,\chi)$ (M may be another capital letter) to denote a Dirichlet polynomial of the form

$$M(s,\chi) = \sum_{m\sim M} \frac{a(m)\chi(m)}{m^s},$$

where $a(m)$ is a complex number with $a(m) = O(1)$, and χ is a character $\bmod q$.

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2. Some preliminary work. Let χ be a character $\bmod q$ and let χ_0 be a principal character. Set $E_0 = 1$ if $\chi = \chi_0$, and $E_0 = 0$ if $\chi \ne \chi_0$.

We divide the characters $\bmod q$ into two classes. We call χ a *good character* if $L(s,\chi)$ has no zeros in the region

$$1 - \frac{24E\log\log Y}{\varepsilon\log Y} \le \sigma \le 1, \qquad |t| \le 6Y. \tag{1}$$

Otherwise, we call χ a *bad character*.

By Siegel's theorem and the zero free region of the L-function (see pp. 255 and 257 of [15]), we know that $L(s,\chi) \ne 0$ in the region

$$\sigma > 1 - \frac{c(\mu)}{\max(q^{\mu}, \log^{\frac{4}{5}}(|t|+2))},$$

where μ (>0) is arbitrary and $c(\mu) > 0$.

Take $\mu = \varepsilon/(500E)$. If a bad character exists, then

$$q \ge \log^{\frac{300E}{\varepsilon}} N. \tag{2}$$

Moreover, by the estimation for the zero density (see [14] and [5]),

$$\sum_{\chi\,(\mathrm{mod}\,q)} N(\sigma,T,\chi) \ll (qT)^{(\frac{12}{5}+\varepsilon)(1-\sigma)}\log^{14} qT,$$

we know that for any modulus $q \le Q$, the number of bad characters is $O(\log^{\frac{64E}{\varepsilon}} N)$. Let

$$I(a,q) = \left(\frac{a}{q} - \frac{\log^{2E} N}{qY}, \frac{a}{q} + \frac{\log^{2E} N}{qY}\right)$$

and let

$$E_1 = \bigcup_{q \le \log^E N} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} I(a,q), \qquad E_2 = [1/Q, 1+1/Q) - E_1.$$

LEMMA 1. *Assume that* r (≥ 2) *and* q *are positive integers*, $d_r(n) = \sum_{n=n_1\ldots n_r} 1$ *and* $x^\varepsilon < y \le x$. *Then*

$$\sum_{x-y<n\le x} d_r^q(n) \ll y(\log x)^{r^q-1}.$$

Proof. See Theorem 2 of [20].

Let

$$\tau(\chi) = \sum_{r=1}^{q} \chi(r) e\left(\frac{r}{q}\right). \tag{3}$$

LEMMA 2. $\tau(\chi_0) = \mu(q)$. *For any character* $\chi \bmod q$, $|\tau(\chi)| \le \sqrt{q}$.

Proof. See pp. 24 and 28 of [15].

LEMMA 3. *For any complex numbers* $a(n)$,

$$\int_{-\eta}^{\eta} \Big| \sum_n a(n) e(nt) \Big|^2 dt \ll \eta^2 \int_{-\infty}^{\infty} \Big| \sum_{x<n\le x+1/(2\eta)} a(n) \Big|^2 dx.$$

Proof. See Lemma 1 of [1].

LEMMA 4. *Suppose* $T \ge 1$, χ *is a character* $\bmod q$, *and* $\sum_{m\sim M} |a(m)|^2 \ll M \log^c T$. *Then*

$$\sum_{\chi \,(\mathrm{mod}\, q)} \int_{-T}^{T} \Big| \sum_{m\sim M} \frac{a(m)\chi(m)}{m^{\frac12+it}} \Big|^2 dt \ll (M+qT)\log^c T.$$

Proof. See Theorem 3 on p. 38 of [15].

3. Mean value estimate (I)

LEMMA 5. *Assume that* $Y^\delta \ll H \ll Y^{\frac{16}{135}}$, $MH = Y$, $q \le Q$, χ *is a character* $\bmod q$, $M(s,\chi)$ *is a Dirichlet polynomial and*

$$H(s,\chi) = \sum_{h\sim H} \frac{\Lambda(h)\chi(h)}{h^s}.$$

Let $\eta = qQ/Y$, $b = 1 + 1/\log N$ *and* $T_0 = \log^{\frac{E}{\delta^2}} Y$. *Then for* $T_0 \le T \le Y$, *we have*

$$\min^2\left(\eta, \frac{1}{T}\right) \sum_{\chi\,(\mathrm{good})} \int_T^{2T} |M(b+it,\chi)H(b+it,\chi)|^2\,dt \ll \eta^2 \log^{-6E} N.$$

Proof. Let $s = b + it$ and let χ be a good character mod$q$. By (1) and the zero free region of the L-function, we know that for $|t| \le 2Y$,

$$\sum_{c_1H<h\le c_2H} \frac{\Lambda(h)\chi(h)}{h^s} = E_0 \frac{(c_2H)^{1-s} - (c_1H)^{1-s}}{1-s} + O(\log^{-\frac{2E}{\delta^2}} Y). \tag{4}$$

So, for $T_0 \le |t| \le 2Y$,

$$H(s,\chi) \ll \log^{-\frac{E}{\delta^2}} Y. \tag{5}$$

According to the discussion in [2], there are $O(\log^2 Y)$ sets $S(V,W)$, where $S(V,W) = \{t_j(\chi) : \chi$ runs through all good characters $\bmod q$, $j = 1,\ldots,J(\chi)$, $|t_r(\chi) - t_s(\chi)| \ge 1$ $(r \ne s)\}$. For $t_j(\chi) \in S(V,W)$,

$$V \le M^{\frac12}|M(b+it_j(\chi),\chi)| < 2V,$$
$$W \le H^{\frac12}|H(b+it_j(\chi),\chi)| < 2W,$$

where $Y^{-1} \le M^{-\frac12}V$, $Y^{-1} \le H^{-\frac12}W$ and $V \ll M^{\frac12}$, $W \ll H^{\frac12}\log^{-\frac{E}{\delta^2}} Y$. Thus

$$\sum_{\chi\,(\mathrm{good})} \int_T^{2T} |M(b+it,\chi)H(b+it,\chi)|^2\,dt \ll V^2W^2Y^{-1}\log^2 Y|S(V,W)|, \tag{6}$$

where $S(V,W)$ is one of the sets with the above properties.

Assume $Y^{\frac{1}{k+1}} \le H \le Y^{\frac1k}$, where k is a positive integer, $k \ge 8$ and $k\delta \ll 1$. Applying the mean value estimate (see Theorem 3 on p. 632 of [16]) and Lemma 1 to $M(s,\chi)$ and $H^k(s,\chi)$, we have

$$|S(V,W)| \ll V^{-2}(M + qT)\log^d Y,$$
$$|S(V,W)| \ll W^{-2k}(H^k + qT)\log^d Y,$$

where $d = c/\delta^2$. Applying the Halász method (see Theorem 6 on p. 650 of [16]) to $M(s,\chi)$ and $H^k(s,\chi)$, we have

$$|S(V,W)| \ll (V^{-2}M + V^{-6}qTM)\log^d Y,$$
$$|S(V,W)| \ll (W^{-2k}H^k + W^{-6k}qTH^k)\log^d Y.$$

Thus,

$$V^2W^2|S(V,W)| \ll V^2W^2F\log^d Y,$$

where

$$F = \min\{V^{-2}(M+qT), V^{-2}M + V^{-6}qTM, W^{-2k}(H^k+qT), W^{-2k}H^k + W^{-6k}qTH^k\}.$$

It will be proved that

$$\min{}^2\left(\eta, \frac{1}{T}\right)V^2W^2F \ll \eta^2 Y \log^{-\frac{E}{2\delta^2}} N. \tag{7}$$

We consider four cases.

(a) $F \le 2V^{-2}M$, $2W^{-2k}H^k$. Then

$$\begin{aligned} V^2W^2F &\ll V^2W^2\min\{V^{-2}M, W^{-2k}H^k\} \\ &\le V^2W^2(V^{-2}M)^{1-\frac{1}{2k}}(W^{-2k}H^k)^{\frac{1}{2k}} \\ &= V^{\frac{1}{k}}WM^{1-\frac{1}{2k}}H^{\frac{1}{2}} \ll Y\log^{-\frac{E}{2\delta^2}} N. \end{aligned}$$

(b) $F > 2V^{-2}M,\ 2W^{-2k}H^k$. Then

$$\begin{aligned} V^2W^2F &\ll V^2W^2\min\{V^{-2}qT, V^{-6}qTM, W^{-2k}qT, W^{-6k}qTH^k\} \\ &\le V^2W^2(V^{-2})^{1-\frac{3}{2k}}(V^{-6}M)^{\frac{1}{2k}}(W^{-2k})^{\frac{1}{k}}qT = qTM^{\frac{1}{2k}}. \end{aligned}$$

Since $k \ge 8$, we have $H \ge Y^{\frac{1}{k+1}} \ge Y^{1-\frac{k}{9}}$, $M^{\frac{1}{2k}} \ll Y^{\frac{1}{18}}$ and so

$$\min{}^2\left(\eta, \frac{1}{T}\right)V^2W^2F \ll \frac{\eta}{T}Y^{\frac{1}{18}}qT \ll \eta^2Y^{1-\varepsilon}.$$

(c) $F \le 2V^{-2}M,\ F > 2W^{-2k}H^k$. Then

$$\begin{aligned} V^2W^2F &\ll V^2W^2\min\{V^{-2}M, W^{-2k}qT, W^{-6k}qTH^k\} \\ &\le V^2W^2(V^{-2}M)^{1-\frac{1}{3k}}(W^{-6k}qTH^k)^{\frac{1}{3k}} \\ &\ll MH^{\frac{1}{3}}(qT)^{\frac{1}{3k}}, \end{aligned}$$

since $V \ll M^{\frac{1}{2}}$. As $H \ge Y^{\frac{1}{k+1}} \ge (\frac{Y}{Q})^{\frac{1}{2k}}Y^{2\varepsilon}$, we have

$$\min{}^2\left(\eta, \frac{1}{T}\right)V^2W^2F \ll \eta^{2-\frac{1}{3k}}T^{-\frac{1}{3k}}Y\left(\frac{Y}{Q}\right)^{-\frac{1}{3k}}(qT)^{\frac{1}{3k}}Y^{-\varepsilon} \ll \eta^2Y^{1-\varepsilon}.$$

(d) $F > 2V^{-2}M$, $F \le 2W^{-2k}H^k$. Then

$$\begin{aligned} V^2W^2F &\ll V^2W^2\min\{V^{-2}qT, V^{-6}qTM, W^{-2k}H^k\} \\ &\le V^2W^2(V^{-2}qT)^{1-\frac{3}{2k}}(V^{-6}qTM)^{\frac{1}{2k}}(W^{-2k}H^k)^{\frac{1}{k}} \\ &= (qT)^{1-\frac{1}{k}}HM^{\frac{1}{2k}}. \end{aligned}$$

If $k \ge 9$, then $H \le Y^{\frac{1}{k}} \le Y^{1-\frac{17(k-1)}{9(2k-1)}}$, $M \gg Y^{\frac{17(k-1)}{9(2k-1)}}$, and so

$$\min^2\left(\eta, \frac{1}{T}\right) V^2 W^2 F \ll \eta^2 Y \left(\frac{Y}{Q}\right)^{1-\frac{1}{k}} Y^{-\frac{17}{18}(1-\frac{1}{k})} \ll \eta^2 Y^{1-\varepsilon}.$$

If $k = 8$, then $Y^{\frac{1}{9}} \le H \ll Y^{\frac{16}{135}}$, $M \gg Y^{\frac{119}{135}}$, and so

$$\min^2\left(\eta, \frac{1}{T}\right) V^2 W^2 F \ll \eta^2 Y \left(\frac{Y}{Q}\right)^{\frac{7}{8}} Y^{-\frac{119}{144}} \ll \eta^2 Y^{1-\varepsilon}.$$

Combining the above, we obtain (7). Hence, Lemma 5 follows.

LEMMA 6. *Assume that* $M \ll Y^{\frac{19}{36}}$, $ML = Y$, $q \le Q$, χ *is a character* mod$q$, $M(s,\chi)$ *is a Dirichlet polynomial, and*

$$F(s,\chi) = M(s,\chi) \sum_{l \sim L} \frac{\chi(l)}{l^s}.$$

Let $\eta = qQ/Y$, $b = 1 + 1/\log N$ *and* $T_1 = \sqrt{L/q}$. *Then for* $T_1 \le T \le Y$, *we have*

$$\min^2\left(\eta, \frac{1}{T}\right) \sum_{\chi \,(\mathrm{mod}\, q)} \int_T^{2T} |F(b+it,\chi)|^2 \, dt \ll \eta^2 Y^{-\varepsilon}.$$

Proof. Perron's formula yields

$$\begin{aligned}
\sum_{l \sim L} \frac{\chi(l)}{l^{b+it}} &= \frac{1}{2\pi i} \int_{-4iT}^{4iT} L(b+it+s,\chi) \frac{(c_2L)^s - (c_1L)^s}{s} \, ds \\
&\quad + O\left(\frac{Y^\varepsilon}{T}\right) + O\left(\frac{1}{L}\right) \\
&= \frac{1}{2\pi i} \int_{\frac{1}{2}-b-4iT}^{\frac{1}{2}-b+4iT} L(b+it+s,\chi) \frac{(c_2L)^s - (c_1L)^s}{s} \, ds \\
&\quad + O\left(\frac{Y^\varepsilon}{T}\right) + O\left(\frac{1}{L}\right) + O\left(\frac{L^{-\frac{1}{2}}(qT)^{\frac{1}{4}+\varepsilon}}{T}\right),
\end{aligned}$$

$$\begin{aligned}
\left|\sum_{l \sim L} \frac{\chi(l)}{l^{b+it}}\right|^2 &\ll L^{-1} \log T \int_{-6T}^{6T} \left|L\left(\frac{1}{2} + iu, \chi\right)\right|^2 \frac{du}{1+|t-u|} \\
&\quad + O\left(\frac{Y^{2\varepsilon}}{T^2}\right) + O\left(\frac{1}{L^2}\right) + O\left(\frac{L^{-1}(qT)^{\frac{1}{2}+2\varepsilon}}{T^2}\right).
\end{aligned}$$

By Lemma 4 and the mean value estimate for the fourth power of the L-function, we have

$$\sum_{\chi\,(\mathrm{mod}\,q)}\int_T^{2T}|F(b+it,\chi)|^2\,dt$$

$$\ll L^{-1}\log N\sum_{\chi\,(\mathrm{mod}\,q)}\int_T^{2T}|M(b+it,\chi)|^2\,dt\Big(\int_{-6T}^{6T}\Big|L\Big(\frac12+iu,\chi\Big)\Big|^2\frac{du}{1+|t-u|}\Big)$$

$$+O\Big(\frac{Y^{2\varepsilon}}{T^2}+\frac{1}{L^2}+\frac{L^{-1}(qT)^{\frac12+2\varepsilon}}{T^2}\Big)\sum_{\chi\,(\mathrm{mod}\,q)}\int_T^{2T}|M(b+it,\chi)|^2\,dt$$

$$\ll L^{-1}\log^2 N\Big(\sum_{\chi\,(\mathrm{mod}\,q)}\int_T^{2T}|M(b+it,\chi)|^4\,dt\Big)^{\frac12}$$

$$\times\Big(\sum_{\chi\,(\mathrm{mod}\,q)}\int_T^{2T}dt\int_{-6T}^{6T}\frac{|L(\frac12+iu,\chi)|^4}{1+|t-u|}\,du\Big)^{\frac12}$$

$$+O\Big(\frac{Y^{2\varepsilon}}{T^2}+\frac{1}{L^2}+\frac{L^{-1}(qT)^{\frac12+2\varepsilon}}{T^2}\Big)\Big(1+\frac{qT}{M}\Big)$$

$$\ll L^{-1}\Big(1+\frac{qT}{M^2}\Big)^{\frac12}(qT)^{\frac12}\log^{10}N+Y^{-2\varepsilon}.$$

Hence,

$$\min{}^2\Big(\eta,\frac1T\Big)\sum_{\chi\,(\mathrm{mod}\,q)}\int_T^{2T}|F(b+it,\chi)|^2\,dt$$

$$\ll \min{}^2\Big(\eta,\frac1T\Big)Y^{-1}(Y^{\frac{19}{18}}+qT)^{\frac12}(qT)^{\frac12}\log^{10}N+\eta^2Y^{-2\varepsilon}$$

$$\ll \eta^2Y^{-1}\Big(Y^{\frac{19}{18}}+\frac{Y}{Q}\Big)^{\frac12}\Big(\frac{Y}{Q}\Big)^{\frac12}\log^{10}N+\eta^2Y^{-2\varepsilon}\ll\eta^2Y^{-2\varepsilon}.$$

Thus Lemma 6 follows.

4. Mean value estimate (II)

LEMMA 7. *Assume that* $MHK=Y$, $q\le Q$, χ *is a character* $\mathrm{mod}\,q$, $M(s,\chi)$, $H(s,\chi)$ *and* $K(s,\chi)$ *are Dirichlet polynomials and* $G(s,\chi)=M(s,\chi)H(s,\chi)K(s,\chi)$. *Let* $\eta=qQ/Y$, $b=1+1/\log N$, $T_0=\log^{\frac{E}{\delta^2}}Y$. *Assume further that for* $T_0\le|t|\le 2Y$, $M(b+it,\chi)\ll\log^{-\frac{E}{\delta^2}}Y$ *and*

$H(b+it,\chi) \ll \log^{-\frac{E}{\delta^2}} Y$. Moreover, suppose that M and H satisfy one of the following four conditions:

1) $MH \ll Y^{\frac{142}{261}}$, $Y^{\frac{17}{99}} \ll H$, $M^{29}/H \ll Y^{\frac{91}{9}}$, $Y^{\frac{17}{57}} \ll M$, $H^{29}/M \ll Y^{\frac{19}{3}}$, $Y^{\frac{17}{33}} \ll M^{\frac{12}{11}}H$;

2) $MH \ll Y^{\frac{47}{81}}$, $Y^{\frac{16}{135}} \ll H$, $M^{\frac{29}{19}}H \ll Y^{\frac{89}{114}}$, $Y^{\frac{68}{81}} \ll M^2H$, $H^4/M \ll Y^{\frac{1}{2}}$, $Y^{\frac{85}{147}} \ll M^{\frac{58}{49}}H$;

3) $MH \ll Y^{\frac{113}{198}}$, $Y^{\frac{17}{90}} \ll H$, $M^6H \ll Y^{\frac{20}{9}}$, $Y^{\frac{17}{63}} \ll M$, $M^{\frac{1}{8}}H \ll Y^{\frac{7}{24}}$, $Y^{\frac{17}{30}} \ll M^{\frac{6}{5}}H$;

4) $MH \ll Y^{\frac{25}{42}}$, $Y^{\frac{16}{135}} \ll H$, $M^{\frac{23}{15}}H \ll Y^{\frac{7}{9}}$, $Y^{\frac{17}{21}} \ll M^2H$, $H^3/M \ll Y^{\frac{7}{18}}$, $Y^{\frac{68}{135}} \ll M^{\frac{14}{15}}H$, $Y^{\frac{5}{9}} \ll MH$.

Then for $T_0 \le T \le Y$, we have

$$\min^2\left(\eta, \frac{1}{T}\right) \sum_{\chi\,(\text{good})} \int_T^{2T} |G(b+it,\chi)|^2\,dt \ll \eta^2 \log^{-6E} N. \tag{8}$$

Proof. We only show that for $T = 1/\eta = Y/(qQ)$,

$$I = \sum_{\chi\,(\text{good})} \int_T^{2T} |G(b+it,\chi)|^2\,dt \ll \log^{-6E} N. \tag{9}$$

I. First, we assume condition 1). On applying the mean value estimate and Halász method to $M^3(s,\chi)$, $H^5(s,\chi)$ and $K^2(s,\chi)$, we get

$$I \ll U^2V^2W^2Y^{-1}F\log^c N,$$

where

$$\begin{aligned}F = \min\{&V^{-6}(M^3+qT), V^{-6}M^3 + V^{-18}qTM^3, W^{-10}(H^5+qT),\\ &W^{-10}H^5 + W^{-30}qTH^5, U^{-4}(K^2+qT), U^{-4}K^2 + U^{-12}qTK^2\}.\end{aligned}$$

We discuss the following cases:

(a) $F \le 2V^{-6}M^3, 2W^{-10}H^5, 2U^{-4}K^2$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-6}M^3, W^{-10}H^5, U^{-4}K^2\}\\ &\le U^2V^2W^2(V^{-6}M^3)^{\frac{5}{16}}(W^{-10}H^5)^{\frac{1}{5}}(U^{-4}K^2)^{\frac{39}{80}}\\ &= V^{\frac{1}{8}}M^{\frac{15}{16}}U^{\frac{1}{20}}K^{\frac{39}{40}}H \ll Y\log^{-7E} N.\end{aligned}$$

(b) $F \le 2V^{-6}M^3, 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-6}M^3, W^{-10}H^5, U^{-4}qT, U^{-12}qTK^2\}\\ &\le U^2V^2W^2(V^{-6}M^3)^{\frac{1}{3}}(W^{-10}H^5)^{\frac{1}{5}}(U^{-4}qT)^{\frac{9}{20}}(U^{-12}qTK^2)^{\frac{1}{60}}\\ &= (qT)^{\frac{7}{15}}MHK^{\frac{1}{30}} \ll Y^{1-\varepsilon}.\end{aligned}$$

(c) $F \le 2V^{-6}M^3, F > 2W^{-10}H^5, F \le 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-6}M^3, W^{-10}qT, W^{-30}qTH^5, U^{-4}K^2\} \\ &\le U^2V^2W^2(V^{-6}M^3)^{\frac{1}{3}}(W^{-10}qT)^{\frac{3}{20}}(W^{-30}qTH^5)^{\frac{1}{60}}(U^{-4}K^2)^{\frac{1}{2}} \\ &= (qT)^{\frac{1}{6}}MKH^{\frac{1}{12}} \ll Y^{1-\varepsilon}. \end{aligned}$$

(d) $F \le 2V^{-6}M^3, F > 2W^{-10}H^5, 2U^{-4}K^2$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\quad \ll U^2V^2W^2 \min\{V^{-6}M^3, W^{-10}qT, W^{-30}qTH^5, U^{-4}qT, U^{-12}qTK^2\} \\ &\quad \le U^2V^2W^2(V^{-6}M^3)^{\frac{1}{3}}(W^{-10}qT)^{\frac{1}{5}}(U^{-4}qT)^{\frac{9}{20}}(U^{-12}qTK^2)^{\frac{1}{60}} \\ &\quad = (qT)^{\frac{2}{3}}MK^{\frac{1}{30}} \ll Y^{1-\varepsilon}. \end{aligned}$$

(e) $F > 2V^{-6}M^3, F \le 2W^{-10}H^5, 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-6}qT, V^{-18}qTM^3, W^{-10}H^5, U^{-4}K^2\} \\ &\le U^2V^2W^2(V^{-6}qT)^{\frac{17}{60}}(V^{-18}qTM^3)^{\frac{1}{60}}(W^{-10}H^5)^{\frac{1}{5}}(U^{-4}K^2)^{\frac{1}{2}} \\ &= (qT)^{\frac{3}{10}}M^{\frac{1}{20}}HK \ll Y^{1-\varepsilon}. \end{aligned}$$

(f) $F > 2V^{-6}M^3, F \le 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\quad \ll U^2V^2W^2 \min\{V^{-6}qT, V^{-18}qTM^3, W^{-10}H^5, U^{-4}qT, U^{-12}qTK^2\} \\ &\quad \le U^2V^2W^2(V^{-6}qT)^{\frac{1}{3}}(W^{-10}H^5)^{\frac{1}{5}}(U^{-4}qT)^{\frac{9}{20}}(U^{-12}qTK^2)^{\frac{1}{60}} \\ &\quad = (qT)^{\frac{4}{5}}HK^{\frac{1}{30}} \ll Y^{1-\varepsilon}. \end{aligned}$$

(g) $F > 2V^{-6}M^3, 2W^{-10}H^5, F \le 2U^{-4}K^2$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\quad \ll U^2V^2W^2 \min\{V^{-6}qT, V^{-18}qTM^3, W^{-10}qT, W^{-30}qTH^5, U^{-4}K^2\} \\ &\quad \le U^2V^2W^2(V^{-6}qT)^{\frac{1}{3}}(W^{-10}qT)^{\frac{3}{20}}(W^{-30}qTH^5)^{\frac{1}{60}}(U^{-4}K^2)^{\frac{1}{2}} \\ &\quad = (qT)^{\frac{1}{2}}H^{\frac{1}{12}}K \ll Y^{1-\varepsilon}. \end{aligned}$$

(h) $F > 2V^{-6}M^3, 2W^{-10}H^5, 2U^{-4}K^2$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\quad \ll U^2V^2W^2 \min\{V^{-6}, V^{-18}M^3, W^{-10}, W^{-30}H^5, U^{-4}, U^{-12}K^2\}qT \\ &\quad \le U^2V^2W^2(V^{-6})^{\frac{17}{60}}(V^{-18}M^3)^{\frac{1}{60}}(W^{-10})^{\frac{1}{5}}(U^{-4})^{\frac{1}{2}}qT \\ &\quad = qTM^{\frac{1}{20}} \ll Y^{1-\varepsilon}, \end{aligned}$$

since $M \ll Y$.

II. Next, we assume condition 2). On applying the mean value estimate and Halász method to $M^2(s,\chi)H(s,\chi)$, $H^5(s,\chi)$ and $K^2(s,\chi)$, we get

$$I \ll U^2V^2W^2Y^{-1}F\log^c N,$$

where

$$\begin{aligned}F = \min\{&V^{-4}W^{-2}(M^2H+qT), V^{-4}W^{-2}M^2H+V^{-12}W^{-6}qTM^2H,\\ &W^{-10}(H^5+qT), W^{-10}H^5+W^{-30}qTH^5, U^{-4}(K^2+qT),\\ &U^{-4}K^2+U^{-12}qTK^2\}.\end{aligned}$$

We consider several cases:

(a) $F\le 2V^{-4}W^{-2}M^2H, 2W^{-10}H^5, 2U^{-4}K^2$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}W^{-2}M^2H, W^{-10}H^5, U^{-4}K^2\}\\ &\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac12}(W^{-10}H^5)^{\frac1{12}}(U^{-4}K^2)^{\frac5{12}}\\ &= W^{\frac16}H^{\frac{11}{12}}U^{\frac13}K^{\frac56}M \ll Y\log^{-7E}N.\end{aligned}$$

(b) $F\le 2V^{-4}W^{-2}M^2H, 2W^{-10}H^5, F>2U^{-4}K^2$. Then

$$\begin{aligned}&U^2V^2W^2F\\ &\quad\ll U^2V^2W^2\min\{V^{-4}W^{-2}M^2H, W^{-10}H^5, U^{-4}qT, U^{-12}qTK^2\}\\ &\quad\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac12}(W^{-10}H^5)^{\frac1{10}}(U^{-4}qT)^{\frac7{20}}(U^{-12}qTK^2)^{\frac1{20}}\\ &\quad= (qT)^{\frac25}MHK^{\frac1{10}} \ll Y^{1-\varepsilon}.\end{aligned}$$

(c) $F\le 2V^{-4}W^{-2}M^2H, F>2W^{-10}H^5, F\le 2U^{-4}K^2$. Then

$$\begin{aligned}&U^2V^2W^2F\\ &\qquad\ll U^2V^2W^2\min\{V^{-4}W^{-2}M^2H, W^{-10}qT, W^{-30}qTH^5, U^{-4}K^2\}\\ &\qquad\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac12}(W^{-30}qTH^5)^{\frac1{30}}(U^{-4}K^2)^{\frac7{15}}\\ &\qquad= (qT)^{\frac1{30}}U^{\frac2{15}}K^{\frac{14}{15}}MH^{\frac23}\\ &\qquad\ll (qT)^{\frac1{30}}MH^{\frac23}K \ll Y^{1-\varepsilon}.\end{aligned}$$

(d) $F\le 2V^{-4}W^{-2}M^2H, F>2W^{-10}H^5, 2U^{-4}K^2$. Then

$$\begin{aligned}&U^2V^2W^2F\\ &\quad\ll U^2V^2W^2\min\{V^{-4}W^{-2}M^2H, W^{-10}qT, W^{-30}qTH^5,\\ &\qquad\qquad U^{-4}qT, U^{-12}qTK^2\}\\ &\quad\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac12}(W^{-30}qTH^5)^{\frac1{30}}(U^{-4}qT)^{\frac9{20}}(U^{-12}qTK^2)^{\frac1{60}}\\ &\quad= (qT)^{\frac12}MH^{\frac23}K^{\frac1{30}} \ll Y^{1-\varepsilon}.\end{aligned}$$

(e) $F > 2V^{-4}W^{-2}M^2H, F \le 2W^{-10}H^5, 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F \\
&\ll U^2V^2W^2 \min\{V^{-4}W^{-2}qT, V^{-12}W^{-6}qTM^2H, W^{-10}H^5, U^{-4}K^2\} \\
&\le U^2V^2W^2(V^{-4}W^{-2}qT)^{\frac{7}{20}}(V^{-12}W^{-6}qTM^2H)^{\frac{1}{20}}(W^{-10}H^5)^{\frac{1}{10}}(U^{-4}K^2)^{\frac{1}{2}} \\
&= (qT)^{\frac{2}{5}}M^{\frac{1}{10}}H^{\frac{11}{20}}K \ll Y^{1-\varepsilon}.
\end{aligned}$$

(f) $F > 2V^{-4}W^{-2}M^2H, F \le 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F \\
&\quad \ll U^2V^2W^2 \min\{V^{-4}W^{-2}qT, V^{-12}W^{-6}qTM^2H, W^{-10}H^5, U^{-4}qT, \\
&\qquad\qquad U^{-12}qTK^2\} \\
&\quad \le U^2V^2W^2(V^{-4}W^{-2}qT)^{\frac{1}{2}}(W^{-10}H^5)^{\frac{1}{10}}(U^{-4}qT)^{\frac{7}{20}}(U^{-12}qTK^2)^{\frac{1}{20}} \\
&\quad = (qT)^{\frac{9}{10}}H^{\frac{1}{2}}K^{\frac{1}{10}} \ll Y^{1-\varepsilon}.
\end{aligned}$$

(g) $F > 2V^{-4}W^{-2}M^2H, 2W^{-10}H^5, F \le 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F \\
&\ll U^2V^2W^2 \min\{V^{-4}W^{-2}qT, V^{-12}W^{-6}qTM^2H, W^{-10}qT, W^{-30}qTH^5, \\
&\qquad\qquad U^{-4}K^2\} \\
&\le U^2V^2W^2(V^{-4}W^{-2}qT)^{\frac{9}{20}}(V^{-12}W^{-6}qTM^2H)^{\frac{1}{60}}(W^{-30}qTH^5)^{\frac{1}{30}} \\
&\quad \times (U^{-4}K^2)^{\frac{1}{2}} \\
&= (qT)^{\frac{1}{2}}M^{\frac{1}{30}}H^{\frac{11}{60}}K \ll Y^{1-\varepsilon}.
\end{aligned}$$

(h) $F > 2V^{-4}W^{-2}M^2H, 2W^{-10}H^5, 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-4}W^{-2}, V^{-12}W^{-6}M^2H, W^{-10}, W^{-30}H^5, \\
&\qquad U^{-4}, U^{-12}K^2\}qT \\
&\le U^2V^2W^2(V^{-4}W^{-2})^{\frac{1}{2}}(W^{-10})^{\frac{1}{10}}(U^{-4})^{\frac{7}{20}}(U^{-12}K^2)^{\frac{1}{20}}qT \\
&= qTK^{\frac{1}{10}} \ll Y^{1-\varepsilon},
\end{aligned}$$

since $Y^{\frac{4}{9}} \ll MH$ (the latter follows from $Y^{\frac{16}{135}} \ll H$ and $Y^{\frac{68}{81}} \ll M^2H$).

III. Now, we assume condition 3). On applying the mean value estimate and Halász method to $M^3(s,\chi)$, $H^4(s,\chi)$ and $K^2(s,\chi)$, we get

$$I \ll U^2V^2W^2Y^{-1}F\log^c N,$$

where

$$\begin{aligned}
F = \min\{&V^{-6}(M^3+qT), V^{-6}M^3 + V^{-18}qTM^3, W^{-8}(H^4+qT), W^{-8}H^4 \\
&+ W^{-24}qTH^4, U^{-4}(K^2+qT), U^{-4}K^2 + U^{-12}qTK^2\}.
\end{aligned}$$

Consider the following cases:

(a) $F \le 2V^{-6}M^3, 2W^{-8}H^4, 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-6}M^3, W^{-8}H^4, U^{-4}K^2\} \\ &\le U^2V^2W^2(V^{-6}M^3)^{\frac{3}{10}}(W^{-8}H^4)^{\frac{1}{4}}(U^{-4}K^2)^{\frac{9}{20}} \\ &= V^{\frac{1}{5}}M^{\frac{9}{10}}U^{\frac{1}{5}}K^{\frac{9}{10}}H \ll Y\log^{-7E} N. \end{aligned}$$

(b) $F \le 2V^{-6}M^3, 2W^{-8}H^4, F > 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-6}M^3, W^{-8}H^4, U^{-4}qT, U^{-12}qTK^2\} \\ &\le U^2V^2W^2(V^{-6}M^3)^{\frac{1}{3}}(W^{-8}H^4)^{\frac{1}{4}}(U^{-4}qT)^{\frac{3}{8}}(U^{-12}qTK^2)^{\frac{1}{24}} \\ &= (qT)^{\frac{5}{12}}MHK^{\frac{1}{12}} \ll Y^{1-\varepsilon}. \end{aligned}$$

(c) $F \le 2V^{-6}M^3, F > 2W^{-8}H^4, F \le 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-6}M^3, W^{-8}qT, W^{-24}qTH^4, U^{-4}K^2\} \\ &\le U^2V^2W^2(V^{-6}M^3)^{\frac{1}{3}}(W^{-8}qT)^{\frac{1}{8}}(W^{-24}qTH^4)^{\frac{1}{24}}(U^{-4}K^2)^{\frac{1}{2}} \\ &= (qT)^{\frac{1}{6}}MKH^{\frac{1}{6}} \ll Y^{1-\varepsilon}. \end{aligned}$$

(d) $F \le 2V^{-6}M^3, F > 2W^{-8}H^4, 2U^{-4}K^2$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\ll U^2V^2W^2 \min\{V^{-6}M^3, W^{-8}qT, W^{-24}qTH^4, U^{-4}qT, U^{-12}qTK^2\} \\ &\le U^2V^2W^2(V^{-6}M^3)^{\frac{1}{3}}(W^{-8}qT)^{\frac{1}{8}}(W^{-24}qTH^4)^{\frac{1}{24}}(U^{-4}qT)^{\frac{1}{2}} \\ &= (qT)^{\frac{2}{3}}MH^{\frac{1}{6}} \ll Y^{1-\varepsilon}. \end{aligned}$$

(e) $F > 2V^{-6}M^3, F \le 2W^{-8}H^4, 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-6}qT, V^{-18}qTM^3, W^{-8}H^4, U^{-4}K^2\} \\ &\le U^2V^2W^2(V^{-6}qT)^{\frac{5}{24}}(V^{-18}qTM^3)^{\frac{1}{24}}(W^{-8}H^4)^{\frac{1}{4}}(U^{-4}K^2)^{\frac{1}{2}} \\ &= (qT)^{\frac{1}{4}}M^{\frac{1}{8}}HK \ll Y^{1-\varepsilon}. \end{aligned}$$

(f) $F > 2V^{-6}M^3, F \le 2W^{-8}H^4, F > 2U^{-4}K^2$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\ll U^2V^2W^2 \min\{V^{-6}qT, V^{-18}qTM^3, W^{-8}H^4, U^{-4}qT, U^{-12}qTK^2\} \\ &\le U^2V^2W^2(V^{-6}qT)^{\frac{5}{24}}(V^{-18}qTM^3)^{\frac{1}{24}}(W^{-8}H^4)^{\frac{1}{4}}(U^{-4}qT)^{\frac{1}{2}} \\ &= (qT)^{\frac{3}{4}}M^{\frac{1}{8}}H \ll Y^{1-\varepsilon}. \end{aligned}$$

(g) $F > 2V^{-6}M^3, 2W^{-8}H^4, F \le 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F & \\ &\ll U^2V^2W^2 \min\{V^{-6}qT, V^{-18}qTM^3, W^{-8}qT, W^{-24}qTH^4, U^{-4}K^2\} \\ &\le U^2V^2W^2(V^{-6}qT)^{\frac{1}{3}}(W^{-8}qT)^{\frac{1}{8}}(W^{-24}qTH^4)^{\frac{1}{24}}(U^{-4}K^2)^{\frac{1}{2}} \\ &= (qT)^{\frac{1}{2}}H^{\frac{1}{6}}K \ll Y^{1-\varepsilon}. \end{aligned}$$

(h) $F > 2V^{-6}M^3, 2W^{-8}H^4, 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F & \\ &\ll U^2V^2W^2 \min\{V^{-6}, V^{-18}M^3, W^{-8}, W^{-24}H^4, U^{-4}, U^{-12}K^2\}qT \\ &\le U^2V^2W^2(V^{-6})^{\frac{5}{24}}(V^{-18}M^3)^{\frac{1}{24}}(W^{-8})^{\frac{1}{4}}(U^{-4})^{\frac{1}{2}}qT \\ &= qTM^{\frac{1}{8}} \ll Y^{1-\varepsilon}, \end{aligned}$$

since $M \ll Y^{\frac{4}{9}}$ (the latter follows from $MH \ll Y^{\frac{113}{198}}$ and $Y^{\frac{17}{90}} \ll H$).

IV. Lastly, condition 4) is assumed. On applying the mean value estimate and Halász method to $M^2(s,\chi)H(s,\chi)$, $H^4(s,\chi)$ and $K^2(s,\chi)$, we get

$$I \ll U^2V^2W^2Y^{-1}F \log^c N,$$

where

$$\begin{aligned} F = \min\{&V^{-4}W^{-2}(M^2H + qT), V^{-4}W^{-2}M^2H + V^{-12}W^{-6}qTM^2H, \\ &W^{-8}(H^4 + qT), W^{-8}H^4 + W^{-24}qTH^4, U^{-4}(K^2 + qT), \\ &U^{-4}K^2 + U^{-12}qTK^2\}. \end{aligned}$$

We consider the following cases:

(a) $F \le 2V^{-4}W^{-2}M^2H, 2W^{-8}H^4, 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-4}W^{-2}M^2H, W^{-8}H^4, U^{-4}K^2\} \\ &\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(W^{-8}H^4)^{\frac{1}{12}}(U^{-4}K^2)^{\frac{5}{12}} \\ &= W^{\frac{1}{3}}H^{\frac{5}{6}}U^{\frac{1}{3}}K^{\frac{5}{6}}M \ll Y \log^{-7E} N. \end{aligned}$$

(b) $F \le 2V^{-4}W^{-2}M^2H, 2W^{-8}H^4, F > 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F & \\ &\ll U^2V^2W^2 \min\{V^{-4}W^{-2}M^2H, W^{-8}H^4, U^{-4}qT, U^{-12}qTK^2\} \\ &\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(W^{-8}H^4)^{\frac{1}{8}}(U^{-4}qT)^{\frac{5}{16}}(U^{-12}qTK^2)^{\frac{1}{16}} \\ &= (qT)^{\frac{3}{8}}MHK^{\frac{1}{8}} \ll Y^{1-\varepsilon}. \end{aligned}$$

(c) $F \le 2V^{-4}W^{-2}M^2H, F > 2W^{-8}H^4, F \le 2U^{-4}K^2$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}W^{-2}M^2H, W^{-8}qT, W^{-24}qTH^4, U^{-4}K^2\}\\ &\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(W^{-24}qTH^4)^{\frac{1}{24}}(U^{-4}K^2)^{\frac{11}{24}}\\ &= (qT)^{\frac{1}{24}}U^{\frac{1}{6}}K^{\frac{11}{12}}MH^{\frac{2}{3}}\\ &\ll (qT)^{\frac{1}{24}}MH^{\frac{2}{3}}K \ll Y^{1-\varepsilon}.\end{aligned}$$

(d) $F \le 2V^{-4}W^{-2}M^2H, F > 2W^{-8}H^4, 2U^{-4}K^2$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}W^{-2}M^2H, W^{-8}qT, W^{-24}qTH^4, U^{-4}qT,\\ &\qquad U^{-12}qTK^2\}\\ &\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(W^{-24}qTH^4)^{\frac{1}{24}}(U^{-4}qT)^{\frac{7}{16}}(U^{-12}qTK^2)^{\frac{1}{48}}\\ &= (qT)^{\frac{1}{2}}MH^{\frac{2}{3}}K^{\frac{1}{24}} \ll Y^{1-\varepsilon}.\end{aligned}$$

(e) $F > 2V^{-4}W^{-2}M^2H, F \le 2W^{-8}H^4, 2U^{-4}K^2$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}W^{-2}qT, V^{-12}W^{-6}qTM^2H, W^{-8}H^4, U^{-4}K^2\}\\ &\le U^2V^2W^2(V^{-4}W^{-2}qT)^{\frac{5}{16}}(V^{-12}W^{-6}qTM^2H)^{\frac{1}{16}}(W^{-8}H^4)^{\frac{1}{8}}(U^{-4}K^2)^{\frac{1}{2}}\\ &= (qT)^{\frac{3}{8}}M^{\frac{1}{8}}H^{\frac{9}{16}}K \ll Y^{1-\varepsilon}.\end{aligned}$$

(f) $F > 2V^{-4}W^{-2}M^2H, F \le 2W^{-8}H^4, F > 2U^{-4}K^2$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}W^{-2}qT, V^{-12}W^{-6}qTM^2H, W^{-8}H^4, U^{-4}qT,\\ &\qquad U^{-12}qTK^2\}\\ &\le U^2V^2W^2(V^{-4}W^{-2}qT)^{\frac{1}{2}}(W^{-8}H^4)^{\frac{1}{8}}(U^{-4}qT)^{\frac{5}{16}}(U^{-12}qTK^2)^{\frac{1}{16}}\\ &= (qT)^{\frac{7}{8}}H^{\frac{1}{2}}K^{\frac{1}{8}} \ll Y^{1-\varepsilon}.\end{aligned}$$

(g) $F > 2V^{-4}W^{-2}M^2H, 2W^{-8}H^4, F \le 2U^{-4}K^2$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}W^{-2}qT, V^{-12}W^{-6}qTM^2H, W^{-8}qT,\\ &\qquad W^{-24}qTH^4, U^{-4}K^2\}\\ &\le U^2V^2W^2(V^{-4}W^{-2}qT)^{\frac{5}{16}}(V^{-12}W^{-6}qTM^2H)^{\frac{1}{16}}(W^{-8}qT)^{\frac{1}{8}}(U^{-4}K^2)^{\frac{1}{2}}\\ &= (qT)^{\frac{1}{2}}M^{\frac{1}{8}}H^{\frac{1}{16}}K \ll Y^{1-\varepsilon}.\end{aligned}$$

(h) $F > 2V^{-4}W^{-2}M^2H, 2W^{-8}H^4, 2U^{-4}K^2$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-4}W^{-2}, V^{-12}W^{-6}M^2H, W^{-8}, W^{-24}H^4,\\ &\qquad U^{-4}, U^{-12}K^2\}qT\\ &\le U^2V^2W^2(V^{-4}W^{-2})^{\frac{1}{2}}(W^{-8})^{\frac{1}{8}}(U^{-4})^{\frac{5}{16}}(U^{-12}K^2)^{\frac{1}{16}}qT\\ &= qTK^{\frac{1}{8}} \ll Y^{1-\varepsilon}.\end{aligned}$$

Combining the above, we obtain (9). Hence, Lemma 7 follows.

LEMMA 8. *Under the assumption of Lemma* 7, M *and* H *lie in one of the following regions*:

(i) $Y^{\frac{17}{63}} \ll M \ll Y^{\frac{17}{57}}, \quad M^{-\frac{6}{5}}Y^{\frac{17}{30}} \ll H \ll M^{-\frac{1}{8}}Y^{\frac{7}{24}};$

(ii) $Y^{\frac{17}{57}} \ll M \ll Y^{\frac{17}{54}}, \quad M^{-\frac{12}{11}}Y^{\frac{17}{33}} \ll H \ll M^{-\frac{1}{8}}Y^{\frac{7}{24}};$

(iii) $Y^{\frac{17}{54}} \ll M \ll Y^{\frac{221}{693}}, \quad Y^{\frac{17}{99}} \ll H \ll M^{-\frac{1}{8}}Y^{\frac{7}{24}};$

(iv) $Y^{\frac{221}{693}} \ll M \ll Y^{\frac{109}{330}}, \quad Y^{\frac{17}{99}} \ll H \ll M^{-1}Y^{\frac{113}{198}};$

(v) $Y^{\frac{109}{330}} \ll M \ll Y^{\frac{151}{450}}, \quad Y^{\frac{17}{99}} \ll H \ll M^{-6}Y^{\frac{20}{9}};$

(vi) $Y^{\frac{151}{450}} \ll M \ll Y^{\frac{34}{99}}, \quad Y^{\frac{17}{99}} \ll H \ll M^{\frac{1}{4}}Y^{\frac{1}{8}};$

(vii) $Y^{\frac{151}{450}} \ll M \ll Y^{\frac{34}{99}}, \quad M^{-1}Y^{\frac{5}{9}} \ll H \ll M^{\frac{1}{3}}Y^{\frac{7}{54}};$

(viii) $Y^{\frac{34}{99}} \ll M \ll Y^{\frac{25}{72}}, \quad M^{-\frac{58}{49}}Y^{\frac{85}{147}} \ll H \ll M^{\frac{1}{4}}Y^{\frac{1}{8}};$

(ix) $Y^{\frac{25}{72}} \ll M \ll Y^{\frac{10}{27}}, \quad M^{-\frac{58}{49}}Y^{\frac{85}{147}} \ll H \ll M^{-\frac{23}{15}}Y^{\frac{7}{9}};$

(x) $Y^{\frac{10}{27}} \ll M \ll Y^{\frac{617}{1620}}, \quad M^{-\frac{58}{49}}Y^{\frac{85}{147}} \ll H \ll M^{-1}Y^{\frac{47}{81}};$

(xi) $Y^{\frac{617}{1620}} \ll M \ll Y^{\frac{3041}{7830}}, \quad M^{-\frac{58}{49}}Y^{\frac{85}{147}} \ll H \ll M^{-\frac{29}{19}}Y^{\frac{89}{114}};$

(xii) $Y^{\frac{3041}{7830}} \ll M \ll Y^{\frac{3397}{7830}}, \quad Y^{\frac{16}{135}} \ll H \ll M^{-\frac{29}{19}}Y^{\frac{89}{114}}.$

Then (8) *holds for* $T_0 \le T \le Y$.

Proof. In the regions:

$$Y^{\frac{17}{57}} \ll M \ll Y^{\frac{17}{54}}, \quad M^{-\frac{12}{11}}Y^{\frac{17}{33}} \ll H \ll M^{\frac{1}{29}}Y^{\frac{19}{87}};$$
$$Y^{\frac{17}{54}} \ll M \ll Y^{\frac{34}{99}}, \quad Y^{\frac{17}{99}} \ll H \ll M^{-1}Y^{\frac{142}{261}},$$

we apply Lemma 7 with condition 1).

In the regions:

$$Y^{\frac{175}{522}} \ll M \ll Y^{\frac{34}{99}}, \quad M^{-1}Y^{\frac{142}{261}} \ll H \ll M^{\frac{1}{4}}Y^{\frac{1}{8}};$$
$$Y^{\frac{34}{99}} \ll M \ll Y^{\frac{59}{162}}, \quad M^{-\frac{58}{49}}Y^{\frac{85}{147}} \ll H \ll M^{\frac{1}{4}}Y^{\frac{1}{8}};$$
$$Y^{\frac{59}{162}} \ll M \ll Y^{\frac{617}{1620}}, \quad M^{-\frac{58}{49}}Y^{\frac{85}{147}} \ll H \ll M^{-1}Y^{\frac{47}{81}};$$

$$Y^{\frac{617}{1620}} \ll M \ll Y^{\frac{3041}{7830}}, \qquad M^{-\frac{58}{49}} Y^{\frac{85}{147}} \ll H \ll M^{-\frac{29}{19}} Y^{\frac{89}{114}};$$
$$Y^{\frac{3041}{7830}} \ll M \ll Y^{\frac{3397}{7830}}, \qquad Y^{\frac{16}{135}} \ll H \ll M^{-\frac{29}{19}} Y^{\frac{89}{114}},$$

we apply Lemma 7 with condition 2).

In the regions:

$$Y^{\frac{17}{63}} \ll M \ll Y^{\frac{17}{57}}, \qquad M^{-\frac{6}{5}} Y^{\frac{17}{30}} \ll H \ll M^{-\frac{1}{8}} Y^{\frac{7}{24}};$$
$$Y^{\frac{17}{57}} \ll M \ll Y^{\frac{17}{54}}, \qquad M^{\frac{1}{29}} Y^{\frac{19}{87}} \ll H \ll M^{-\frac{1}{8}} Y^{\frac{7}{24}};$$
$$Y^{\frac{17}{54}} \ll M \ll Y^{\frac{221}{693}}, \qquad M^{-1} Y^{\frac{142}{261}} \ll H \ll M^{-\frac{1}{8}} Y^{\frac{7}{24}};$$
$$Y^{\frac{221}{693}} \ll M \ll Y^{\frac{109}{330}}, \qquad M^{-1} Y^{\frac{142}{261}} \ll H \ll M^{-1} Y^{\frac{113}{198}};$$
$$Y^{\frac{109}{330}} \ll M \ll Y^{\frac{175}{522}}, \qquad M^{-1} Y^{\frac{142}{261}} \ll H \ll M^{-6} Y^{\frac{20}{9}};$$
$$Y^{\frac{175}{522}} \ll M \ll Y^{\frac{151}{450}}, \qquad M^{\frac{1}{4}} Y^{\frac{1}{8}} \ll H \ll M^{-6} Y^{\frac{20}{9}},$$

we apply Lemma 7 with condition 3).

In the regions:

$$Y^{\frac{151}{450}} \ll M \ll Y^{\frac{34}{99}}, \qquad M^{-1} Y^{\frac{5}{9}} \ll H \ll M^{\frac{1}{3}} Y^{\frac{7}{54}};$$
$$Y^{\frac{25}{72}} \ll M \ll Y^{\frac{59}{162}}, \qquad M^{\frac{1}{4}} Y^{\frac{1}{8}} \ll H \ll M^{-\frac{23}{15}} Y^{\frac{7}{9}};$$
$$Y^{\frac{59}{162}} \ll M \ll Y^{\frac{10}{27}}, \qquad M^{-1} Y^{\frac{47}{81}} \ll H \ll M^{-\frac{23}{15}} Y^{\frac{7}{9}},$$

we apply Lemma 7 with condition 4).

Putting together the above regions, we get Lemma 8.

LEMMA 9. *Under the assumption of Lemma* 7, *suppose that M and H also satisfy one of the following four conditions*:

1) $MH \ll Y^{\frac{40}{57}}$, $Y^{\frac{17}{99}} \ll H$, $M \ll Y^{\frac{1}{2}}$, $Y^{\frac{119}{261}} \ll M$, $H^{19}/M \ll Y^{\frac{35}{9}}$, $Y^{\frac{68}{99}} \ll M^{\frac{12}{11}} H$;

2) $M^2 H \ll Y^{\frac{94}{81}}$, $M^{\frac{58}{9}} H \ll Y^{\frac{89}{27}}$, $Y^{\frac{34}{81}} \ll M$, $M^{\frac{1}{5}} H \ll Y^{\frac{3}{10}}$, $Y^{\frac{17}{12}} \ll M^{\frac{29}{10}} H$, $Y^{\frac{16}{135}} \ll H$;

3) $MH \ll Y^{\frac{46}{63}}$, $Y^{\frac{17}{90}} \ll H$, $M^6 H \ll Y^{\frac{19}{6}}$, $Y^{\frac{85}{198}} \ll M$, $H^7/M \ll Y^{\frac{4}{3}}$, $Y^{\frac{34}{45}} \ll M^{\frac{6}{5}} H$;

4) $M^2 H \ll Y^{\frac{25}{21}}$, $M^{\frac{46}{7}} H \ll Y^{\frac{10}{3}}$, $Y^{\frac{17}{42}} \ll M$, $M^{\frac{1}{4}} H \ll Y^{\frac{25}{72}}$, $Y^{\frac{17}{12}} \ll M^{\frac{23}{8}} H$, $M \ll Y^{\frac{4}{9}}$.

Then (8) *holds for* $T_0 \le T \le Y$.

Proof. We only show that (9) holds for $T = 1/\eta = Y/(qQ)$.

I. First, we assume condition 1). We apply the mean value estimate and Halász method to $M^2(s,\chi)$, $H^5(s,\chi)$ and $K^3(s,\chi)$ to get

$$I \ll U^2 V^2 W^2 Y^{-1} F \log^c N,$$

where

$$F = \min\{V^{-4}(M^2+qT), V^{-4}M^2+V^{-12}qTM^2, W^{-10}(H^5+qT),\\ W^{-10}H^5+W^{-30}qTH^5, U^{-6}(K^3+qT), U^{-6}K^3+U^{-18}qTK^3\}.$$

We consider several cases:

(a) $F \le 2V^{-4}M^2, 2W^{-10}H^5, 2U^{-6}K^3$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-10}H^5, U^{-6}K^3\}\\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac12}(W^{-10}H^5)^{\frac16}(U^{-6}K^3)^{\frac13}\\ &= W^{\frac13}H^{\frac56}MK \ll Y\log^{-7E}N.\end{aligned}$$

(b) $F \le 2V^{-4}M^2, 2W^{-10}H^5, F > 2U^{-6}K^3$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-10}H^5, U^{-6}qT, U^{-18}qTK^3\}\\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac12}(W^{-10}H^5)^{\frac15}(U^{-6}qT)^{\frac{17}{60}}(U^{-18}qTK^3)^{\frac{1}{60}}\\ &= (qT)^{\frac{3}{10}}MHK^{\frac{1}{20}} \ll Y^{1-\varepsilon}.\end{aligned}$$

(c) $F \le 2V^{-4}M^2, F > 2W^{-10}H^5, F \le 2U^{-6}K^3$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-10}qT, W^{-30}qTH^5, U^{-6}K^3\}\\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac12}(W^{-10}qT)^{\frac{3}{20}}(W^{-30}qTH^5)^{\frac{1}{60}}(U^{-6}K^3)^{\frac13}\\ &= (qT)^{\frac16}MKH^{\frac{1}{12}} \ll Y^{1-\varepsilon}.\end{aligned}$$

(d) $F \le 2V^{-4}M^2, F > 2W^{-10}H^5, 2U^{-6}K^3$. Then

$$\begin{aligned}&U^2V^2W^2F\\ &\quad\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-10}qT, W^{-30}qTH^5, U^{-6}qT, U^{-18}qTK^3\}\\ &\quad\le U^2V^2W^2(V^{-4}M^2)^{\frac12}(W^{-10}qT)^{\frac15}(U^{-6}qT)^{\frac{17}{60}}(U^{-18}qTK^3)^{\frac{1}{60}}\\ &\quad= (qT)^{\frac12}MK^{\frac{1}{20}} \ll Y^{1-\varepsilon},\end{aligned}$$

since $M^{19}/H \ll Y^{\frac{86}{9}}$ (the latter follows from $M \ll Y^{\frac12}$).

(e) $F > 2V^{-4}M^2, F \le 2W^{-10}H^5, 2U^{-6}K^3$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}qT, V^{-12}qTM^2, W^{-10}H^5, U^{-6}K^3\}\\ &\le U^2V^2W^2(V^{-4}qT)^{\frac{9}{20}}(V^{-12}qTM^2)^{\frac{1}{60}}(W^{-10}H^5)^{\frac15}(U^{-6}K^3)^{\frac13}\\ &= (qT)^{\frac{7}{15}}M^{\frac{1}{30}}HK \ll Y^{1-\varepsilon}.\end{aligned}$$

(f) $F > 2V^{-4}M^2, F \le 2W^{-10}H^5, F > 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F&\\
&\ll U^2V^2W^2\min\{V^{-4}qT, V^{-12}qTM^2, W^{-10}H^5, U^{-6}qT, U^{-18}qTK^3\}\\
&\le U^2V^2W^2(V^{-4}qT)^{\frac{1}{2}}(W^{-10}H^5)^{\frac{1}{5}}(U^{-6}qT)^{\frac{17}{60}}(U^{-18}qTK^3)^{\frac{1}{60}}\\
&= (qT)^{\frac{4}{5}}HK^{\frac{1}{20}} \ll Y^{1-\varepsilon}.
\end{aligned}$$

(g) $F > 2V^{-4}M^2, 2W^{-10}H^5, F \le 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F&\\
&\ll U^2V^2W^2\min\{V^{-4}qT, V^{-12}qTM^2, W^{-10}qT, W^{-30}qTH^5, U^{-6}K^3\}\\
&\le U^2V^2W^2(V^{-4}qT)^{\frac{1}{2}}(W^{-10}qT)^{\frac{3}{20}}(W^{-30}qTH^5)^{\frac{1}{60}}(U^{-6}K^3)^{\frac{1}{3}}\\
&= (qT)^{\frac{2}{3}}H^{\frac{1}{12}}K \ll Y^{1-\varepsilon}.
\end{aligned}$$

(h) $F > 2V^{-4}M^2, 2W^{-10}H^5, 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F&\\
&\ll U^2V^2W^2\min\{V^{-4}, V^{-12}M^2, W^{-10}, W^{-30}H^5, U^{-6}, U^{-18}K^3\}qT\\
&\le U^2V^2W^2(V^{-4})^{\frac{1}{2}}(W^{-10})^{\frac{1}{5}}(U^{-6})^{\frac{17}{60}}(U^{-18}K^3)^{\frac{1}{60}}qT\\
&= qTK^{\frac{1}{20}} \ll Y^{1-\varepsilon},
\end{aligned}$$

since $K \ll Y$.

II. Next, assume condition 2). We apply the mean value estimate and Halász method to $M^2(s,\chi)$, $H^5(s,\chi)$ and $K^2(s,\chi)H(s,\chi)$ to get

$$I \ll U^2V^2W^2Y^{-1}F\log^c N,$$

where

$$\begin{aligned}
F = \min\{&V^{-4}(M^2+qT), V^{-4}M^2 + V^{-12}qTM^2, W^{-10}(H^5+qT),\\
&W^{-10}H^5 + W^{-30}qTH^5, U^{-4}W^{-2}(K^2H+qT),\\
&U^{-4}W^{-2}K^2H + U^{-12}W^{-6}qTK^2H\}.
\end{aligned}$$

Consider the following cases:

(a) $F \le 2V^{-4}M^2, 2W^{-10}H^5, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-10}H^5, U^{-4}W^{-2}K^2H\}\\
&\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(U^{-4}W^{-2}K^2H)^{\frac{1}{2}}\\
&= WH^{\frac{1}{2}}MK \ll Y\log^{-7E} N.
\end{aligned}$$

(b) $F \le 2V^{-4}M^2, 2W^{-10}H^5, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}
&U^2V^2W^2F \\
&\ll U^2V^2W^2 \min\{V^{-4}M^2, W^{-10}H^5, U^{-4}W^{-2}qT, U^{-12}W^{-6}qTK^2H\} \\
&\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(W^{-10}H^5)^{\frac{1}{10}}(U^{-4}W^{-2}qT)^{\frac{7}{20}}(U^{-12}W^{-6}qTK^2H)^{\frac{1}{20}} \\
&= (qT)^{\frac{2}{5}}MH^{\frac{11}{20}}K^{\frac{1}{10}} \ll Y^{1-\varepsilon}.
\end{aligned}$$

(c) $F \le 2V^{-4}M^2, F > 2W^{-10}H^5, F \le 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}
&U^2V^2W^2F \\
&\qquad \ll U^2V^2W^2 \min\{V^{-4}M^2, W^{-10}qT, W^{-30}qTH^5, U^{-4}W^{-2}K^2H\} \\
&\qquad \le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(U^{-4}W^{-2}K^2H)^{\frac{1}{2}} \\
&\qquad = WH^{\frac{1}{2}}MK \ll Y\log^{-7E} N.
\end{aligned}$$

(d) $F \le 2V^{-4}M^2, F > 2W^{-10}H^5, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-4}M^2, W^{-10}qT, W^{-30}qTH^5, U^{-4}W^{-2}qT, \\
&\qquad\qquad U^{-12}W^{-6}qTK^2H\} \\
&\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(W^{-30}qTH^5)^{\frac{1}{30}}(U^{-4}W^{-2}qT)^{\frac{9}{20}} \\
&\quad \times (U^{-12}W^{-6}qTK^2H)^{\frac{1}{60}} \\
&= (qT)^{\frac{1}{2}}MH^{\frac{11}{60}}K^{\frac{1}{30}} \ll Y^{1-\varepsilon}.
\end{aligned}$$

(e) $F > 2V^{-4}M^2, F \le 2W^{-10}H^5, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}
&U^2V^2W^2F \\
&\quad \ll U^2V^2W^2 \min\{V^{-4}qT, V^{-12}qTM^2, W^{-10}H^5, U^{-4}W^{-2}K^2H\} \\
&\quad \le U^2V^2W^2(V^{-4}qT)^{\frac{7}{20}}(V^{-12}qTM^2)^{\frac{1}{20}}(W^{-10}H^5)^{\frac{1}{10}}(U^{-4}W^{-2}K^2H)^{\frac{1}{2}} \\
&\quad = (qT)^{\frac{2}{5}}M^{\frac{1}{10}}HK \ll Y^{1-\varepsilon}.
\end{aligned}$$

(f) $F > 2V^{-4}M^2, F \le 2W^{-10}H^5, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}
&U^2V^2W^2F \\
&\quad \ll U^2V^2W^2 \min\{V^{-4}qT, V^{-12}qTM^2, W^{-10}H^5, U^{-4}W^{-2}qT, \\
&\qquad\qquad U^{-12}W^{-6}qTK^2H\} \\
&\quad \le U^2V^2W^2(V^{-4}qT)^{\frac{7}{20}}(V^{-12}qTM^2)^{\frac{1}{20}}(W^{-10}H^5)^{\frac{1}{10}}(U^{-4}W^{-2}qT)^{\frac{1}{2}} \\
&\quad = (qT)^{\frac{9}{10}}M^{\frac{1}{10}}H^{\frac{1}{2}} \ll Y^{1-\varepsilon}.
\end{aligned}$$

(g) $F > 2V^{-4}M^2, 2W^{-10}H^5, F \le 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\quad \ll U^2V^2W^2\min\{V^{-4}qT, V^{-12}qTM^2, W^{-10}qT, W^{-30}qTH^5, \\ &\qquad\qquad U^{-4}W^{-2}K^2H\} \\ &\quad \le U^2V^2W^2(V^{-4}qT)^{\frac{9}{20}}(V^{-12}qTM^2)^{\frac{1}{60}}(W^{-30}qTH^5)^{\frac{1}{30}}(U^{-4}W^{-2}K^2H)^{\frac{1}{2}} \\ &\quad = (qT)^{\frac{1}{2}}M^{\frac{1}{30}}H^{\frac{2}{3}}K \ll Y^{1-\varepsilon}. \end{aligned}$$

(h) $F > 2V^{-4}M^2, 2W^{-10}H^5, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}, V^{-12}M^2, W^{-10}, W^{-30}H^5, U^{-4}W^{-2}, \\ &\qquad U^{-12}W^{-6}K^2H\}qT \\ &\le U^2V^2W^2(V^{-4})^{\frac{7}{20}}(V^{-12}M^2)^{\frac{1}{20}}(W^{-10})^{\frac{1}{10}}(U^{-4}W^{-2})^{\frac{1}{2}}qT \\ &= qTM^{\frac{1}{10}} \ll Y^{1-\varepsilon}, \end{aligned}$$

since $M \ll Y^{\frac{5}{9}}$ (the latter follows from $Y^{\frac{16}{135}} \ll H$ and $M^2H \ll Y^{\frac{94}{81}}$).

III. Now, assume condition 3). We apply the mean value estimate and Halász method to $M^2(s,\chi)$, $H^4(s,\chi)$ and $K^3(s,\chi)$ to get

$$I \ll U^2V^2W^2Y^{-1}F\log^c N,$$

where

$$\begin{aligned} F = \min\{&V^{-4}(M^2+qT), V^{-4}M^2 + V^{-12}qTM^2, W^{-8}(H^4+qT), \\ &W^{-8}H^4 + W^{-24}qTH^4, U^{-6}(K^3+qT), U^{-6}K^3 + U^{-18}qTK^3\}. \end{aligned}$$

Consider the following cases:

(a) $F \le 2V^{-4}M^2, 2W^{-8}H^4, 2U^{-6}K^3$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-8}H^4, U^{-6}K^3\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(W^{-8}H^4)^{\frac{1}{6}}(U^{-6}K^3)^{\frac{1}{3}} \\ &= W^{\frac{2}{3}}H^{\frac{2}{3}}MK \ll Y\log^{-7E} N. \end{aligned}$$

(b) $F \le 2V^{-4}M^2, 2W^{-8}H^4, F > 2U^{-6}K^3$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-8}H^4, U^{-6}qT, U^{-18}qTK^3\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(W^{-8}H^4)^{\frac{1}{4}}(U^{-6}qT)^{\frac{5}{24}}(U^{-18}qTK^3)^{\frac{1}{24}} \\ &= (qT)^{\frac{1}{4}}MHK^{\frac{1}{8}} \ll Y^{1-\varepsilon}. \end{aligned}$$

(c) $F \le 2V^{-4}M^2, F > 2W^{-8}H^4, F \le 2U^{-6}K^3$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-8}qT, W^{-24}qTH^4, U^{-6}K^3\}\\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac12}(W^{-8}qT)^{\frac18}(W^{-24}qTH^4)^{\frac1{24}}(U^{-6}K^3)^{\frac13}\\ &= (qT)^{\frac16}MKH^{\frac16} \ll Y^{1-\varepsilon}.\end{aligned}$$

(d) $F \le 2V^{-4}M^2, F > 2W^{-8}H^4, 2U^{-6}K^3$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-8}qT, W^{-24}qTH^4,\\ &\qquad U^{-6}qT, U^{-18}qTK^3\}\\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac12}(W^{-8}qT)^{\frac18}(W^{-24}qTH^4)^{\frac1{24}}(U^{-6}qT)^{\frac13}\\ &= (qT)^{\frac12}MH^{\frac16} \ll Y^{1-\varepsilon}.\end{aligned}$$

(e) $F > 2V^{-4}M^2, F \le 2W^{-8}H^4, 2U^{-6}K^3$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}qT, V^{-12}qTM^2, W^{-8}H^4, U^{-6}K^3\}\\ &\le U^2V^2W^2(V^{-4}qT)^{\frac38}(V^{-12}qTM^2)^{\frac1{24}}(W^{-8}H^4)^{\frac14}(U^{-6}K^3)^{\frac13}\\ &= (qT)^{\frac5{12}}M^{\frac1{12}}HK \ll Y^{1-\varepsilon}.\end{aligned}$$

(f) $F > 2V^{-4}M^2, F \le 2W^{-8}H^4, F > 2U^{-6}K^3$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}qT, V^{-12}qTM^2, W^{-8}H^4,\\ &\qquad U^{-6}qT, U^{-18}qTK^3\}\\ &\le U^2V^2W^2(V^{-4}qT)^{\frac12}(W^{-8}H^4)^{\frac14}(U^{-6}qT)^{\frac5{24}}(U^{-18}qTK^3)^{\frac1{24}}\\ &= (qT)^{\frac34}HK^{\frac18} \ll Y^{1-\varepsilon}.\end{aligned}$$

(g) $F > 2V^{-4}M^2, 2W^{-8}H^4, F \le 2U^{-6}K^3$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}qT, V^{-12}qTM^2, W^{-8}qT,\\ &\qquad W^{-24}qTH^4, U^{-6}K^3\}\\ &\le U^2V^2W^2(V^{-4}qT)^{\frac12}(W^{-8}qT)^{\frac18}(W^{-24}qTH^4)^{\frac1{24}}(U^{-6}K^3)^{\frac13}\\ &= (qT)^{\frac23}H^{\frac16}K \ll Y^{1-\varepsilon}.\end{aligned}$$

(h) $F > 2V^{-4}M^2, 2W^{-8}H^4, 2U^{-6}K^3$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}, V^{-12}M^2, W^{-8}, W^{-24}H^4, U^{-6},\\ &\qquad U^{-18}K^3\}qT\\ &\le U^2V^2W^2(V^{-4})^{\frac12}(W^{-8})^{\frac14}(U^{-6})^{\frac5{24}}(U^{-18}K^3)^{\frac1{24}}qT\\ &= qTK^{\frac18} \ll Y^{1-\varepsilon},\end{aligned}$$

since $Y^{\frac59} \ll MH$ (the latter follows from $Y^{\frac{17}{90}} \ll H$ and $Y^{\frac{85}{198}} \ll M$).

IV. Lastly, we assume condition 4). We apply the mean value estimate and Halász method to $M^2(s,\chi)$, $H^4(s,\chi)$ and $K^2(s,\chi)H(s,\chi)$ to get

$$I \ll U^2V^2W^2Y^{-1}F\log^c N,$$

where

$$\begin{aligned}F = \min\{&V^{-4}(M^2+qT), V^{-4}M^2+V^{-12}qTM^2, W^{-8}(H^4+qT),\\ &W^{-8}H^4+W^{-24}qTH^4, U^{-4}W^{-2}(K^2H+qT),\\ &U^{-4}W^{-2}K^2H+U^{-12}W^{-6}qTK^2H\}.\end{aligned}$$

Consider the following cases:

(a) $F \le 2V^{-4}M^2, 2W^{-8}H^4, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-8}H^4, U^{-4}W^{-2}K^2H\}\\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac12}(U^{-4}W^{-2}K^2H)^{\frac12}\\ &= WH^{\frac12}MK \ll Y\log^{-7E}N.\end{aligned}$$

(b) $F \le 2V^{-4}M^2, 2W^{-8}H^4, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}&U^2V^2W^2F\\ &\quad\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-8}H^4, U^{-4}W^{-2}qT, U^{-12}W^{-6}qTK^2H\}\\ &\quad\le U^2V^2W^2(V^{-4}M^2)^{\frac12}(W^{-8}H^4)^{\frac18}(U^{-4}W^{-2}qT)^{\frac{5}{16}}(U^{-12}W^{-6}qTK^2H)^{\frac{1}{16}}\\ &\quad= (qT)^{\frac38}MH^{\frac{9}{16}}K^{\frac18} \ll Y^{1-\varepsilon}.\end{aligned}$$

(c) $F \le 2V^{-4}M^2, F > 2W^{-8}H^4, F \le 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-8}qT, W^{-24}qTH^4, U^{-4}W^{-2}K^2H\}\\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac12}(U^{-4}W^{-2}K^2H)^{\frac12}\\ &= WH^{\frac12}MK \ll Y\log^{-7E}N.\end{aligned}$$

(d) $F \le 2V^{-4}M^2, F > 2W^{-8}H^4, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-8}qT, W^{-24}qTH^4, U^{-4}W^{-2}qT,\\ &\qquad U^{-12}W^{-6}qTK^2H\}\\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac12}(W^{-24}qTH^4)^{\frac{1}{24}}(U^{-4}W^{-2}qT)^{\frac{7}{16}}\\ &\quad\times(U^{-12}W^{-6}qTK^2H)^{\frac{1}{48}}\\ &= (qT)^{\frac12}MH^{\frac{3}{16}}K^{\frac{1}{24}} \ll Y^{1-\varepsilon}.\end{aligned}$$

(e) $F > 2V^{-4}M^2, F \le 2W^{-8}H^4, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}qT, V^{-12}qTM^2, W^{-8}H^4, U^{-4}W^{-2}K^2H\} \\ &\le U^2V^2W^2(V^{-4}qT)^{\frac{5}{16}}(V^{-12}qTM^2)^{\frac{1}{16}}(W^{-8}H^4)^{\frac{1}{8}}(U^{-4}W^{-2}K^2H)^{\frac{1}{2}} \\ &= (qT)^{\frac{3}{8}}M^{\frac{1}{8}}HK \ll Y^{1-\varepsilon}. \end{aligned}$$

(f) $F > 2V^{-4}M^2, F \le 2W^{-8}H^4, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}qT, V^{-12}qTM^2, W^{-8}H^4, U^{-4}W^{-2}qT, \\ &\qquad U^{-12}W^{-6}qTK^2H\} \\ &\le U^2V^2W^2(V^{-4}qT)^{\frac{5}{16}}(V^{-12}qTM^2)^{\frac{1}{16}}(W^{-8}H^4)^{\frac{1}{8}}(U^{-4}W^{-2}qT)^{\frac{1}{2}} \\ &= (qT)^{\frac{7}{8}}M^{\frac{1}{8}}H^{\frac{1}{2}} \ll Y^{1-\varepsilon}. \end{aligned}$$

(g) $F > 2V^{-4}M^2, 2W^{-8}H^4, F \le 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}qT, V^{-12}qTM^2, W^{-8}qT, W^{-24}qTH^4, \\ &\qquad U^{-4}W^{-2}K^2H\} \\ &\le U^2V^2W^2(V^{-4}qT)^{\frac{7}{16}}(V^{-12}qTM^2)^{\frac{1}{48}}(W^{-24}qTH^4)^{\frac{1}{24}}(U^{-4}W^{-2}K^2H)^{\frac{1}{2}} \\ &= (qT)^{\frac{1}{2}}M^{\frac{1}{24}}H^{\frac{2}{3}}K \ll Y^{1-\varepsilon}. \end{aligned}$$

(h) $F > 2V^{-4}M^2, 2W^{-8}H^4, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}, V^{-12}M^2, W^{-8}, W^{-24}H^4, U^{-4}W^{-2}, \\ &\qquad U^{-12}W^{-6}K^2H\}qT \\ &\le U^2V^2W^2(V^{-4})^{\frac{5}{16}}(V^{-12}M^2)^{\frac{1}{16}}(W^{-8})^{\frac{1}{8}}(U^{-4}W^{-2})^{\frac{1}{2}}qT \\ &= qTM^{\frac{1}{8}} \ll Y^{1-\varepsilon}. \end{aligned}$$

Combining the above, we obtain (9). Hence, Lemma 9 follows.

LEMMA 10. *Under the assumption of Lemma* 7, *suppose that* M *and* H *lie in one of the following regions*:

(i) $Y^{\frac{11}{27}} \ll M \ll Y^{\frac{34}{81}}, \quad M^{-\frac{23}{8}}Y^{\frac{17}{12}} \ll H \ll M^{-\frac{1}{4}}Y^{\frac{25}{72}}$;

(ii) $Y^{\frac{34}{81}} \ll M \ll Y^{\frac{49}{114}}, \quad M^{-\frac{29}{10}}Y^{\frac{17}{12}} \ll H \ll M^{-\frac{1}{4}}Y^{\frac{25}{72}}$;

(iii) $Y^{\frac{49}{114}} \ll M \ll Y^{\frac{4}{9}}, \quad M^{-\frac{29}{10}}Y^{\frac{17}{12}} \ll H \ll M^{\frac{1}{7}}Y^{\frac{4}{21}}$;

(iv) $Y^{\frac{4}{9}} \ll M \ll Y^{\frac{701}{1566}}, \quad M^{-\frac{29}{10}}Y^{\frac{17}{12}} \ll H \ll M^{-\frac{1}{5}}Y^{\frac{3}{10}}$;

(v) $Y^{\frac{4}{9}} \ll M \ll Y^{\frac{701}{1566}}, \quad M^{-\frac{6}{5}}Y^{\frac{34}{45}} \ll H \ll M^{\frac{1}{7}}Y^{\frac{4}{21}}$;

(vi) $Y^{\frac{701}{1566}} \ll M \ll Y^{\frac{41}{90}}, \quad Y^{\frac{16}{135}} \ll H \ll M^{-\frac{1}{5}} Y^{\frac{3}{10}};$

(vii) $Y^{\frac{701}{1566}} \ll M \ll Y^{\frac{41}{90}}, \quad M^{-\frac{6}{5}} Y^{\frac{34}{45}} \ll H \ll M^{\frac{1}{7}} Y^{\frac{4}{21}};$

(viii) $Y^{\frac{41}{90}} \ll M \ll Y^{\frac{17}{36}}, \quad Y^{\frac{16}{135}} \ll H \ll M^{\frac{1}{7}} Y^{\frac{4}{21}};$

(ix) $Y^{\frac{17}{36}} \ll M \ll Y^{\frac{16}{33}}, \quad Y^{\frac{16}{135}} \ll H \ll M^{-1} Y^{\frac{46}{63}};$

(x) $Y^{\frac{16}{33}} \ll M \ll Y^{\frac{307}{630}}, \quad Y^{\frac{16}{135}} \ll H \ll M^{-\frac{58}{9}} Y^{\frac{89}{27}};$

(xi) $Y^{\frac{16}{33}} \ll M \ll Y^{\frac{307}{630}}, \quad Y^{\frac{17}{99}} \ll H \ll M^{-1} Y^{\frac{46}{63}};$

(xii) $Y^{\frac{307}{630}} \ll M \ll Y^{\frac{281}{570}}, \quad Y^{\frac{16}{135}} \ll H \ll M^{-\frac{58}{9}} Y^{\frac{89}{27}};$

(xiii) $Y^{\frac{307}{630}} \ll M \ll Y^{\frac{281}{570}}, \quad Y^{\frac{17}{99}} \ll H \ll M^{-6} Y^{\frac{19}{6}};$

(xiv) $Y^{\frac{281}{570}} \ll M \ll Y^{\frac{429}{870}}, \quad Y^{\frac{16}{135}} \ll H \ll M^{-\frac{58}{9}} Y^{\frac{89}{27}};$

(xv) $Y^{\frac{429}{870}} \ll M \ll Y^{\frac{1}{2}}, \quad Y^{\frac{17}{99}} \ll H \ll M^{-1} Y^{\frac{40}{57}}.$

Then (8) *holds for* $T_0 \le T \le Y$.

Proof. In the regions:

$$Y^{\frac{119}{261}} \ll M \ll Y^{\frac{17}{36}}, \quad M^{-\frac{12}{11}} Y^{\frac{68}{99}} \ll H \ll M^{\frac{1}{19}} Y^{\frac{35}{171}};$$
$$Y^{\frac{17}{36}} \ll M \ll Y^{\frac{281}{570}}, \quad Y^{\frac{17}{99}} \ll H \ll M^{-1} Y^{\frac{40}{57}};$$
$$Y^{\frac{429}{870}} \ll M \ll Y^{\frac{1}{2}}, \quad Y^{\frac{17}{99}} \ll H \ll M^{-1} Y^{\frac{40}{57}},$$

we apply Lemma 9 with condition 1).

In the regions:

$$Y^{\frac{34}{81}} \ll M \ll Y^{\frac{701}{1566}}, \quad M^{-\frac{29}{10}} Y^{\frac{17}{12}} \ll H \ll M^{-\frac{1}{5}} Y^{\frac{3}{10}};$$
$$Y^{\frac{701}{1566}} \ll M \ll Y^{\frac{119}{261}}, \quad Y^{\frac{16}{135}} \ll H \ll M^{-\frac{1}{5}} Y^{\frac{3}{10}};$$
$$Y^{\frac{119}{261}} \ll M \ll Y^{\frac{17}{36}}, \quad Y^{\frac{16}{135}} \ll H \ll M^{-\frac{12}{11}} Y^{\frac{68}{99}};$$
$$Y^{\frac{17}{36}} \ll M \ll Y^{\frac{16}{33}}, \quad Y^{\frac{16}{135}} \ll H \ll Y^{\frac{17}{99}};$$
$$Y^{\frac{16}{33}} \ll M \ll Y^{\frac{429}{870}}, \quad Y^{\frac{16}{135}} \ll H \ll M^{-\frac{58}{9}} Y^{\frac{89}{27}},$$

we apply Lemma 9 with condition 2).

In the regions:

$$Y^{\frac{49}{114}} \ll M \ll Y^{\frac{41}{90}}, \quad M^{-\frac{6}{5}} Y^{\frac{34}{45}} \ll H \ll M^{\frac{1}{7}} Y^{\frac{4}{21}};$$
$$Y^{\frac{41}{90}} \ll M \ll Y^{\frac{119}{261}}, \quad M^{-\frac{1}{5}} Y^{\frac{3}{10}} \ll H \ll M^{\frac{1}{7}} Y^{\frac{4}{21}};$$
$$Y^{\frac{119}{261}} \ll M \ll Y^{\frac{17}{36}}, \quad M^{\frac{1}{19}} Y^{\frac{35}{171}} \ll H \ll M^{\frac{1}{7}} Y^{\frac{4}{21}};$$
$$Y^{\frac{17}{36}} \ll M \ll Y^{\frac{307}{630}}, \quad M^{-1} Y^{\frac{40}{57}} \ll H \ll M^{-1} Y^{\frac{46}{63}};$$
$$Y^{\frac{307}{630}} \ll M \ll Y^{\frac{281}{570}}, \quad M^{-1} Y^{\frac{40}{57}} \ll H \ll M^{-6} Y^{\frac{19}{6}},$$

we apply Lemma 9 with condition 3).

In the regions:

$$Y^{\frac{11}{27}} \ll M \ll Y^{\frac{34}{81}}, \quad M^{-\frac{23}{8}} Y^{\frac{17}{12}} \ll H \ll M^{-\frac{1}{4}} Y^{\frac{25}{72}};$$
$$Y^{\frac{34}{81}} \ll M \ll Y^{\frac{49}{114}}, \quad M^{-\frac{1}{5}} Y^{\frac{3}{10}} \ll H \ll M^{-\frac{1}{4}} Y^{\frac{25}{72}};$$
$$Y^{\frac{49}{114}} \ll M \ll Y^{\frac{4}{9}}, \quad M^{-\frac{1}{5}} Y^{\frac{3}{10}} \ll H \ll M^{-\frac{6}{5}} Y^{\frac{34}{45}},$$

we apply Lemma 9 with condition 4).

Putting together the above regions, we get Lemma 10.

5. Mean value estimate (III)

LEMMA 11. *Under the assumption of Lemma* 7, *suppose that* M *and* H *lie in one of the following regions*:

(i) $Y^{\frac{34}{99}} \ll M \ll Y^{\frac{1649}{4752}}, \quad M^{-2} Y^{\frac{85}{99}} \ll H \ll M^{-\frac{58}{49}} Y^{\frac{85}{147}};$

(ii) $Y^{\frac{1649}{4752}} \ll M \ll Y^{\frac{1823}{4725}}, \quad M^{-\frac{70}{59}} Y^{\frac{34}{59}} \ll H \ll M^{-\frac{58}{49}} Y^{\frac{85}{147}};$

(iii) $Y^{\frac{1823}{4725}} \ll M \ll Y^{\frac{3041}{7830}}, \quad Y^{\frac{16}{135}} \ll H \ll M^{-\frac{58}{49}} Y^{\frac{85}{147}};$

(iv) $Y^{\frac{11}{27}} \ll M \ll Y^{\frac{49}{114}}, \quad M^{-\frac{29}{19}} Y^{\frac{89}{114}} \ll H \ll M^{-\frac{35}{23}} Y^{\frac{18}{23}}.$

Then (8) *holds for* $T_0 \le T \le Y$.

Proof. First we show that (9) holds for $T = 1/\eta = Y/(qQ)$, providing M and H satisfy the following conditions:

$$MH \ll Y^{\frac{113}{198}}, \quad M^{\frac{35}{23}} H \ll Y^{\frac{18}{23}}, \quad Y^{\frac{85}{99}} \ll M^2 H,$$
$$H^5/M \ll Y^{\frac{11}{18}}, \quad Y^{\frac{34}{59}} \ll M^{\frac{70}{59}} H, \quad Y^{\frac{16}{135}} \ll H.$$

We apply the mean value estimate and Halász method to $M^2(s,\chi) \times H(s,\chi)$, $H^6(s,\chi)$ and $K^2(s,\chi)$ to get

$$I \ll U^2 V^2 W^2 Y^{-1} F \log^c N,$$

where

$$F = \min\{V^{-4}W^{-2}(M^2H + qT), V^{-4}W^{-2}M^2H + V^{-12}W^{-6}qTM^2H, W^{-12}(H^6 + qT), W^{-12}H^6 + W^{-36}qTH^6, U^{-4}(K^2 + qT), U^{-4}K^2 + U^{-12}qTK^2\}.$$

Consider the following cases:

(a) $F \le 2V^{-4}W^{-2}M^2H, 2W^{-12}H^6, 2U^{-4}K^2$. Then

$$U^2V^2W^2F \ll U^2V^2W^2 \min\{V^{-4}W^{-2}M^2H, W^{-12}H^6,\ U^{-4}K^2\}$$
$$\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(U^{-4}K^2)^{\frac{1}{2}}$$
$$= WH^{\frac{1}{2}}MK \ll Y \log^{-7E} N.$$

(b) $F \le 2V^{-4}W^{-2}M^2H, 2W^{-12}H^6, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\quad \ll U^2V^2W^2 \min\{V^{-4}W^{-2}M^2H, W^{-12}H^6, U^{-4}qT, U^{-12}qTK^2\}\\
&\quad \le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(W^{-12}H^6)^{\frac{1}{12}}(U^{-4}qT)^{\frac{3}{8}}(U^{-12}qTK^2)^{\frac{1}{24}}\\
&\quad = (qT)^{\frac{5}{12}}MHK^{\frac{1}{12}} \ll Y^{1-\varepsilon}.
\end{aligned}$$

(c) $F \le 2V^{-4}W^{-2}M^2H, F > 2W^{-12}H^6, F \le 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-4}W^{-2}M^2H, W^{-12}qT, W^{-36}qTH^6, U^{-4}K^2\}\\
&\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(U^{-4}K^2)^{\frac{1}{2}}\\
&= WH^{\frac{1}{2}}MK \ll Y \log^{-7E} N.
\end{aligned}$$

(d) $F \le 2V^{-4}W^{-2}M^2H, F > 2W^{-12}H^6, 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\quad \ll U^2V^2W^2 \min\{V^{-4}W^{-2}M^2H, W^{-12}qT, W^{-36}qTH^6, U^{-4}qT,\\
&\qquad\qquad U^{-12}qTK^2\}\\
&\quad \le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(W^{-36}qTH^6)^{\frac{1}{36}}(U^{-4}qT)^{\frac{11}{24}}(U^{-12}qTK^2)^{\frac{1}{72}}\\
&\quad = (qT)^{\frac{1}{2}}MH^{\frac{2}{3}}K^{\frac{1}{36}} \ll Y^{1-\varepsilon}.
\end{aligned}$$

(e) $F > 2V^{-4}W^{-2}M^2H, F \le 2W^{-12}H^6, 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\quad \ll U^2V^2W^2 \min\{V^{-4}W^{-2}qT, V^{-12}W^{-6}qTM^2H, W^{-12}H^6, U^{-4}K^2\}\\
&\quad \le U^2V^2W^2(V^{-4}W^{-2}qT)^{\frac{3}{8}}(V^{-12}W^{-6}qTM^2H)^{\frac{1}{24}}(W^{-12}H^6)^{\frac{1}{12}}(U^{-4}K^2)^{\frac{1}{2}}\\
&\quad = (qT)^{\frac{5}{12}}M^{\frac{1}{12}}H^{\frac{13}{24}}K \ll Y^{1-\varepsilon}.
\end{aligned}$$

(f) $F > 2V^{-4}W^{-2}M^2H, F \le 2W^{-12}H^6, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\quad \ll U^2V^2W^2 \min\{V^{-4}W^{-2}qT, V^{-12}W^{-6}qTM^2H, W^{-12}H^6,\\
&\qquad\qquad U^{-4}qT, U^{-12}qTK^2\}\\
&\quad \le U^2V^2W^2(V^{-4}W^{-2}qT)^{\frac{1}{2}}(W^{-12}H^6)^{\frac{1}{12}}(U^{-4}qT)^{\frac{3}{8}}(U^{-12}qTK^2)^{\frac{1}{24}}\\
&\quad = (qT)^{\frac{11}{12}}H^{\frac{1}{2}}K^{\frac{1}{12}} \ll Y^{1-\varepsilon}.
\end{aligned}$$

(g) $F > 2V^{-4}W^{-2}M^2H, 2W^{-12}H^6, F \le 2U^{-4}K^2$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}W^{-2}qT, V^{-12}W^{-6}qTM^2H, W^{-12}qT,\\ &\qquad W^{-36}qTH^6, U^{-4}K^2\}\\ &\le U^2V^2W^2(V^{-4}W^{-2}qT)^{\frac{11}{24}}(V^{-12}W^{-6}qTM^2H)^{\frac{1}{72}}\\ &\quad\times(W^{-36}qTH^6)^{\frac{1}{36}}(U^{-4}K^2)^{\frac12}\\ &= (qT)^{\frac12}M^{\frac{1}{36}}H^{\frac{13}{72}}K \ll Y^{1-\varepsilon}.\end{aligned}$$

(h) $F > 2V^{-4}W^{-2}M^2H, 2W^{-12}H^6, 2U^{-4}K^2$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}W^{-2}, V^{-12}W^{-6}M^2H, W^{-12},\\ &\qquad W^{-36}H^6, U^{-4}, U^{-12}K^2\}qT\\ &\le U^2V^2W^2(V^{-4}W^{-2})^{\frac12}(W^{-12})^{\frac{1}{12}}(U^{-4})^{\frac38}(U^{-12}K^2)^{\frac{1}{24}}qT\\ &= qTK^{\frac{1}{12}} \ll Y^{1-\varepsilon},\end{aligned}$$

since $Y^{\frac13} \ll MH$ (the latter follows from $Y^{\frac{85}{99}} \ll M^2H$ and $Y^{\frac{16}{135}} \ll H$).

In every region, our conditions are satisfied. So the proof of Lemma 11 is complete.

LEMMA 12. *Under the assumption of Lemma* 7, *suppose that M and H lie in one of the following regions*:

(i) $Y^{\frac{3397}{7830}} \ll M \ll Y^{\frac49}, \quad M^{-\frac{35}{12}}Y^{\frac{17}{12}} \ll H \ll M^{-\frac{29}{10}}Y^{\frac{17}{12}}$;

(ii) $Y^{\frac{211}{432}} \ll M \ll Y^{\frac{281}{570}}, \quad M^{-\frac{58}{9}}Y^{\frac{89}{27}} \ll H \ll M^{-\frac{70}{11}}Y^{\frac{36}{11}}$.

Then (8) *holds for* $T_0 \le T \le Y$.

Proof. First we show that (9) holds for $T = 1/\eta = Y/(qQ)$, providing that M and H satisfy the following conditions:

$$M^2H \ll Y^{\frac{113}{99}}, \quad M^{\frac{70}{11}}H \ll Y^{\frac{36}{11}}, \quad Y^{\frac{85}{198}} \ll M,$$
$$M^{\frac16}H \ll Y^{\frac{29}{108}}, \quad Y^{\frac{17}{12}} \ll M^{\frac{35}{12}}H, \quad M \ll Y^{\frac12}.$$

We apply the mean value estimate and Halász method to $M^2(s,\chi)$, $H^6(s,\chi)$ and $K^2(s,\chi)H(s,\chi)$ to get

$$I \ll U^2V^2W^2Y^{-1}F\log^c N,$$

where

$$\begin{aligned}F = \min\{&V^{-4}(M^2+qT), V^{-4}M^2+V^{-12}qTM^2, W^{-12}(H^6+qT),\\ &W^{-12}H^6+W^{-36}qTH^6, U^{-4}W^{-2}(K^2H+qT),\\ &U^{-4}W^{-2}K^2H+U^{-12}W^{-6}qTK^2H\}.\end{aligned}$$

Consider the following cases:

(a) $F \le 2V^{-4}M^2, 2W^{-12}H^6, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-12}H^6,\ U^{-4}W^{-2}K^2H\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(U^{-4}W^{-2}K^2H)^{\frac{1}{2}} \\ &= WH^{\frac{1}{2}}MK \ll Y\log^{-7E}N. \end{aligned}$$

(b) $F \le 2V^{-4}M^2, 2W^{-12}H^6, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-12}H^6, U^{-4}W^{-2}qT, U^{-12}W^{-6}qTK^2H\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(W^{-12}H^6)^{\frac{1}{12}}(U^{-4}W^{-2}qT)^{\frac{3}{8}}(U^{-12}W^{-6}qTK^2H)^{\frac{1}{24}} \\ &= (qT)^{\frac{5}{12}}MH^{\frac{13}{24}}K^{\frac{1}{12}} \ll Y^{1-\varepsilon}. \end{aligned}$$

(c) $F \le 2V^{-4}M^2, F > 2W^{-12}H^6, F \le 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-12}qT, W^{-36}qTH^6, \\ &\qquad U^{-4}W^{-2}K^2H\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(U^{-4}W^{-2}K^2H)^{\frac{1}{2}} \\ &= WH^{\frac{1}{2}}MK \ll Y\log^{-7E}N. \end{aligned}$$

(d) $F \le 2V^{-4}M^2, F > 2W^{-12}H^6, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-12}qT, W^{-36}qTH^6, U^{-4}W^{-2}qT, \\ &\qquad U^{-12}W^{-6}qTK^2H\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(W^{-36}qTH^6)^{\frac{1}{36}}(U^{-4}W^{-2}qT)^{\frac{11}{24}} \\ &\quad \times (U^{-12}W^{-6}qTK^2H)^{\frac{1}{72}} \\ &= (qT)^{\frac{1}{2}}MH^{\frac{13}{72}}K^{\frac{1}{36}} \ll Y^{1-\varepsilon}. \end{aligned}$$

(e) $F > 2V^{-4}M^2, F \le 2W^{-12}H^6, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\ll U^2V^2W^2\min\{V^{-4}qT, V^{-12}qTM^2, W^{-12}H^6, U^{-4}W^{-2}K^2H\} \\ &\le U^2V^2W^2(V^{-4}qT)^{\frac{3}{8}}(V^{-12}qTM^2)^{\frac{1}{24}}(W^{-12}H^6)^{\frac{1}{12}}(U^{-4}W^{-2}K^2H)^{\frac{1}{2}} \\ &= (qT)^{\frac{5}{12}}M^{\frac{1}{12}}HK \ll Y^{1-\varepsilon}. \end{aligned}$$

(f) $F > 2V^{-4}M^2, F \le 2W^{-12}H^6, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}qT, V^{-12}qTM^2, W^{-12}H^6, U^{-4}W^{-2}qT,\\ &\qquad U^{-12}W^{-6}qTK^2H\}\\ &\le U^2V^2W^2(V^{-4}qT)^{\frac{3}{8}}(V^{-12}qTM^2)^{\frac{1}{24}}(W^{-12}H^6)^{\frac{1}{12}}\\ &\quad\times (U^{-4}W^{-2}qT)^{\frac{1}{2}}\\ &= (qT)^{\frac{11}{12}}M^{\frac{1}{12}}H^{\frac{1}{2}} \ll Y^{1-\varepsilon}. \end{aligned}$$

(g) $F > 2V^{-4}M^2, 2W^{-12}H^6, F \le 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} &U^2V^2W^2F\\ &\quad\ll U^2V^2W^2\min\{V^{-4}qT, V^{-12}qTM^2, W^{-12}qT, W^{-36}qTH^6,\\ &\qquad\qquad U^{-4}W^{-2}K^2H\}\\ &\quad\le U^2V^2W^2(V^{-4}qT)^{\frac{11}{24}}(V^{-12}qTM^2)^{\frac{1}{72}}(W^{-36}qTH^6)^{\frac{1}{36}}(U^{-4}W^{-2}K^2H)^{\frac{1}{2}}\\ &\quad= (qT)^{\frac{1}{2}}M^{\frac{1}{36}}H^{\frac{2}{3}}K \ll Y^{1-\varepsilon}. \end{aligned}$$

(h) $F > 2V^{-4}M^2, 2W^{-12}H^6, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}, V^{-12}M^2, W^{-12}, W^{-36}H^6, U^{-4}W^{-2},\\ &\qquad U^{-12}W^{-6}K^2H\}qT\\ &\le U^2V^2W^2(V^{-4})^{\frac{3}{8}}(V^{-12}M^2)^{\frac{1}{24}}(W^{-12})^{\frac{1}{12}}(U^{-4}W^{-2})^{\frac{1}{2}}qT\\ &= qTM^{\frac{1}{12}} \ll Y^{1-\varepsilon}. \end{aligned}$$

In every region, our conditions are satisfied, so the proof of Lemma 12 is complete.

LEMMA 13. *Assume that* $PQRK = Y$, $q \le Q$, χ *is a character* $\bmod q$, $P(s,\chi)$, $Q(s,\chi)$, $R(s,\chi)$ *and* $K(s,\chi)$ *are Dirichlet polynomials and* $G(s,\chi) = P(s,\chi)Q(s,\chi)R(s,\chi)K(s,\chi)$. *Let* $\eta = qQ/Y$, $b = 1 + 1/\log N$, $T_0 = \log^{\frac{E}{\delta^2}} Y$. *Assume further that for* $T_0 \le |t| \le 2Y$, $P(b+it,\chi)Q(b+it,\chi) \ll \log^{-\frac{E}{\delta^2}} Y$ *and* $R(b+it,\chi) \ll \log^{-\frac{E}{\delta^2}} Y$. *Moreover, assume that*

$$Y^{\frac{16}{135}} \ll R \ll Q$$

and that P *and* Q *lie in one of the following regions*:

(i) $\quad Y^{\frac{41}{180}} \ll P \ll Y^{\frac{8}{33}}, \qquad P^{-1}Y^{\frac{41}{90}} \ll Q \ll P;$

(ii) $\quad Y^{\frac{8}{33}} \ll P \ll Y^{\frac{37}{150}}, \qquad P^{-1}Y^{\frac{41}{90}} \ll Q \ll P^{-1}Y^{\frac{16}{33}};$

(iii) $\quad Y^{\frac{37}{150}} \ll P \ll Y^{\frac{31}{99}}, \qquad P^{-1}Y^{\frac{701}{1566}} \ll Q \ll P^{-1}Y^{\frac{16}{33}};$

(iv) $Y^{\frac{31}{99}} \ll P \ll Y^{\frac{859}{2610}}, \quad P^{-1}Y^{\frac{701}{1566}} \ll Q \ll P^{-\frac{58}{67}}Y^{\frac{89}{201}};$

(v) $Y^{\frac{859}{2610}} \ll P \ll Y^{\frac{10}{27}}, \quad Y^{\frac{16}{135}} \ll Q \ll P^{-\frac{58}{67}}Y^{\frac{89}{201}}.$

Then (8) *holds for* $T_0 \le T \le Y$.

Proof. Let $m = pq$, $h = r$.

(a) On applying Lemma 10 with region (vi), we see that (8) holds under the conditions

$$P^{-1}Y^{\frac{701}{1566}} \ll Q \ll P^{-1}Y^{\frac{41}{90}}, \quad Y^{\frac{16}{135}} \ll R \ll Q \ll (PQ)^{-\frac{1}{5}}Y^{\frac{3}{10}},$$

which can be written as

$$P^{-1}Y^{\frac{701}{1566}} \ll Q \ll P^{-1}Y^{\frac{41}{90}}, \quad Y^{\frac{16}{135}} \ll Q \ll P^{-\frac{1}{6}}Y^{\frac{1}{4}}, \quad Y^{\frac{16}{135}} \ll R \ll Q.$$

In the regions:

$$Y^{\frac{37}{150}} \ll P \ll Y^{\frac{859}{2610}}, \quad P^{-1}Y^{\frac{701}{1566}} \ll Q \ll P^{-1}Y^{\frac{41}{90}};$$
$$Y^{\frac{859}{2610}} \ll P \ll Y^{\frac{91}{270}}, \quad Y^{\frac{16}{135}} \ll Q \ll P^{-1}Y^{\frac{41}{90}},$$

the above conditions on P and Q are satisfied.

(b) On applying Lemma 10 with region (viii), we see that (8) holds under the conditions

$$P^{-1}Y^{\frac{41}{90}} \ll Q \ll P^{-1}Y^{\frac{17}{36}}, \quad Y^{\frac{16}{135}} \ll R \ll Q \ll (PQ)^{\frac{1}{7}}Y^{\frac{4}{21}},$$

which can be written as

$$P^{-1}Y^{\frac{41}{90}} \ll Q \ll P^{-1}Y^{\frac{17}{36}}, \quad Y^{\frac{16}{135}} \ll Q \ll P^{\frac{1}{6}}Y^{\frac{2}{9}}, \quad Y^{\frac{16}{135}} \ll R \ll Q.$$

In the regions:

$$Y^{\frac{41}{180}} \ll P \ll Y^{\frac{17}{72}}, \quad P^{-1}Y^{\frac{41}{90}} \ll Q \ll P;$$
$$Y^{\frac{17}{72}} \ll P \ll Y^{\frac{91}{270}}, \quad P^{-1}Y^{\frac{41}{90}} \ll Q \ll P^{-1}Y^{\frac{17}{36}};$$
$$Y^{\frac{91}{270}} \ll P \ll Y^{\frac{191}{540}}, \quad Y^{\frac{16}{135}} \ll Q \ll P^{-1}Y^{\frac{17}{36}},$$

the above conditions on P and Q are satisfied.

(c) On applying Lemma 10 with region (ix), we see that (8) holds under the conditions

$$P^{-1}Y^{\frac{17}{36}} \ll Q \ll P^{-1}Y^{\frac{16}{33}}, \quad Y^{\frac{16}{135}} \ll R \ll Q \ll (PQ)^{-1}Y^{\frac{46}{63}},$$

which can be written as

$$P^{-1}Y^{\frac{17}{36}} \ll Q \ll P^{-1}Y^{\frac{16}{33}}, \quad Y^{\frac{16}{135}} \ll Q \ll P^{-\frac{1}{2}}Y^{\frac{23}{63}}, \quad Y^{\frac{16}{135}} \ll R \ll Q.$$

In the regions:

$$Y^{\frac{17}{72}} \ll P \ll Y^{\frac{8}{33}}, \quad P^{-1}Y^{\frac{17}{36}} \ll Q \ll P;$$
$$Y^{\frac{8}{33}} \ll P \ll Y^{\frac{191}{540}}, \quad P^{-1}Y^{\frac{17}{36}} \ll Q \ll P^{-1}Y^{\frac{16}{33}};$$
$$Y^{\frac{191}{540}} \ll P \ll Y^{\frac{544}{1485}}, \quad Y^{\frac{16}{135}} \ll Q \ll P^{-1}Y^{\frac{16}{33}},$$

the above conditions on P and Q are satisfied.

(d) On applying Lemma 10 with regions (x), (xii) and (xiv), we see that (8) holds under the conditions

$$P^{-1}Y^{\frac{16}{33}} \ll Q \ll P^{-1}Y^{\frac{429}{870}}, \quad Y^{\frac{16}{135}} \ll R \ll Q \ll (PQ)^{-\frac{58}{9}}Y^{\frac{89}{27}},$$

which can be written as

$$P^{-1}Y^{\frac{16}{33}} \ll Q \ll P^{-1}Y^{\frac{429}{870}}, \quad Y^{\frac{16}{135}} \ll Q \ll P^{-\frac{58}{67}}Y^{\frac{89}{201}}, \quad Y^{\frac{16}{135}} \ll R \ll Q.$$

In the regions:

$$Y^{\frac{31}{99}} \ll P \ll Y^{\frac{544}{1485}}, \quad P^{-1}Y^{\frac{16}{33}} \ll Q \ll P^{-\frac{58}{67}}Y^{\frac{89}{201}};$$
$$Y^{\frac{544}{1485}} \ll P \ll Y^{\frac{10}{27}}, \quad Y^{\frac{16}{135}} \ll Q \ll P^{-\frac{58}{67}}Y^{\frac{89}{201}},$$

the above conditions on P and Q are satisfied.

Putting together the above regions, we get Lemma 13.

LEMMA 14. *Assume that* $PQRL = Y$, $q \le Q$, χ *is a character* $\bmod q$, $P(s,\chi)$, $Q(s,\chi)$ *and* $R(s,\chi)$ *are Dirichlet polynomials*, *and*

$$F(s,\chi) = P(s,\chi)Q(s,\chi)R(s,\chi)\sum_{l\sim L}\frac{\chi(l)}{l^s}.$$

Let $\eta = qQ/Y$, $b = 1 + 1/\log N$, $T_1 = \sqrt{L/q}$. *Moreover assume that*

$$Y^{\frac{16}{135}} \ll R \ll Q$$

and that P *and* Q *lie in one of the following regions*:

(i) $\quad Y^{\frac{16}{135}} \ll P \ll Y^{\frac{89}{462}}, \quad Y^{\frac{16}{135}} \ll Q \ll P;$

(ii) $\quad Y^{\frac{89}{462}} \ll P \ll Y^{\frac{17}{54}}, \quad Y^{\frac{16}{135}} \ll Q \ll P^{-\frac{29}{48}}Y^{\frac{89}{288}}.$

Then for $T_1 \le T \le Y$, *we have*

$$\min^2\left(\eta, \frac{1}{T}\right) \sum_{\chi \,(\mathrm{mod}\, q)} \int_T^{2T} |F(b+it,\chi)|^2\, dt \ll \eta^2 Y^{-\varepsilon}. \tag{10}$$

P r o o f. Let $m = pq$ and $n = r$. An application of Lemma 6 yields that (10) holds under the following condition:

(a) $\quad M \ll Y^{\frac{3041}{7830}}, \quad N \ll M^{-1}Y^{\frac{19}{36}}.$

By the van der Corput method, it can be shown that for $T_1 \le |t| \le Y$,

$$\sum_{l\sim L}\frac{\chi(l)}{l^{b+it}} \ll \sum_{a=1}^{q}\frac{1}{q}\left|\sum_{l_1\sim L/q}\frac{1}{(l_1+a/q)^{b+it}}\right| \ll Y^{-\delta_0},$$

where δ_0 is a small positive constant.

Using a similar discussion to that for Lemmas 7 and 8 with regions (i)–(vi) and (viii)–(xii), we can see that (10) holds under the conditions:

(b) $Y^{\frac{17}{63}} \ll M \ll Y^{\frac{221}{693}}, \quad M^{-1}Y^{\frac{19}{36}} \ll N \ll M^{-\frac{1}{8}}Y^{\frac{7}{24}};$

(c) $Y^{\frac{221}{693}} \ll M \ll Y^{\frac{109}{330}}, \quad M^{-1}Y^{\frac{19}{36}} \ll N \ll M^{-1}Y^{\frac{113}{198}};$

(d) $Y^{\frac{109}{330}} \ll M \ll Y^{\frac{151}{450}}, \quad M^{-1}Y^{\frac{19}{36}} \ll N \ll M^{-6}Y^{\frac{20}{9}};$

(e) $Y^{\frac{151}{450}} \ll M \ll Y^{\frac{25}{72}}, \quad M^{-1}Y^{\frac{19}{36}} \ll N \ll M^{\frac{1}{4}}Y^{\frac{1}{8}};$

(f) $Y^{\frac{25}{72}} \ll M \ll Y^{\frac{10}{27}}, \quad M^{-1}Y^{\frac{19}{36}} \ll N \ll M^{-\frac{23}{15}}Y^{\frac{7}{9}};$

(g) $Y^{\frac{10}{27}} \ll M \ll Y^{\frac{617}{1620}}, \quad M^{-1}Y^{\frac{19}{36}} \ll N \ll M^{-1}Y^{\frac{47}{81}};$

(h) $Y^{\frac{617}{1620}} \ll M \ll Y^{\frac{3041}{7830}}, \quad M^{-1}Y^{\frac{19}{36}} \ll N \ll M^{-\frac{29}{19}}Y^{\frac{89}{114}};$

(i) $Y^{\frac{3041}{7830}} \ll M \ll Y^{\frac{3397}{7830}}, \quad Y^{\frac{16}{135}} \ll N \ll M^{-\frac{29}{19}}Y^{\frac{89}{114}}.$

In the regions:

$$Y^{\frac{16}{135}} \ll P \ll Y^{\frac{17}{126}}, \quad Y^{\frac{16}{135}} \ll Q \ll P;$$
$$Y^{\frac{17}{126}} \ll P \ll Y^{\frac{143}{945}}, \quad Y^{\frac{16}{135}} \ll Q \ll P^{-1}Y^{\frac{17}{63}},$$

condition (a) is satisfied.

In the regions:

$$Y^{\frac{17}{126}} \ll P \ll Y^{\frac{143}{945}}, \quad P^{-1}Y^{\frac{17}{63}} \ll Q \ll P;$$
$$Y^{\frac{143}{945}} \ll P \ll Y^{\frac{221}{1386}}, \quad Y^{\frac{16}{135}} \ll Q \ll P;$$
$$Y^{\frac{221}{1386}} \ll P \ll Y^{\frac{2083}{10395}}, \quad Y^{\frac{16}{135}} \ll Q \ll P^{-1}Y^{\frac{221}{693}},$$

either condition (a) or (b) is satisfied.

In the regions:

$$Y^{\frac{221}{1386}} \ll P \ll Y^{\frac{109}{660}}, \quad P^{-1}Y^{\frac{221}{693}} \ll Q \ll P;$$
$$Y^{\frac{109}{660}} \ll P \ll Y^{\frac{2083}{10395}}, \quad P^{-1}Y^{\frac{221}{693}} \ll Q \ll P^{-1}Y^{\frac{109}{330}};$$
$$Y^{\frac{2083}{10395}} \ll P \ll Y^{\frac{629}{2970}}, \quad Y^{\frac{16}{135}} \ll Q \ll P^{-1}Y^{\frac{109}{330}},$$

either condition (a) or (c) is satisfied.

In the regions:

$$Y^{\frac{109}{660}} \ll P \ll Y^{\frac{151}{900}}, \quad P^{-1}Y^{\frac{109}{330}} \ll Q \ll P;$$
$$Y^{\frac{151}{900}} \ll P \ll Y^{\frac{629}{2970}}, \quad P^{-1}Y^{\frac{109}{330}} \ll Q \ll P^{-1}Y^{\frac{151}{450}};$$
$$Y^{\frac{629}{2970}} \ll P \ll Y^{\frac{293}{1350}}, \quad Y^{\frac{16}{135}} \ll Q \ll P^{-1}Y^{\frac{151}{450}},$$

either condition (a) or (d) is satisfied.

In the regions:

$$Y^{\frac{151}{900}} \ll P \ll Y^{\frac{25}{144}}, \qquad P^{-1}Y^{\frac{151}{450}} \ll Q \ll P;$$
$$Y^{\frac{25}{144}} \ll P \ll Y^{\frac{293}{1350}}, \qquad P^{-1}Y^{\frac{151}{450}} \ll Q \ll P^{-1}Y^{\frac{25}{72}};$$
$$Y^{\frac{293}{1350}} \ll P \ll Y^{\frac{247}{1080}}, \qquad Y^{\frac{16}{135}} \ll Q \ll P^{-1}Y^{\frac{25}{72}},$$

either condition (a) or (e) is satisfied.

In the regions:

$$Y^{\frac{25}{144}} \ll P \ll Y^{\frac{5}{27}}, \qquad P^{-1}Y^{\frac{25}{72}} \ll Q \ll P;$$
$$Y^{\frac{5}{27}} \ll P \ll Y^{\frac{247}{1080}}, \qquad P^{-1}Y^{\frac{25}{72}} \ll Q \ll P^{-1}Y^{\frac{10}{27}};$$
$$Y^{\frac{247}{1080}} \ll P \ll Y^{\frac{34}{135}}, \qquad Y^{\frac{16}{135}} \ll Q \ll P^{-1}Y^{\frac{10}{27}},$$

either condition (a) or (f) is satisfied.

In the regions:

$$Y^{\frac{5}{27}} \ll P \ll Y^{\frac{617}{3240}}, \qquad P^{-1}Y^{\frac{10}{27}} \ll Q \ll P;$$
$$Y^{\frac{617}{3240}} \ll P \ll Y^{\frac{34}{135}}, \qquad P^{-1}Y^{\frac{10}{27}} \ll Q \ll P^{-1}Y^{\frac{617}{1620}};$$
$$Y^{\frac{34}{135}} \ll P \ll Y^{\frac{85}{324}}, \qquad Y^{\frac{16}{135}} \ll Q \ll P^{-1}Y^{\frac{617}{1620}},$$

either condition (a) or (g) is satisfied.

In the regions:

$$Y^{\frac{617}{3240}} \ll P \ll Y^{\frac{89}{462}}, \qquad P^{-1}Y^{\frac{617}{1620}} \ll Q \ll P;$$
$$Y^{\frac{89}{462}} \ll P \ll Y^{\frac{85}{324}}, \qquad P^{-1}Y^{\frac{617}{1620}} \ll Q \ll P^{-\frac{29}{48}}Y^{\frac{89}{288}};$$
$$Y^{\frac{85}{324}} \ll P \ll Y^{\frac{17}{54}}, \qquad Y^{\frac{16}{135}} \ll Q \ll P^{-\frac{29}{48}}Y^{\frac{89}{288}},$$

either condition (a) or (h) or (i) is satisfied.

Putting together the above regions, we get Lemma 14.

6. The remainder term in the sieve method

LEMMA 15. *Suppose that* $MH \ll Y^{1-6\varepsilon}/Q$, $MHL = Y$, $a(m) = O(1)$, $b(h) = O(1)$, $q \le Q$, χ *is a character* $\bmod q$, $M(s,\chi)$ *and* $H(s,\chi)$ *are Dirichlet polynomials*, *and*

$$F(s,\chi) = M(s,\chi)H(s,\chi)\sum_{l\sim L}\frac{\chi(l)}{l^s}.$$

Let $\eta = qQ/Y$, $b = 1 + 1/\log N$ *and* $T_1 = \sqrt{L/q}$. *Assume further that for* $T_1 \le T \le Y$,

$$\min^2\left(\eta, \frac{1}{T}\right) \sum_{\chi \,(\mathrm{mod}\, q)} \int_T^{2T} |F(b+it,\chi)|^2\, dt \ll \eta^2 Y^{-\varepsilon}. \tag{11}$$

Then for $N < n \le N + A$, except for $O(A\log^{-B} N)$ values, we have

$$\sum_{\substack{m\sim M,\, h\sim H\\ (m,n)=(h,n)=1}} a(m)b(h)\Bigg(\sum_{\substack{mhl+p=n\\ N-Y<p\le N\\ mhl\le 2Y}} 1 - \frac{1}{\varphi(mh)}\cdot\frac{Y}{\log N}\Bigg) = O(Y\log^{-B} N).$$

Proof. Set

$$\begin{aligned}\Sigma_1 &= \sum_{\substack{m\sim M,\, h\sim H\\ (m,n)=(h,n)=1}} a(m)b(h)\sum_{\substack{mhl+p=n\\ N-Y<p\le N\\ mhl\le 2Y}} 1\\ &= \int_{\frac{1}{Q}}^{1+\frac{1}{Q}} \sum_{\substack{mhl\le 2Y\\ m\sim M,\, h\sim H\\ (m,n)=(h,n)=1}} a(m)b(h)e(\theta mhl)\sum_{N-Y<p\le N} e(p\theta)e(-n\theta)\,d\theta\end{aligned}$$

and

$$\begin{aligned}S(\theta,n) &= \sum_{\substack{mhl\le 2Y\\ m\sim M,\, h\sim H\\ (m,n)=(h,n)=1}} a(m)b(h)e(\theta mhl) \qquad (12)\\ &= \sum_{\substack{mhl\le 2Y\\ m\sim M,\, h\sim H}} a(m)b(h)e(\theta mhl)\sum_{\substack{d_1|n\\ d_1|m}}\mu(d_1)\sum_{\substack{d_2|n\\ d_2|m}}\mu(d_2)\\ &= \sum_{d_1|n,\, d_2|n}\mu(d_1)\mu(d_2)\sum_{\substack{mhl\le 2Y\\ d_1|m,\, d_2|h}} a(m)b(h)e(\theta mhl)\\ &= \sum_{\substack{d_1|n,\, d_2|n\\ d_1,\, d_2\le \log^{12B} N}} + \sum_{d_1|n,\, d_2|n}^{\ *} = S_1(\theta,n) + S_2(\theta,n),\end{aligned}$$

where "$*$" denotes that one of d_1, d_2 is larger than $\log^{12B} N$. Let

$$S_3(\theta) = \sum_{N-Y<p\le N} e(p\theta).$$

Then

$$\Big|\int_{E_2} S_2(\theta,n)S_3(\theta)e(-n\theta)\,d\theta\Big|^2 \le \int_0^1 |S_2(\theta,n)|^2\,d\theta\int_0^1 |S_3(\theta)|^2\,d\theta$$

$$\ll d^2(n)Y\int_0^1 {\sum_{d_1|n,\,d_2|n}}^{*} \Big|\sum_{\substack{mhl\le 2Y\\ d_1|m,\,d_2|h}} a(m)b(h)e(\theta mhl)\Big|^2\,d\theta$$

$$\ll d^2(n)Y {\sum_{d_1|n,\,d_2|n}}^{*} \sum_{\substack{r\le 2Y\\ d_1|r,\,d_2|r}} d_3^2(r).$$

By Lemma 1, the last expression is

$$\ll d_3^6(n)Y\log^8 N {\sum_{d_1|n,\,d_2|n}}^{*}\left(\frac{Y}{[d_1,d_2]}+Y^{\varepsilon}\right) \ll d_3^8(n)Y^2\log^{-11B} N.$$

By Lemma 1, except for $O(A\log^{-B} N)$ values, $d_3(n)\le \log^{B+2} N$. So,

$$\int_{E_2} S_2(\theta,n)S_3(\theta)e(-n\theta)\,d\theta = O(Y\log^{-B} N). \tag{13}$$

Next,

$$\Big|\int_{E_2} S_1(\theta,n)S_3(\theta)e(-n\theta)\,d\theta\Big| \tag{14}$$
$$\le \sum_{d_1\le \log^{12B} N}\ \sum_{d_2\le \log^{12B} N}\Big|\int_{E_2}\sum_{\substack{mhl\le 2Y\\ d_1|m,\,d_2|h}} a(m)b(h)e(\theta mhl)S_3(\theta)e(-n\theta)\,d\theta\Big|.$$

Let

$$S_4(\theta)=\sum_{\substack{mhl\le 2Y\\ m\sim M,\,h\sim H}} a(m)b(h)e(\theta mhl).$$

In the following we shall estimate

$$\Sigma_2=\sum_{N<n\le N+A}\Big|\int_{E_2} S_4(\theta)S_3(\theta)e(-n\theta)\,d\theta\Big|^2. \tag{15}$$

We have

$$\Sigma_2=\sum_{N<n\le N+A}\int_{E_2} S_4(\xi)S_3(\xi)e(-n\xi)\,d\xi\int_{E_2}\overline{S_4(\alpha)S_3(\alpha)}e(n\alpha)\,d\alpha$$
$$\ll \int_{E_2}|S_4(\xi)S_3(\xi)|\,d\xi\left(\int_{E_2}|S_4(\alpha)S_3(\alpha)|\min\left(A,\frac{1}{\|\alpha-\xi\|}\right)d\alpha\right).$$

Now,

$$\int_{E_2}|S_4(\alpha)S_3(\alpha)|\min\left(A,\frac{1}{\|\alpha-\xi\|}\right)d\alpha$$

$$\le \Big(\int_{E_2} |S_3(\alpha)|^2\,d\alpha\Big)^{\frac12}\Big(\int_{E_2} |S_4(\alpha)|^2 \min^2\Big(A, \frac{1}{\|\alpha-\xi\|}\Big)\,d\alpha\Big)^{\frac12}$$

$$\ll Y^{\frac12} \sup_{\xi\in[0,1]} \Big(\int_{E_2} |S_4(\alpha)|^2 \min^2\Big(A, \frac{1}{\|\alpha-\xi\|}\Big)\,d\alpha\Big)^{\frac12}.$$

Consequently,

$$\varSigma_2 \ll Y^{\frac32}\log^4 N \sup_{\xi\in[0,1]} \Big(\int_{E_2} |S_4(\alpha)|^2 \min^2\Big(A, \frac{1}{\|\alpha-\xi\|}\Big)\,d\alpha\Big)^{\frac12}. \tag{16}$$

In order to prove

$$\varSigma_2 \ll AY^2 \log^{-52B} N,$$

we have to prove that

$$\int_{D(\xi)} |S_4(\alpha)|^2\,d\alpha \ll Y\log^{-120B} N \tag{17}$$

uniformly for $\xi\in[0,1]$, where

$$D(\xi) = \Big(\xi - \frac{\log^{128B} N}{A}, \xi + \frac{\log^{128B} N}{A}\Big) \cap E_2.$$

Let

$$J(a,q) = \Big(\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\Big),$$

$$\varOmega(a,q) = \begin{cases} J(a,q) - I(a,q) & \text{if } q \le \log^E N,\\ J(a,q) & \text{if } q > \log^E N.\end{cases}$$

Note that $1/(qQ) \ge 1/Q^2 = (4\log^{128B} N)/A$. By a simple argument (see Section 5 of [17]), $D(\xi)$ can be covered by at most two $\varOmega(a,q)$. The proof of (17) reduces to showing that

$$\max_{\substack{q\le Q\\ (a,q)=1}} \int_{\varOmega(a,q)} |S_4(\alpha)|^2\,d\alpha \ll Y\log^{-120B} N. \tag{18}$$

Let $\alpha = a/q + \beta$ with $\beta\in\varPhi(q)$, where

$$\varPhi(q) = \begin{cases} \Big\{\beta : \dfrac{\log^{2E} N}{qY} \le |\beta| \le \dfrac{1}{qQ}\Big\} & \text{if } q \le \log^E N,\\ \Big\{\beta : |\beta| \le \dfrac{1}{qQ}\Big\} & \text{if } q > \log^E N.\end{cases}$$

Let $\varGamma = \max(q^{2\varepsilon}, \log^{260B} N)$ and $\varGamma \ge q^{\varepsilon}\log^{130B} N$.

Assume that $d_1 = (m,q)$, $d_2 = (h,q)$, $q_1 = q/d_1$, $q_2 = q/d_2$, $M_1 = M/d_1$, $H_1 = H/d_2$, $d' = d_1d_2/(q, d_1d_2)$, $q' = q/(q, d_1d_2)$, and $d = (l, q')$. Then

$$(19)\quad S_4(\alpha) = \sum_{\substack{mhl\le 2Y\\ d_1,d_2,d\le \Gamma}} a(m)b(h)e(\alpha mhl) + {\sum_{mhl\le 2Y}}' a(m)b(h)e(\alpha mhl) = S_5(\alpha) + S_6(\alpha),$$

where "$'$" denotes that one of d_1, d_2, d is larger than Γ.

Let $\sum''$ be the part of $\sum'$ with $d_1 > \Gamma$. Then

$$\int_{\Omega(a,q)} \Big| {\sum_{mhl\le 2Y}}'' a(m)b(h)e(\alpha mhl)\Big|^2 d\alpha \le \int_0^1 \Big| {\sum_{mhl\le 2Y}}'' a(m)b(h)e(\alpha mhl)\Big|^2 d\alpha$$
$$\ll \sum_{\substack{k\mid q\\ k>\Gamma}} \sum_{\substack{r\le 2Y\\ k\mid r}} d_3^2(r) \ll Y\log^8 N \sum_{\substack{k\mid q\\ k>\Gamma}} \frac{d_3^2(k)}{k} \ll Y\log^8 N\cdot \frac{q^{\varepsilon}}{\Gamma} \ll Y\log^{-120B} N.$$

Hence,

$$(20)\qquad \int_{\Omega(a,q)} |S_6(\alpha)|^2\, d\alpha \ll Y\log^{-120B} N.$$

Next,

$$S_5(\alpha) = \sum_{\substack{d_1\mid q\\ d_1\le\Gamma}} \sum_{\substack{d_2\mid q\\ d_2\le\Gamma}} \sum_{\substack{m_1\sim M_1\\ (m_1,q_1)=1}} a(m_1d_1) \sum_{\substack{h_1\sim H_1\\ (h_1,q_2)=1}} b(h_1d_2) \times \sum_{\substack{m_1h_1l\le 2Y/(d_1d_2)\\ (l,q')\le\Gamma}} e\Big(\frac{ad'm_1h_1l}{q'}\Big) e(\beta d_1d_2m_1h_1l)$$
$$= \sum_{\substack{d_1\mid q\\ d_1\le\Gamma}} \sum_{\substack{d_2\mid q\\ d_2\le\Gamma}} \sum_{\substack{d\mid q'\\ d\le\Gamma}} \sum_{\substack{m_1\sim M_1\\ (m_1,q_1)=1}} a(m_1d_1) \sum_{\substack{h_1\sim H_1\\ (h_1,q_2)=1}} b(h_1d_2) \times \sum_{\substack{m_1h_1l_1\le 2Y/(dd_1d_2)\\ (l_1,q'')=1}} e\Big(\frac{ad'm_1h_1l_1}{q''}\Big) e(\beta dd_1d_2m_1h_1l_1),$$

where $q''=q'/d$ and $l_1=l/d$. We have $q' \mid q_1$ and $(m_1,q_1)=1$, so $(m_1,q')=1$;

$q' \mid q_2$ and $(h_1, q_2) = 1$, so $(h_1, q') = 1$; and $(ad', q') = 1$. Therefore

(21)
$$\int\limits_{\Omega(a,q)} |S_5(\alpha)|^2\, d\alpha \ll q^{\varepsilon} \sum_{\substack{d_1 \mid q \\ d_1 \le \Gamma}} \sum_{\substack{d_2 \mid q \\ d_2 \le \Gamma}} \sum_{\substack{d \mid q' \\ d \le \Gamma}} \int\limits_{\Phi(q)} \Bigg| \sum_{\substack{m_1 \sim M_1 \\ (m_1, q_1) = 1}} a(m_1 d_1) \sum_{\substack{h_1 \sim H_1 \\ (h_1, q_2) = 1}} b(h_1 d_2) \times \sum_{\substack{m_1 h_1 l_1 \le 2Y/(d d_1 d_2) \\ (l_1, q'') = 1}} e\bigg(\frac{ad' m_1 h_1 l_1}{q''}\bigg) e(\beta d d_1 d_2 m_1 h_1 l_1) \Bigg|^2 d\beta.$$

In the following we first assume that $d_1 = d_2 = d = 1$ and set

$$S_7(\beta) = \sum_{\substack{mhl \le 2Y \\ m \sim M,\, h \sim H \\ (mhl, q) = 1}} a(m) b(h) e\bigg(\frac{amhl}{q}\bigg) e(\beta mhl).$$

We proceed to prove

(22)
$$\int\limits_{\Phi(q)} |S_7(\beta)|^2\, d\beta \ll \frac{Y}{q \log^E N} + Y^{1-\frac{\varepsilon}{2}}.$$

We have

$$\begin{aligned} S_7(\beta) &= \frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a) \tau(\overline{\chi}) \sum_{\substack{mhl \le 2Y \\ m \sim M,\, h \sim H}} a(m)\chi(m) b(h)\chi(h)\chi(l) e(\beta mhl) \\ &= \frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a) \tau(\overline{\chi}) W(\chi, \beta) \\ &\quad + \frac{\mu(q)}{q} \sum_{\substack{m \sim M \\ (m,q)=1}} \frac{a(m)}{m} \sum_{\substack{h \sim H \\ (h,q)=1}} \frac{b(h)}{h} \sum_{r \le 2Y} e(\beta r) \\ &= S_8(\beta) + S_9(\beta), \end{aligned}$$

where

$$\begin{aligned} W(\chi, \beta) = &\sum_{\substack{mhl \le 2Y \\ m \sim M,\, h \sim H}} a(m)\chi(m) b(h)\chi(h)\chi(l) e(\beta mhl) \\ &- E_0 \frac{\varphi(q)}{q} \sum_{\substack{m \sim M \\ (m,q)=1}} \frac{a(m)}{m} \sum_{\substack{h \sim H \\ (h,q)=1}} \frac{b(h)}{h} \sum_{r \le 2Y} e(\beta r). \end{aligned}$$

When $q \le \log^E N$,

$$\int_{\Phi(q)} |S_9(\beta)|^2\,d\beta \ll \frac{1}{q^2}\int_{\Phi(q)} \Big|\sum_{r\le 2Y} e(\beta r)\Big|^2\,d\beta \ll \frac{1}{q^2}\int_{\log^{2E} N/(qY)}^{\infty} \frac{d\beta}{\beta^2} \ll \frac{Y}{q\log^E N}.$$

When $q > \log^E N$,

$$\int_{\Phi(q)} |S_9(\beta)|^2\,d\beta \ll \frac{1}{q^2}\int_0^1 \Big|\sum_{r\le 2Y} e(\beta r)\Big|^2\,d\beta \ll \frac{Y}{q^2} \ll \frac{Y}{q\log^E N}.$$

So, we always have

$$\int_{\Phi(q)} |S_9(\beta)|^2\,d\beta \ll \frac{Y}{q\log^E N}. \tag{23}$$

By Lemma 2,

$$\int_{\Phi(q)} |S_8(\beta)|^2\,d\beta \ll \log N \sum_{\chi\,(\mathrm{mod}\ q)} \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} |W(\chi,\beta)|^2\,d\beta, \tag{24}$$

and

$$W(\chi,\beta) = \sum_{r\le 2Y} (\lambda(r)\chi(r) - E_0 I)e(\beta r),$$

where

$$\lambda(r) = \sum_{\substack{mhl=r\\ m\sim M,\, h\sim H}} a(m)b(h), \qquad I = \frac{\varphi(q)}{q}\sum_{\substack{m\sim M\\ (m,q)=1}} \frac{a(m)}{m} \sum_{\substack{h\sim H\\ (h,q)=1}} \frac{b(h)}{h}.$$

By Lemma 3,

$$\begin{aligned}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} &|W(\chi,\beta)|^2\,d\beta\\ &\ll \frac{1}{(qQ)^2}\int_{-\infty}^{\infty} \Big|\sum_{\substack{x<r\le x+qQ/2\\ 1\le r\le 2Y}} (\lambda(r)\chi(r) - E_0 I)\Big|^2\,dx\\ &= \frac{1}{(qQ)^2}\int_{(qQ)^2}^{2Y} \Big|\sum_{x<r\le x+qQ/2} (\lambda(r)\chi(r) - E_0 I)\Big|^2\,dx + O(Q^4\log^4 N).\end{aligned}$$

The contribution of the term $O(Q^4\log^4 N)$ to (24) is $O(Y^{1-\varepsilon})$.

By Theorem 2 on p. 34 of [15],

$$\sum_{\chi\,(\mathrm{mod}\,q)}\Big|\sum_{x<r\le x+qQ/2}\lambda(r)\chi(r)\Big|^2\ll qQ\sum_{x<r\le x+qQ/2}|\lambda(r)|^2\ll(qQ)^2\log^8N$$

and

$$\sum_{\chi\,(\mathrm{mod}\,q)}\Big|\sum_{x<r\le x+qQ/2}E_0I\Big|^2\ll(qQ)^2.$$

Hence,

$$\frac{1}{(qQ)^2}\int\limits_{(qQ)^2}^{Y^{1-2\varepsilon}}\sum_{\chi\,(\mathrm{mod}\,q)}\Big|\sum_{x<r\le x+qQ/2}(\lambda(r)\chi(r)-E_0I)\Big|^2\,dx\ll Y^{1-\varepsilon}.$$

By the discussion in Lemma 6 of [19] (see also (24) of [17]), we have

$$\begin{aligned}\frac{1}{(qQ)^2}&\int\limits_{Y^{1-2\varepsilon}}^{2Y}\sum_{\chi\,(\mathrm{mod}\,q)}\Big|\sum_{x<r\le x+qQ/2}(\lambda(r)\chi(r)-E_0I)\Big|^2\,dx\\&\ll\frac{\log N}{(qQ)^2}\sup_{Y^{1-2\varepsilon}\le V\le 2Y}\ \sup_{qQ/(20V)\le\eta\le 20qQ/V}\\&\qquad\int\limits_{V}^{3V}\sum_{\chi\,(\mathrm{mod}\,q)}\Big|\sum_{x<r\le x+\eta x}(\lambda(r)\chi(r)-E_0I)\Big|^2\,dx.\end{aligned}$$

We only estimate for $V=Y$ and $\eta=qQ/Y$ the quantity

$$\Sigma_3=\frac{1}{(qQ)^2}\sum_{\chi\,(\mathrm{mod}\,q)}\int\limits_{Y}^{3Y}\Big|\sum_{x<r\le x+\eta x}(\lambda(r)\chi(r)-E_0I)\Big|^2\,dx.\tag{25}$$

The others can be dealt with in the same way.

Assume $MH\le QY^{-\varepsilon}/\sqrt{q}$. The trivial estimation yields

$$\begin{aligned}\sum_{x<r\le x+\eta x}&\lambda(r)\chi(r)\\&=\sum_{m\sim M}a(m)\chi(m)\sum_{h\sim H}b(h)\chi(h)\sum_{x/(mh)<l\le(x+\eta x)/(mh)}\chi(l)\\&=\sum_{m\sim M}a(m)\chi(m)\sum_{h\sim H}b(h)\chi(h)\Big(E_0\frac{\varphi(q)}{q}\cdot\frac{\eta x}{mh}+O(q)\Big)\\&=\eta xE_0I+O(qMH).\end{aligned}$$

Hence,

$$\Sigma_3\ll\frac{1}{(qQ)^2}qY(qMH)^2\ll Y^{1-\varepsilon}.$$

We can assume

$$MH > \frac{QY^{-\varepsilon}}{\sqrt{q}}. \tag{26}$$

Let $b = 1 + 1/\log N$, $MHL = Y$. Perron's formula yields

$$\sum_{x<r\le x+\eta x} \lambda(r)\chi(r) = \frac{1}{2\pi i}\int_{b-iY}^{b+iY} F(s,\chi)\frac{(1+\eta)^s-1}{s}x^s\,ds + O(Y^{\varepsilon}). \tag{27}$$

If $s = b+it$ and $|t| \le c_1L/q$, by Theorem 1 on p. 447 of [16], we have

$$\sum_{c_1L<l\le c_2L} \frac{\chi(l)}{l^s} = E_0\frac{\varphi(q)}{q}\cdot\frac{(c_2L)^{1-s}-(c_1L)^{1-s}}{1-s} + O\left(\frac{q}{L}\right).$$

Moreover,

$$\frac{(1+\eta)^s-1}{s}x^s = \eta x^s + O(|s|\eta^2 x), \qquad \frac{(1+\eta)^s-1}{s}x^s \ll \eta x.$$

Let $T_1 = \sqrt{L/q}$. Then

$$\begin{aligned}\frac{1}{2\pi i}&\int_{b-iT_1}^{b+iT_1} F(s,\chi)\frac{(1+\eta)^s-1}{s}x^s\,ds\\ &= \eta E_0\cdot\frac{\varphi(q)}{q}\cdot\frac{1}{2\pi i}\int_{b-iT_1}^{b+iT_1} M(s,\chi)H(s,\chi)\frac{(c_2L)^{1-s}-(c_1L)^{1-s}}{1-s}x^s\,ds\\ &\quad + O(S_1) + O(S_2),\end{aligned}$$

where

$$S_1 = \frac{q}{L}\eta Y\int_{-T_1}^{T_1} |M(b+it,\chi)H(b+it,\chi)|\,dt,$$

$$S_2 = \eta^2 Y\int_{-T_1}^{T_1} |M(b+it,\chi)H(b+it,\chi)|\,dt.$$

By Lemma 4, the contribution of S_1 to (25) is

$$\begin{aligned}&\ll \frac{1}{(qQ)^2}Y\left(\frac{q}{L}\cdot qQ\right)^2\sum_{\chi\,(\mathrm{mod}\,q)} T_1\int_{-T_1}^{T_1}|M(b+it,\chi)H(b+it,\chi)|^2\,dt\\ &\ll YT_1\left(\frac{q}{L}\right)^2\left(1+\frac{qT_1}{MH}\right)\log^3 N \ll Y^{1-\varepsilon}.\end{aligned}$$

The contribution of S_2 to (25) is

$$\ll \frac{1}{(qQ)^2} Y\left(\frac{qQ}{Y}\cdot qQ\right)^2 \sum_{\chi \,(\mathrm{mod}\, q)} T_1 \int_{-T_1}^{T_1} |M(b+it,\chi)H(b+it,\chi)|^2\, dt$$
$$\ll YT_1\left(\frac{qQ}{Y}\right)^2\left(1+\frac{qT_1}{MH}\right)\log^3 N \ll Y^{1-\varepsilon}.$$

By Perron's formula again,

$$\begin{aligned}\eta E_0\cdot\frac{\varphi(q)}{q}\cdot\frac{1}{2\pi i}\int_{b-iT_1}^{b+iT_1} & M(s,\chi)H(s,\chi)\frac{(c_2L)^{1-s}-(c_1L)^{1-s}}{1-s}x^s\, ds \\ &= \eta x E_0\frac{\varphi(q)}{q}\sum_{m\sim M}\frac{a(m)\chi(m)}{m}\sum_{h\sim H}\frac{b(h)\chi(h)}{h} \\ &\quad + O\left(\frac{\eta x E_0 Y^{\varepsilon}}{T_1}\right) + O\left(\frac{\eta x E_0 Y^{\varepsilon}}{MH}\right) \\ &= \eta x E_0 I + O\left(\frac{\eta x E_0 Y^{\varepsilon}}{T_1}\right) + O\left(\frac{\eta x E_0 Y^{\varepsilon}}{MH}\right).\end{aligned}$$

The contribution of the term $O(\eta x E_0 Y^{\varepsilon}/T_1)$ to (25) is

$$\ll \frac{1}{(qQ)^2} Y\left(\frac{qQY^{\varepsilon}}{T_1}\right)^2 \ll Y^{1-\varepsilon}.$$

The contribution of the term $O(\eta x E_0 Y^{\varepsilon}/(MH))$ to (25) is

$$\ll \frac{1}{(qQ)^2} Y\left(\frac{qQY^{\varepsilon}}{MH}\right)^2 \ll Y^{1-\varepsilon}.$$

Combining the above, we have

$$\begin{aligned}(28)\quad \Sigma_3 &\ll \frac{\log^2 N}{(qQ)^2} \\ &\times \max_{T_1\le|T|\le Y}\sum_{\chi\,(\mathrm{mod}\,q)}\int_Y^{3Y}\left|\int_{b+iT}^{b+2iT} F(s,\chi)\frac{(1+\eta)^s-1}{s}x^s\, ds\right|^2 dx + O(Y^{1-\varepsilon}).\end{aligned}$$

Let

$$\varrho(s) = \frac{(1+\eta)^s-1}{s}.$$

If $s=b+it$ and $|\mathrm{Im}(s)|\sim T$, then $\varrho(s)\ll \min(\eta,1/T)$. Thus

$$\int_Y^{3Y}\Big|\int_{b+iT}^{b+2iT}F(s,\chi)\varrho(s)x^s\,ds\Big|^2\,dx$$
$$=\int_Y^{3Y}dx\int_{b+iT}^{b+2iT}ds_1\int_{b+iT}^{b+2iT}F(s_1,\chi)\overline{F(s_2,\chi)}\varrho(s_1)\overline{\varrho(s_2)}x^{s_1+\bar s_2}\overline{ds_2}$$
$$\ll \min^2\left(\eta,\frac{1}{T}\right)\int_{b+iT}^{b+2iT}|ds_1|\int_{b+iT}^{b+2iT}|F(s_1,\chi)F(s_2,\chi)|\Big|\int_Y^{3Y}x^{s_1+\bar s_2}\,dx\Big||ds_2|$$
$$\ll \min^2\left(\eta,\frac{1}{T}\right)Y^3\int_{b+iT}^{b+2iT}|ds_1|\int_{b+iT}^{b+2iT}\frac{|F(s_1,\chi)|^2+|F(s_2,\chi)|^2}{|1+s_1+\bar s_2|}|ds_2|$$
$$\ll \min^2\left(\eta,\frac{1}{T}\right)Y^3\log N\int_T^{2T}|F(b+it,\chi)|^2\,dt.$$

Consequently

$$\Sigma_3\ll\frac{Y^3\log^3 N}{(qQ)^2}\times\max_{T_1\le T\le Y}\min^2\left(\eta,\frac{1}{T}\right)\sum_{\chi\,(\mathrm{mod}\,q)}\int_T^{2T}|F(b+it,\chi)|^2\,dt+O(Y^{1-\varepsilon}).\tag{29}$$

By the assumption in Lemma 15, $\Sigma_3\ll Y^{1-\frac{\varepsilon}{2}}$. Hence, (22) holds.

In the same way, we can prove

$$\int_{\Phi(q)}\Big|\sum_{\substack{m_1\sim M_1\\(m_1,q_1)=1}}a(m_1d_1)\sum_{\substack{h_1\sim H_1\\(h_1,q_2)=1}}b(h_1d_2)\sum_{\substack{m_1h_1l_1\le 2Y/(dd_1d_2)\\(l_1,q'')=1}}e\left(\frac{ad'm_1h_1l_1}{q''}\right)\times e(\beta dd_1d_2m_1h_1l_1)\Big|^2\,d\beta\ll\frac{q^{6\varepsilon}Y}{q''\log^{\frac{E}{2}}N}+Y^{1-\frac{\varepsilon}{2}}.$$

From $q''\ge q^{1-6\varepsilon}\log^{-780B}N$ and (21), it follows that

$$\int_{\Omega(a,q)}|S_5(\alpha)|^2\,d\alpha\ll Y\log^{-120B}N.$$

Hence, (18) holds. We have

$$\Sigma_2\ll AY^2\log^{-52B}N.$$

In the same way, it can be proved that

$$\sum_{N<n\le N+A}\Big|\int_{E_2}S_1(\theta,n)S_3(\theta)e(-n\theta)\,d\theta\Big|^2\ll AY^2\log^{-4B}N.$$

So, except for $O(A\log^{-B} N)$ values, we always have

$$\int_{E_2} S_1(\theta,n)S_3(\theta)e(-n\theta)\,d\theta = O(Y\log^{-B} N). \tag{30}$$

If $\theta = a/q+\beta \in E_1$, then

$$S(\theta,n) = \sum_{\substack{kl\le 2Y\\ k\sim K,\,(k,n)=1}} g(k)e(\theta kl),$$

where $K = MH$ and

$$g(k) = \sum_{\substack{mh=k\\ m\sim M,\,h\sim H}} a(m)b(h).$$

If $q\nmid b$, then

$$\sum_{l\le x} e\left(\frac{bl}{q}\right) = O(q).$$

From this and Abel's summation, it follows that

$$\begin{aligned}
\sum_{\substack{kl\le 2Y\\ k\sim K,\,(k,n)=1\\ q\nmid k}} g(k)e(\theta kl) &= \sum_{\substack{k\sim K\\ (k,n)=1,\,q\nmid k}} g(k)\sum_{l\le 2Y/k} e\left(\frac{akl}{q}+\beta kl\right)\\
&\ll \sum_{k\sim K} d(k)\log^{2E} N \ll Y^{1-2\varepsilon},\\
\sum_{\substack{kl\le 2Y\\ k\sim K,\,(k,n)=1\\ q\mid k}} g(k)e(\theta kl) &= \sum_{\substack{k\sim K\\ (k,n)=1,q\mid k}} g(k)\sum_{l\le 2Y/k} e(\beta kl),
\end{aligned}$$

$$\begin{aligned}
\sum_{l\le 2Y/k} e(\beta kl) &= \int_1^{2Y/k} e(\beta kt)\,d[t]\\
&= \int_1^{2Y/k} e(\beta kt)\,dt - \int_1^{2Y/k} e(\beta kt)\,d(\{t\})\\
&= \frac{1}{k}\int_k^{2Y} e(\beta u)\,du + O(\log^{2E} N)\\
&= \frac{1}{k}\sum_{r\le 2Y} e(\beta r) + O(\log^{2E} N),
\end{aligned}$$

hence

$$S(\theta,n)=\sum_{\substack{k\sim K\\(k,n)=1,q\mid k}}\frac{g(k)}{k}\sum_{r\le 2Y}e(\beta r)+O(Y^{1-2\varepsilon}). \tag{31}$$

By the same discussion as in Lemma 10 of [7], we have

$$\sum_{N-Y<p\le N}e(p\theta)=\frac{\mu(q)}{\varphi(q)\log N}\sum_{N-Y<s\le N}e(\beta s)+O(Y\exp(-c\sqrt{\log N})).$$

Set

$$\begin{aligned}\Sigma_4&=\int\limits_{E_1}S(\theta,n)S_3(\theta)e(-n\theta)\,d\theta\\&=\sum_{q\le\log^E N}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int\limits_{-\log^{2E}N/(qY)}^{\log^{2E}N/(qY)}S\Big(\frac{a}{q}+\beta,n\Big)S_3\Big(\frac{a}{q}+\beta\Big)\\&\quad\times e\Big(-\Big(\frac{a}{q}+\beta\Big)n\Big)\,d\beta\\&=\sum_{q\le\log^E N}\sum_{\substack{a=1\\(a,q)=1}}^{q}e\Big(-\frac{an}{q}\Big)\int\limits_{-\log^{2E}N/(qY)}^{\log^{2E}N/(qY)}\bigg\{\bigg(\sum_{\substack{k\sim K\\(k,n)=1,q\mid k}}\frac{g(k)}{k}\bigg)\sum_{r\le 2Y}e(\beta r)\\&\quad\times\frac{\mu(q)}{\varphi(q)\log N}\sum_{N-Y<s\le N}e(\beta s)e(-n\beta)\bigg\}d\beta+O\Big(\frac{Y}{\log^B N}\Big).\end{aligned}$$

Since

$$\int\limits_{-\log^{2E}N/(qY)}^{\log^{2E}N/(qY)}\sum_{r\le 2Y}e(\beta r)\sum_{N-Y<s\le N}e(\beta s)e(-n\beta)\,d\beta=Y+O\Big(\frac{qY}{\log^{2E}N}\Big),$$

we obtain

$$\begin{aligned}\Sigma_4&=\frac{Y}{\log N}\sum_{\substack{q\le\log^E N\\(q,n)=1}}\frac{\mu(q)}{\varphi(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q}e\Big(-\frac{an}{q}\Big)\sum_{\substack{k\sim K\\(k,n)=1,q\mid k}}\frac{g(k)}{k}+O\Big(\frac{Y}{\log^B N}\Big)\\&=\frac{Y}{\log N}\sum_{q\le\log^E N}\frac{\mu^2(q)}{\varphi(q)}\sum_{\substack{k\sim K\\(k,n)=1,q\mid k}}\frac{g(k)}{k}+O\Big(\frac{Y}{\log^B N}\Big).\end{aligned}$$

Set

$$\Omega=\sum_{q\le\log^E N}\frac{\mu^2(q)}{\varphi(q)}\sum_{\substack{k\sim K\\(k,n)=1,q\mid k}}\frac{g(k)}{k}=\sum_{\substack{k\sim K\\(k,n)=1}}\frac{g(k)}{k}\sum_{\substack{q\le\log^E N\\q\mid k}}\frac{\mu^2(q)}{\varphi(q)}.$$

We have

$$\sum_{\substack{k\sim K\\(k,n)=1}}\frac{g(k)}{k}\sum_{\substack{q>\log^E N\\ q|k}}\frac{\mu^2(q)}{\varphi(q)}\ll\frac{1}{\log^{E-1}N}\sum_{k\sim K}\frac{d^2(k)}{k}\ll\frac{1}{\log^{E-4}N}.$$

Consequently,

$$\begin{aligned}\Omega&=\sum_{\substack{k\sim K\\(k,n)=1}}\frac{g(k)}{k}\sum_{q|k}\frac{\mu^2(q)}{\varphi(q)}+O\left(\frac{1}{\log^B N}\right)\\&=\sum_{\substack{k\sim K\\(k,n)=1}}\frac{g(k)}{\varphi(k)}+O\left(\frac{1}{\log^B N}\right)\\&=\sum_{\substack{m\sim M,h\sim H\\(m,n)=(h,n)=1}}\frac{a(m)b(h)}{\varphi(mh)}+O\left(\frac{1}{\log^B N}\right).\end{aligned}$$

Thus

$$\Sigma_4=\frac{Y}{\log N}\sum_{\substack{m\sim M,h\sim H\\(m,n)=(h,n)=1}}\frac{a(m)b(h)}{\varphi(mh)}+O\left(\frac{Y}{\log^B N}\right),$$

and so

$$\Sigma_1=\frac{Y}{\log N}\sum_{\substack{m\sim M,h\sim H\\(m,n)=(h,n)=1}}\frac{a(m)b(h)}{\varphi(mh)}+O\left(\frac{Y}{\log^B N}\right).$$

The proof of Lemma 15 is complete.

LEMMA 16. *Suppose that* $a(p)=O(1)$, $b(q)=O(1)$, $c(r)=O(1)$,

$$Y^{\frac{16}{135}}\ll R\ll Q,$$

and that P *and* Q *lie in one of the following regions*:

(i) $Y^{\frac{16}{135}}\ll P\ll Y^{\frac{89}{462}}$, $\quad Y^{\frac{16}{135}}\ll Q\ll P$;

(ii) $Y^{\frac{89}{462}}\ll P\ll Y^{\frac{17}{54}}$, $\quad Y^{\frac{16}{135}}\ll Q\ll P^{-\frac{29}{48}}Y^{\frac{89}{288}}$.

Then for $N<n\le N+A$, *except for* $O(A\log^{-B}N)$ *values*, *we have*

$$\sum_{\substack{p\sim P\\(p,n)=1}}\sum_{\substack{q\sim Q\\(q,n)=1}}\sum_{\substack{r\sim R\\(r,n)=1}}a(p)b(q)c(r)\Bigg(\sum_{\substack{pqrl+p_1=n\\N-Y<p_1\le N\\pqrl\le 2Y}}1-\frac{1}{\varphi(pqr)}\cdot\frac{Y}{\log N}\Bigg)$$
$$=O(Y\log^{-B}N).$$

Proof. This follows from Lemmas 14 and 15.

LEMMA 17. *Suppose that* $M \ll Y^{\frac{19}{36}}$ *and* $a(m) = O(1)$. *Then for* $N < n \le N + A$, *except for* $O(A\log^{-B} N)$ *values, we have*

$$\sum_{\substack{m\sim M\\(m,n)=1}} a(m)\Bigg(\sum_{\substack{ml+p=n\\N-Y<p\le N\\ml\le 2Y}} 1 - \frac{1}{\varphi(m)}\cdot\frac{Y}{\log N}\Bigg) = O(Y\log^{-B} N).$$

Proof. This follows from Lemmas 6 and 15.

7. Asymptotic formula

LEMMA 18. *Suppose that* $QY^{\varepsilon} \ll HK \ll Y^{1-\delta}$, $MHK = Y$, $0 \le b(h) = O(1)$, *and* $0 \le g(k) = O(1)$. *If* h *has a prime factor* $< Y^{\delta}$, *then* $b(h) = 0$, *and similarly for* $g(k)$. *Assume that* $q \le Q$, χ *is a character* $\bmod q$, $H(s,\chi)$ *and* $K(s,\chi)$ *are Dirichlet polynomials*,

$$M(s,\chi) = \sum_{m\sim M}\frac{\Lambda(m)\chi(m)}{m^s}$$

and $G(s,\chi) = M(s,\chi)H(s,\chi)K(s,\chi)$. *Let* $\eta = qQ/Y$, $b = 1 + 1/\log N$ *and* $T_0 = \log^{\frac{E}{\delta^2}} Y$. *Assume further that for* $T_0 \le T \le Y$,

$$\min{}^2\left(\eta, \frac{1}{T}\right)\sum_{\chi\,(\mathrm{good})}\int_T^{2T}|G(b+it,\chi)|^2\,dt \ll \eta^2\log^{-6E} N. \tag{32}$$

Then for $N < n \le N + A$, *except for* $O(A\log^{-B} N)$ *values, we have*

$$\sum_{\substack{hkp+p_1=n\\N-Y<p_1\le N\\hkp\le 2Y\\h\sim H,\,k\sim K}} b(h)g(k) = \frac{C(n)}{\log N}\left(1 + O\left(\frac{1}{\log N}\right)\right)$$
$$\times\sum_{h\sim H,\,k\sim K} b(h)g(k)\sum_{Y/(hk)<p\le 2Y/(hk)} 1 + O(Y\log^{-B} N),$$

where

$$C(n) = \prod_{p\nmid n}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p\mid n}\left(1 + \frac{1}{p-1}\right).$$

Proof. Set

$$\Sigma_1 = \sum_{\substack{h\sim H\\ k\sim K}} b(h)g(k) \sum_{\substack{hkp+p_1=n\\ N-Y<p_1\le N\\ hkp\le 2Y}} 1$$

$$= \int\limits_{\frac{1}{Q}}^{1+\frac{1}{Q}} \sum_{\substack{hkp\le 2Y\\ h\sim H,\, k\sim K}} b(h)g(k)e(\theta hkp) \sum_{N-Y<p_1\le N} e(p_1\theta)e(-n\theta)\, d\theta.$$

In the following we often use the definitions and the argument of Lemma 15. Write

$$S_1(\theta) = \sum_{\substack{hkp\le 2Y\\ h\sim H,\, k\sim K}} b(h)g(k)e(\theta hkp), \qquad S_2(\theta) = \sum_{N-Y<p_1\le N} e(p_1\theta).$$

In order to prove

$$\sum_{N<n\le N+A} \Big| \int\limits_{E_2} S_1(\theta)S_2(\theta)e(-n\theta)\, d\theta \Big|^2 \ll AY^2 \log^{-52B} N, \tag{33}$$

it suffices to prove

$$\max_{\substack{q\le Q\\ (a,q)=1}} \int\limits_{\Omega(a,q)} |S_1(\alpha)|^2\, d\alpha \ll Y \log^{-120B} N. \tag{34}$$

Using the argument of Lemma 15, we can deal with the part with $(hkp,q) > \Gamma = \max(q^{2\varepsilon}, \log^{260B} N)$ in $S_1(\alpha)$. Since $(hkp,q) \le \Gamma$ and $b(h)g(k) \ne 0$ imply that $(hkp,q)=1$, we need to prove

$$\max_{\substack{q\le Q\\ (a,q)=1}} \int\limits_{\Omega(a,q)} |S_3(\alpha)|^2\, d\alpha \ll Y \log^{-120B} N, \tag{35}$$

where

$$S_3(\alpha) = \sum_{\substack{mhk\le 2Y\\ h\sim H,\, k\sim K\\ (mhk,q)=1}} \Lambda(m)b(h)g(k)e\Big(\frac{amhk}{q}\Big)e(\beta mhk).$$

We have

$$S_3(\alpha) = \frac{1}{\varphi(q)} \sum_{\chi\,(\mathrm{mod}\, q)} \chi(a)\tau(\overline{\chi})$$
$$\times \sum_{\substack{mhk\le 2Y\\ h\sim H,\, k\sim K}} \Lambda(m)\chi(m)b(h)\chi(h)g(k)\chi(k)e(\beta mhk)$$

$$= \frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a)\tau(\overline{\chi})W(\chi,\beta) + \frac{\mu(q)}{\varphi(q)} \sum_{\substack{h\sim H\\(h,q)=1}} \frac{b(h)}{h} \sum_{\substack{k\sim K\\(k,q)=1}} \frac{g(k)}{k} \sum_{r\le 2Y} e(\beta r),$$

where

$$W(\chi,\beta) = \sum_{\substack{mhk\le 2Y\\h\sim H,\, k\sim K}} \Lambda(m)\chi(m)b(h)\chi(h)g(k)\chi(k)e(\beta mhk) - E_0 \sum_{\substack{h\sim H\\(h,q)=1}} \frac{b(h)}{h} \sum_{\substack{k\sim K\\(k,q)=1}} \frac{g(k)}{k} \sum_{r\le 2Y} e(\beta r).$$

Now,

$$W(\chi,\beta) = \sum_{r\le 2Y} (\lambda(r)\chi(r) - E_0 I)e(\beta r),$$

where

$$\lambda(r) = \sum_{\substack{mhk=r\\h\sim H,\, k\sim K}} \Lambda(m)b(h)g(k), \qquad I = \sum_{\substack{h\sim H\\(h,q)=1}} \frac{b(h)}{h} \sum_{\substack{k\sim K\\(k,q)=1}} \frac{g(k)}{k}.$$

By the argument of Lemma 15, we need to prove

$$\frac{1}{\varphi^2(q)} \int_{\Phi(q)} \Big| \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a)\tau(\overline{\chi})W(\chi,\beta)\Big|^2 d\beta \ll Y \log^{-120B} N. \tag{36}$$

By the explanation in Section 2,

$$\frac{1}{\varphi^2(q)} \int_{\Phi(q)} \Big| \sum_{\chi \,(\mathrm{bad})} \chi(a)\tau(\overline{\chi})W(\chi,\beta)\Big|^2 d\beta \ll \frac{q}{\varphi^2(q)} \log^{\frac{64E}{\varepsilon}} N \sum_{\chi\,(\mathrm{bad})} \int_{-\frac12}^{\frac12} |W(\chi,\beta)|^2\, d\beta \ll \frac{Y}{q} \log^{\frac{129E}{\varepsilon}} N \ll \frac{Y}{\log^E N}.$$

Since

$$\frac{1}{\varphi^2(q)} \int_{\Phi(q)} \Big| \sum_{\chi \,(\mathrm{good})} \chi(a)\tau(\overline{\chi})W(\chi,\beta)\Big|^2 d\beta \ll \frac{q^2}{\varphi^2(q)} \sum_{\chi\,(\mathrm{good})} \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} |W(\chi,\beta)|^2\, d\beta,$$

it suffices to prove

$$\sum_{\chi\,(\text{good})}\int\limits_{-\frac{1}{qQ}}^{\frac{1}{qQ}}|W(\chi,\beta)|^2\,d\beta \ll \frac{Y}{\log^E N}.$$

By the explanation of Lemma 15, we only estimate for $\eta = qQ/Y$ the quantity

$$\Sigma_2 = \frac{1}{(qQ)^2}\sum_{\chi\,(\text{good})}\int\limits_{Y}^{3Y}\Big|\sum_{x<r\le x+\eta x}(\lambda(r)\chi(r)-E_0 I)\Big|^2\,dx. \tag{37}$$

Perron's formula yields

$$\sum_{x<r\le x+\eta x}\lambda(r)\chi(r) = \frac{1}{2\pi i}\int\limits_{b-iY}^{b+iY}G(s,\chi)\frac{(1+\eta)^s-1}{s}x^s\,ds + O(Y^{\varepsilon}).$$

If $s=b+it$, $|t|\le 3Y$ and χ is a good character $\bmod q$,

$$\sum_{m\sim M}\frac{\Lambda(m)\chi(m)}{m^s} = E_0\frac{(c_2M)^{1-s}-(c_1M)^{1-s}}{1-s} + O(\log^{-\frac{2E}{\delta^2}} Y)$$

(see (4)).

By the argument of Lemma 15, we have

$$\begin{aligned}\frac{1}{2\pi i}&\int\limits_{b-iT_0}^{b+iT_0}G(s,\chi)\frac{(1+\eta)^s-1}{s}x^s\,ds\\ &= \eta E_0\frac{1}{2\pi i}\int\limits_{b-iT_0}^{b+iT_0}H(s,\chi)K(s,\chi)\frac{(c_2M)^{1-s}-(c_1M)^{1-s}}{1-s}x^s\,ds\\ &\quad + O(S_1)+O(S_2),\end{aligned}$$

where

$$S_1 = \eta Y(\log N)^{-\frac{2E}{\delta^2}}\int\limits_{-T_0}^{T_0}|H(b+it,\chi)K(b+it,\chi)|\,dt,$$

$$S_2 = \eta^2 Y\int\limits_{-T_0}^{T_0}|H(b+it,\chi)K(b+it,\chi)|\,dt.$$

By Lemma 4, the contribution of S_1 to (37) is

$$\begin{aligned}&\ll Y(\log N)^{-\frac{4E}{\delta^2}}\sum_{\chi\,(\mathrm{mod}\,q)}T_0\int\limits_{-T_0}^{T_0}|H(b+it,\chi)K(b+it,\chi)|^2\,dt\\ &\ll YT_0(\log N)^{-\frac{4E}{\delta^2}}\bigg(1+\frac{qT_0}{HK}\bigg)\log^6 N \ll Y\log^{-2E} N.\end{aligned}$$

The contribution of S_2 to (37) is

$$\ll \frac{1}{(qQ)^2}\cdot Y\biggl(\frac{qQ}{Y}\cdot qQ\biggr)^2 \sum_{\chi\,(\mathrm{mod}\,q)} T_0\int_{-T_0}^{T_0}|H(b+it,\chi)K(b+it,\chi)|^2\,dt$$

$$\ll YT_0\biggl(\frac{qQ}{Y}\biggr)^2\biggl(1+\frac{qT_0}{HK}\biggr)\log^6 N \ll Y^{1-\varepsilon}.$$

By Perron's formula again,

$$\begin{aligned}\eta E_0\cdot\frac{1}{2\pi i}\int_{b-iT_0}^{b+iT_0} H(s,\chi)K(s,\chi)\frac{(c_2M)^{1-s}-(c_1M)^{1-s}}{1-s}x^s\,ds \\ &= \eta x\cdot E_0\cdot\sum_{h\sim H}\frac{b(h)\chi(h)}{h}\sum_{k\sim K}\frac{g(k)\chi(k)}{k} \\ &\quad + O\biggl(\frac{\eta x E_0\log^2 Y}{T_0}\biggr)+O\biggl(\frac{\eta x E_0 Y^{\varepsilon}}{HK}\biggr) \\ &= \eta x\cdot E_0 I + O\biggl(\frac{\eta x E_0\log^2 Y}{T_0}\biggr)+O\biggl(\frac{\eta x E_0 Y^{\varepsilon}}{HK}\biggr).\end{aligned}$$

The contribution of the term $O(\eta x E_0\log^2 Y/T_0)$ to (37) is

$$\ll \frac{1}{(qQ)^2}\cdot Y\biggl(\frac{qQ\log^2 Y}{T_0}\biggr)^2 \ll Y\log^{-E} N.$$

The contribution of the term $O(\eta x E_0 Y^{\varepsilon}/(HK))$ to (37) is

$$\ll \frac{1}{(qQ)^2}\cdot Y\biggl(\frac{qQY^{\varepsilon}}{HK}\biggr)^2 \ll Y^{1-\varepsilon}.$$

Combining the above, we have

$$\Sigma_2 \ll \frac{\log^2 N}{(qQ)^2} \tag{38}$$
$$\times \max_{T_0\le|T|\le Y}\sum_{\chi\,(\mathrm{good})}\int_Y^{3Y}\biggl|\int_{b+iT}^{b+2iT}G(s,\chi)\frac{(1+\eta)^s-1}{s}x^s\,ds\biggr|^2\,dx + O(Y\log^{-E}N).$$

By the argument of Lemma 15, we have

$$\Sigma_2 \ll \frac{Y^3\log^4 N}{(qQ)^2}\cdot\max_{T_0\le T\le Y}\min^2\biggl(\eta,\frac{1}{T}\biggr)\sum_{\chi\,(\mathrm{good})}\int_T^{2T}|G(b+it,\chi)|^2\,dt$$
$$+O(Y\log^{-E}N).$$

By the assumption in Lemma 18, $\Sigma_2 \ll Y\log^{-E}N$. Hence, (33) holds. Then for $N < n \le N + A$, except for $O(A\log^{-B} N)$ values,

$$\int\limits_{E_2} S_1(\theta)S_2(\theta)e(-n\theta)\,d\theta = O(Y\log^{-B} N). \tag{39}$$

If $\theta = a/q + \beta \in E_1$, we have

$$S_1(\theta) = \sum_{\substack{pd\le 2Y\\ d\sim D}} l(d)e(\theta pd),$$

where $D = HK$ and

$$l(d) = \sum_{\substack{hk=d\\ h\sim H,\, k\sim K}} b(h)g(k).$$

Hence

$$\begin{aligned} S_1(\theta) &= \sum_{d\sim D} l(d) \sum_{p\le 2Y/d} e\bigg(\frac{apd}{q} + \beta pd\bigg) \\ &= \sum_{d\sim D} l(d) \sum_{\substack{p\le 2Y/d\\ (p,q)=1}} e\bigg(\frac{apd}{q} + \beta pd\bigg) + O(Y^{1-\varepsilon}), \end{aligned}$$

where

$$\sum_{\substack{p\le 2Y/d\\ (p,q)=1}} e\bigg(\frac{apd}{q} + \beta pd\bigg) = \sum_{\substack{l=1\\ (l,q)=1}}^{q} e\bigg(\frac{adl}{q}\bigg) \sum_{\substack{p\le 2Y/d\\ p\equiv l \,(\mathrm{mod}\, q)}} e(\beta pd).$$

From the prime number theorem in the arithmetic progression, it follows that

$$\begin{aligned} \sum_{\substack{p\le 2Y/d\\ p\equiv l \,(\mathrm{mod}\, q)}} e(\beta pd) &= \int\limits_{2}^{2Y/d} e(\beta dt)\,d\pi(t;l,q) \\ &= \frac{1}{\varphi(q)} \int\limits_{2}^{2Y/d} \frac{e(\beta dt)}{\log t}\,dt + O\bigg(\frac{Y}{d\log^{6E} N}\bigg) \\ &= \frac{1}{\varphi(q)}\cdot\frac{1}{d}\cdot \int\limits_{2d}^{2Y} \frac{e(\beta u)}{\log(u/d)}\,du + O\bigg(\frac{Y}{d\log^{6E} N}\bigg). \end{aligned}$$

Note that $q \le \log^E N$ and $(d,q) > 1$ imply $l(d) = 0$. So, $(d,q) = 1$ can be assumed. Consequently,

$$\sum_{\substack{p\le 2Y/d\\ (p,q)=1}} e\bigg(\frac{apd}{q} + \beta pd\bigg)$$

$$= \frac{1}{\varphi(q)} \cdot \frac{1}{d} \sum_{\substack{l=1\\(l,q)=1}}^{q} e\left(\frac{adl}{q}\right) \int_{2d}^{2Y} \frac{e(\beta u)}{\log(u/d)}\,du + O\left(\frac{Y}{d\log^{5E} N}\right)$$

$$= \frac{\mu(q)}{\varphi(q)} \cdot \frac{1}{d} \sum_{2d\le r\le 2Y} \frac{e(\beta r)}{\log(r/d)} + O\left(\frac{Y}{d\log^{5E} N}\right).$$

Therefore

$$S_1(\theta) = \frac{\mu(q)}{\varphi(q)} \sum_{d\sim D} \frac{l(d)}{d} \sum_{2d\le r\le 2Y} \frac{e(\beta r)}{\log(r/d)} + O\left(\frac{Y}{\log^{4E} N}\right).$$

Now, set

$$\begin{aligned} \Sigma_3 = {} & \int_{E_1} S_1(\theta)S_2(\theta)e(-n\theta)\,d\theta \\ = {} & \sum_{q\le \log^E N} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{-\log^{2E} N/(qY)}^{\log^{2E} N/(qY)} S_1\left(\frac{a}{q}+\beta\right) S_2\left(\frac{a}{q}+\beta\right) \\ & \times e\left(-\left(\frac{a}{q}+\beta\right)n\right)d\beta \\ = {} & \sum_{q\le \log^E N} \frac{\mu^2(q)}{\varphi^2(q)} \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{an}{q}\right) \int_{-\log^{2E} N/(qY)}^{\log^{2E} N/(qY)} \sum_{d\sim D} \frac{l(d)}{d} \\ & \times \sum_{2d\le r\le 2Y} \frac{e(\beta r)}{\log(r/d)} \cdot \frac{1}{\log N} \sum_{N-Y\le s\le N} e(\beta s)e(-n\beta)\,d\beta \\ & + O\left(\frac{Y}{\log^B N}\right). \end{aligned}$$

Since

$$\begin{aligned} & \int_{-\log^{2E} N/(qY)}^{\log^{2E} N/(qY)} \sum_{d\sim D} \frac{l(d)}{d} \sum_{2d\le r\le 2Y} \frac{e(\beta r)}{\log(r/d)} \sum_{N-Y<s\le N} e(\beta s)e(-n\beta)\,d\beta \\ & \quad = \int_{-\frac12}^{\frac12} \sum_{d\sim D} \frac{l(d)}{d} \sum_{2d\le r\le 2Y} \frac{e(\beta r)}{\log(r/d)} \sum_{N-Y<s\le N} e(\beta s)e(-n\beta)\,d\beta \\ & \qquad + O\left(\frac{qY}{\log^{2E} N}\right) \end{aligned}$$

$$= \sum_{d\sim D} \frac{l(d)}{d} \sum_{2d\le r\le 2Y} \frac{1}{\log(r/d)} \sum_{\substack{N-Y<s\le N\\ r+s=n}} 1 + O\Big(\frac{qY}{\log^{2E} N}\Big)$$

$$= \sum_{d\sim D} \frac{l(d)}{d} \sum_{2d\le r\le Y} \frac{1}{\log(r/d)} + O\Big(\frac{qY}{\log^{2E} N}\Big),$$

we have

$$\Sigma_3 = \frac{1}{\log N} \sum_{q\le \log^E N} \frac{\mu^2(q)}{\varphi^2(q)} \sum_{\substack{a=1\\ (a,q)=1}}^{q} e\Big(-\frac{an}{q}\Big) \sum_{d\sim D} \frac{l(d)}{d} \sum_{2d\le r\le Y} \frac{1}{\log(r/d)} + O\Big(\frac{Y}{\log^B N}\Big).$$

By the argument in Lemmas 11 and 12 of [7], we find that for $N < n \le N+A$, except for $O(A\log^{-B} N)$ values,

$$\sum_{q\le \log^E N} \frac{\mu^2(q)}{\varphi^2(q)} \sum_{\substack{a=1\\ (a,q)=1}}^{q} e\Big(-\frac{an}{q}\Big) = C(n) + O\Big(\frac{1}{\log^{2B} N}\Big).$$

Hence,

$$\Sigma_3 = C(n)\cdot \frac{Y}{\log N}\Big(1+O\Big(\frac{1}{\log N}\Big)\Big) \sum_{d\sim D} \frac{l(d)}{d\log(Y/d)} + O\Big(\frac{Y}{\log^B N}\Big)$$

$$= \frac{C(n)}{\log N}\Big(1+O\Big(\frac{1}{\log N}\Big)\Big) \sum_{\substack{h\sim H\\ k\sim K}} b(h)g(k) \sum_{Y/(hk)<p\le 2Y/(hk)} 1 + O\Big(\frac{Y}{\log^B N}\Big).$$

The proof of Lemma 18 is complete.

LEMMA 19. *Suppose that* $Y^{\frac{119}{135}} \ll M \ll Y^{1-\delta}$ *and* $0 \le a(m) = O(1)$. *If* m *has a prime factor* $< Y^{\delta}$, $a(m) = 0$. *Then for* $N < n \le N + A$, *except for* $O(A\log^{-B} N)$ *values*, *we have*

$$\sum_{\substack{mp+p_1=n\\ N-Y<p_1\le N\\ mp\le 2Y\\ m\sim M}} a(m) = \frac{C(n)}{\log N}\Big(1+O\Big(\frac{1}{\log N}\Big)\Big) \sum_{m\sim M} a(m) \sum_{Y/m<p\le 2Y/m} 1 + O(Y\log^{-B} N).$$

P r o o f. This follows from Lemmas 5 and 18.

8. Buchstab's function. We define $w(u)$ as the continuous solution of the equations

$$\begin{cases} w(u) = 1/u, & 1 \le u \le 2, \\ (uw(u))' = w(u-1), & u > 2. \end{cases} \tag{40}$$

$w(u)$ is called *Buchstab's function* and plays an important role in finding asymptotic formulas in the sieve method. In particular,

$$w(u) = \frac{1+\log(u-1)}{u}, \qquad 2 \le u \le 3;$$

$$w(u) = \frac{1+\log(u-1)}{u} + \frac{1}{u}\int_2^{u-1} \frac{\log(t-1)}{t}\,dt, \qquad 3 \le u \le 4;$$

$$w(u) = \frac{1+\log(u-1)}{u} + \frac{1}{u}\int_2^{u-1} \frac{\log(t-1)}{t}\,dt + \frac{1}{u}\int_3^{u-1}\frac{dt}{t}\int_2^{t-1}\frac{\log(s-1)}{s}\,ds, \qquad 4 \le u \le 5.$$

LEMMA 20. *For the function $w(u)$, we have the following bounds*:

(i) $w(u) \ge 0.5607$ *for* $u \ge 2.47$,
(ii) $w(u) \le 0.5644$ *for* $u \ge 3$,
(iii) $0.5612 \le w(u) \le 0.5617$ *for* $u \ge 4$.

Proof. It is easy to see that $0.5 \le w(u) \le 1$ for $1 \le u \le 2$. Then we employ induction.

Suppose that $0.5 \le w(u) \le 1$ for $1 \le k \le u \le k+1$. If $k+1 \le u \le k+2$, then (40) yields

$$uw(u) = (k+1)w(k+1) + \int_k^{u-1} w(t)\,dt. \tag{41}$$

Hence, $0.5 \le w(u) \le 1$ for $k+1 \le u \le k+2$. By induction, we obtain $0.5 \le w(u) \le 1$ for $u \ge 1$.

If $u > 2$, (40) yields

$$w'(u) = \frac{w(u-1) - w(u)}{u}. \tag{42}$$

If $2 \le u \le 3$, by calculation, we have

$$\max_{0 \le k \le 10^4} w(2 + 10^{-4}k) \le 0.56716.$$

From (42) and $0.5 \le w(u) \le 1$ for $u \ge 1$, it follows that $|w'(u)| \le \frac{1}{4}$ if $u > 2$. Using Lagrange's mean value theorem, we have $w(u) \le 0.5672$ for $2 \le u \le 3$. By induction, we obtain $0.5 \le w(u) \le 0.5672$ for $u \ge 2$.

If $3 \le u \le 4$, by calculation, we have

$$\max_{0\le k\le 10^4} w(3+10^{-4}k) \le 0.56439, \qquad \min_{0\le k\le 10^4} w(3+10^{-4}k) \ge 0.56081.$$

From (42) and $0.5 \le w(u) \le 0.5672$ for $u \ge 2$, it follows that $|w'(u)| \le 0.0224$ if $u > 3$. The above discussion implies that $0.5607 \le w(u) \le 0.5644$ for $u \ge 3$. By the same discussion we can also get $0.5607 \le w(u)$ for $u \ge 2.47$.

If $4 \le u \le 5$, by calculation, we have

$$\max_{0\le k\le 10^4} w(4+10^{-4}k) \le 0.5616, \qquad \min_{0\le k\le 10^4} w(4+10^{-4}k) \ge 0.5613.$$

The above discussion and the fact that $|w'(u)| \le 0.0224$ for $u > 3$ imply that $0.5612 \le w(u) \le 0.5617$ for $u \ge 4$.

Gathering together the above discussion, we get Lemma 20.

LEMMA 21. *Suppose that* $\mathcal{E} = \{n : t < n \le 2t\}$ *and* $z \le t$. *Let*

$$P(z) = \prod_{p<z} p.$$

Then for sufficiently large t *and* z, *we have*

$$S(\mathcal{E}, z) = \sum_{\substack{t<n\le 2t\\ (n,P(z))=1}} 1 = \left(w\left(\frac{\log t}{\log z}\right) + O(\varepsilon)\right)\frac{t}{\log z}.$$

Proof. See Lemma 5 of [9]. If $(2t)^{\frac12} < z \le t$, it is the prime number theorem.

9. Sieve method. Let n be an even number, $N < n \le N + A$, and set

$$T(n) = \sum_{\substack{p_1+p_2=n\\ N-Y<p_2\le N\\ p_1\le 2Y}} 1. \tag{43}$$

We proceed to show that

$$T(n) \ge 0.011 \cdot \frac{C(n)Y}{\log Y \log N}, \tag{44}$$

where

$$C(n) = \prod_{p\nmid n}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p\mid n}\left(1 + \frac{1}{p-1}\right). \tag{45}$$

Set

$$\mathcal{A} = \{a : a = n - p,\ N - Y < p \le N\}, \qquad \mathcal{P} = \{p : p \nmid n\},$$
$$P(z,n) = \prod_{\substack{p<z\\ p\in\mathcal{P}}} p, \qquad S(\mathcal{A}, z) = \sum_{\substack{a\in\mathcal{A}\\ (a,P(z,n))=1}} 1.$$

Then

$$T(n) = S(\mathcal{A}, (2Y)^{\frac{1}{2}}) + O(Y^{\frac{1}{2}}). \tag{46}$$

Buchstab's identity yields

$$\begin{aligned} S(\mathcal{A}, (2Y)^{\frac{1}{2}}) &= S(\mathcal{A}, Y^{\frac{16}{135}}) - \sum_{Y^{\frac{16}{135}} < p \le (2Y)^{\frac{1}{2}}} S(\mathcal{A}_p, p) \\ &= S(\mathcal{A}, Y^{\frac{16}{135}}) - \sum_{Y^{\frac{16}{135}} < p \le (2Y)^{\frac{1}{2}}} S(\mathcal{A}_p, Y^{\frac{16}{135}}) \\ &\quad + \sum_{Y^{\frac{16}{135}} < p \le (2Y)^{\frac{1}{2}}} \ \sum_{Y^{\frac{16}{135}} < q < \min(p, (\frac{2Y}{p})^{\frac{1}{2}})} S(\mathcal{A}_{pq}, q). \end{aligned} \tag{47}$$

The following lemmas always concern $N < n \le N + A$, except for $O(A \log^{-B} N)$ values. Let

$$\mathcal{B} = \{n : Y < n \le 2Y\}. \tag{48}$$

LEMMA 22. *We have*

$$S(\mathcal{A}, Y^{\frac{16}{135}}) \ge 4.735124 \cdot \frac{C(n)Y}{\log Y \log N}.$$

Proof. We have

$$S(\mathcal{A}, Y^{\frac{16}{135}}) = S(\mathcal{A}, Y^{\delta}) - \sum_{Y^{\delta} < p \le Y^{\frac{16}{135}}} S(\mathcal{A}_p, p). \tag{49}$$

Let $X = Y/\log N$ and

$$\omega(d) = \begin{cases} d/\varphi(d), & (d, n) = 1, \\ 0, & (d, n) > 1, \end{cases} \qquad \widetilde{r}(\mathcal{A}, d) = |\mathcal{A}_d| - \frac{X}{\varphi(d)}.$$

By the argument in Lemma 11 of [8] (see also Theorem 2 on p. 164 and (40) on p. 171 of [15]),

$$W(z) = \prod_{p<z} \left(1 - \frac{\omega(p)}{p}\right) = (1 + O(\varepsilon))C(n)\frac{e^{-\gamma}}{\log z},$$

where γ is Euler's constant.

Let $z = Y^{\delta}$ and $D = Y^{\frac{19}{36}}$. Applying Iwaniec's sieve method (see Theorem 1 of [6]), we have

$$S(\mathcal{A}, Y^{\delta}) \ge \frac{C(n)Y}{\log z \log N} f\left(\frac{\log D}{\log z}\right) - O\left(\frac{\delta C(n)Y}{\log^2 N}\right) - R^{-},$$

where

$$R^{-} = \sum_{\substack{m \le Y^{\frac{19}{36}} \\ (m,n)=1}} a(m)\widetilde{r}(\mathcal{A}, m).$$

Lemma 17 yields $R^- = O(Y\log^{-5} N)$. By Theorem 8 on p. 181 of [15], we have

$$f\left(\frac{\log D}{\log z}\right) = e^{-\gamma} + O(\varepsilon^2),$$

where γ is Euler's constant. Thus,

$$S(\mathcal{A}, Y^\delta) \geq \frac{e^{-\gamma}}{\delta} \cdot \frac{C(n)Y}{\log Y \log N} + O\left(\frac{\delta C(n)Y}{\log^2 N}\right).$$

In the same way,

$$\begin{aligned} S(\mathcal{A}, Y^\delta) &\leq \frac{C(n)Y}{\log z \log N} \cdot F\left(\frac{\log D}{\log z}\right) + O\left(\frac{\delta C(n)Y}{\log^2 N}\right) + \sum_{\substack{m \leq Y^{\frac{19}{36}} \\ (m,n)=1}} b(m)\widetilde{r}(\mathcal{A}, m) \\ &\leq \frac{e^{-\gamma}}{\delta} \cdot \frac{C(n)Y}{\log Y \log N} + O\left(\frac{\delta C(n)Y}{\log^2 N}\right). \end{aligned}$$

So, we have

$$S(\mathcal{A}, Y^\delta) = \frac{e^{-\gamma}}{\delta} \cdot \frac{C(n)Y}{\log Y \log N} + O\left(\frac{\delta C(n)Y}{\log^2 N}\right). \tag{50}$$

Now,

$$\sum_{Y^\delta < p \leq Y^{\frac{16}{135}}} S(\mathcal{A}_p, p) = \sum_{pq+p_1=n} 1,$$

where $pq \leq 2Y$, $N - Y < p_1 \leq N$, $Y^\delta < p \leq Y^{\frac{16}{135}}$, and the least prime factor of q is greater than p. Using Lemmas 19 and 21 and the prime number theorem, we have the asymptotic formula

$$\begin{aligned} &\sum_{Y^\delta < p \leq Y^{\frac{16}{135}}} S(\mathcal{A}_p, p) \\ &= \frac{C(n)}{\log N} \sum_{Y^\delta < p \leq Y^{\frac{16}{135}}} S(\mathcal{B}_p, p) + O\left(\frac{\delta C(n)Y}{\log^2 N}\right) \\ &= \frac{C(n)Y}{\log N} \sum_{Y^\delta < p \leq Y^{\frac{16}{135}}} \frac{1}{p\log p} w\left(\frac{\log(Y/p)}{\log p}\right) + O\left(\frac{\delta C(n)Y}{\log^2 N}\right) \end{aligned} \tag{51}$$

$$= \frac{C(n)Y}{\log Y \log N} \int_{\delta}^{\frac{16}{135}} \frac{1}{t^2} w\left(\frac{1-t}{t}\right) dt + O\left(\frac{\delta C(n)Y}{\log^2 N}\right)$$

$$= \frac{C(n)Y}{\log Y \log N} \int_{\frac{135}{16}}^{\frac{1}{\delta}} w(u-1)\, du + O\left(\frac{\delta C(n)Y}{\log^2 N}\right)$$

$$= \frac{C(n)Y}{\log Y \log N} \left\{ \frac{1}{\delta} w\left(\frac{1}{\delta}\right) - \frac{135}{16} w\left(\frac{135}{16}\right) \right\} + O\left(\frac{\delta C(n)Y}{\log^2 N}\right)$$

$$= \frac{e^{-\gamma}}{\delta} \cdot \frac{C(n)Y}{\log Y \log N} - \frac{135}{16} w\left(\frac{135}{16}\right) \cdot \frac{C(n)Y}{\log Y \log N} + O\left(\frac{\delta C(n)Y}{\log^2 N}\right),$$

since

$$w\left(\frac{1}{\delta}\right) = e^{-\gamma} + O(\varepsilon^2)$$

(see Lemma 12 on p. 179 of [15]). Hence,

$$S(\mathcal{A}, Y^{\frac{16}{135}}) = \frac{135}{16} w\left(\frac{135}{16}\right) \cdot \frac{C(n)Y}{\log Y \log N} + O\left(\frac{\delta C(n)Y}{\log^2 N}\right).$$

By Lemma 20, we get

$$S(\mathcal{A}, Y^{\frac{16}{135}}) \geq \frac{135}{16} \cdot 0.5612 \cdot \frac{C(n)Y}{\log Y \log N} + O\left(\frac{\delta C(n)Y}{\log^2 N}\right)$$
$$\geq 4.735124 \cdot \frac{C(n)Y}{\log Y \log N}.$$

The proof of Lemma 22 is complete.

Lemma 23. *We have*

$$\sum_{Y^{\frac{16}{135}} < p \leq (2Y)^{\frac{1}{2}}} S(\mathcal{A}_p, Y^{\frac{16}{135}}) \leq 6.822470 \cdot \frac{C(n)Y}{\log Y \log N}.$$

Proof. Buchstab's identity yields

$$\sum_{Y^{\frac{16}{135}} < p \leq (2Y)^{\frac{1}{2}}} S(\mathcal{A}_p, Y^{\frac{16}{135}})$$
$$= \sum_{Y^{\frac{16}{135}} < p \leq (2Y)^{\frac{1}{2}}} S(\mathcal{A}_p, Y^{\delta}) - \sum_{Y^{\frac{16}{135}} < p \leq (2Y)^{\frac{1}{2}}} \sum_{Y^{\delta} < q < Y^{\frac{16}{135}}} S(\mathcal{A}_{pq}, q).$$

Using Lemma 17, in the same way as in Lemma 22, we have

$$\sum_{Y^{\frac{16}{135}}<p\le(2Y)^{\frac12}} S(\mathcal{A}_p,Y^{\delta}) = \sum_{Y^{\frac{16}{135}}<p\le(2Y)^{\frac12}} \frac{e^{-\gamma}}{\delta}\cdot\frac{1}{p}\cdot\frac{C(n)Y}{\log Y\log N} + O\biggl(\frac{\delta C(n)Y}{\log^2 N}\biggr)$$
$$= \frac{e^{-\gamma}}{\delta}\cdot\frac{C(n)Y}{\log Y\log N}\int_{\frac{16}{135}}^{\frac12}\frac{dt}{t} + O\biggl(\frac{\delta C(n)Y}{\log^2 N}\biggr).$$

Using Lemmas 19 and 21, in the same way as in Lemma 22, we have

$$\sum_{Y^{\frac{16}{135}}<p\le(2Y)^{\frac12}}\ \sum_{Y^{\delta}<q<Y^{\frac{16}{135}}} S(\mathcal{A}_{pq},q)$$
$$= \frac{C(n)}{\log N}\sum_{Y^{\frac{16}{135}}<p\le(2Y)^{\frac12}}\ \sum_{Y^{\delta}<q\le Y^{\frac{16}{135}}} S(\mathcal{B}_{pq},q) + O\biggl(\frac{\delta C(n)Y}{\log^2 N}\biggr)$$
$$= \frac{C(n)Y}{\log N}\sum_{Y^{\frac{16}{135}}<p\le(2Y)^{\frac12}}\ \sum_{Y^{\delta}<q\le Y^{\frac{16}{135}}} \frac{1}{pq\log q}w\biggl(\frac{\log(Y/(pq))}{\log q}\biggr) + O\biggl(\frac{\delta C(n)Y}{\log^2 N}\biggr)$$
$$= \frac{C(n)Y}{\log Y\log N}\int_{\frac{16}{135}}^{\frac12}\frac{dt}{t}\int_{\delta}^{\frac{16}{135}}\frac{1}{u^2}w\biggl(\frac{1-t-u}{u}\biggr)du + O\biggl(\frac{\delta C(n)Y}{\log^2 N}\biggr)$$
$$= \frac{C(n)Y}{\log Y\log N}\int_{\frac{16}{135}}^{\frac12}\frac{dt}{t(1-t)}\int_{\frac{135}{16}(1-t)}^{\frac{1}{\delta}(1-t)} w(r-1)\,dr + O\biggl(\frac{\delta C(n)Y}{\log^2 N}\biggr)$$
$$= \frac{1}{\delta}\cdot\frac{C(n)Y}{\log Y\log N}\int_{\frac{16}{135}}^{\frac12}\frac{1}{t}w\biggl(\frac{1}{\delta}(1-t)\biggr)dt$$
$$- \frac{135}{16}\cdot\frac{C(n)Y}{\log Y\log N}\int_{\frac{16}{135}}^{\frac12}\frac{1}{t}w\biggl(\frac{135}{16}(1-t)\biggr)dt + O\biggl(\frac{\delta C(n)Y}{\log^2 N}\biggr)$$
$$= \frac{e^{-\gamma}}{\delta}\cdot\frac{C(n)Y}{\log Y\log N}\int_{\frac{16}{135}}^{\frac12}\frac{dt}{t} - \frac{135}{16}\cdot\frac{C(n)Y}{\log Y\log N}\int_{\frac{16}{135}}^{\frac12}\frac{1}{t}w\biggl(\frac{135}{16}(1-t)\biggr)dt$$
$$+ O\biggl(\frac{\delta C(n)Y}{\log^2 N}\biggr).$$

Gathering together the above discussion and applying Lemma 20, we have

$$\begin{aligned}
&\sum_{Y^{\frac{16}{135}}<p\le(2Y)^{\frac12}} S(\mathcal{A}_p, Y^{\frac{16}{135}}) \\
&\qquad = \frac{135}{16}\cdot\frac{C(n)Y}{\log Y\log N}\int_{\frac{16}{135}}^{\frac12}\frac{1}{t}w\Big(\frac{135}{16}(1-t)\Big)\,dt + O\Big(\frac{\delta C(n)Y}{\log^2 N}\Big) \\
&\qquad \le 0.5617\cdot\frac{135}{16}\cdot\log\Big(\frac{135}{32}\Big)\cdot\frac{C(n)Y}{\log Y\log N} + O\Big(\frac{\delta C(n)Y}{\log^2 N}\Big) \\
&\qquad \le 6.822470\cdot\frac{C(n)Y}{\log Y\log N}.
\end{aligned}$$

The proof of Lemma 23 is complete.

We now set

$$\begin{aligned}
\Omega &= \sum_{Y^{\frac{16}{135}}<p\le(2Y)^{\frac12}}\ \sum_{Y^{\frac{16}{135}}<q<\min(p,(\frac{2Y}{p})^{\frac12})} S(\mathcal{A}_{pq}, q) \\
&\ge \sum_{i=1}^{69}\sum_{(p,q)\in D_i} S(\mathcal{A}_{pq}, q) = \sum_{i=1}^{69}\Omega_i,
\end{aligned} \tag{52}$$

where

$$\begin{aligned}
D_1 &= \{(p,q): Y^{\frac{16}{135}}<p\le Y^{\frac{89}{462}},\ Y^{\frac{16}{135}}<q<p\},\\
D_2 &= \{(p,q): Y^{\frac{89}{462}}<p\le Y^{\frac{41}{180}},\ Y^{\frac{16}{135}}<q<p^{-\frac{29}{48}}Y^{\frac{89}{288}}\},\\
D_3 &= \{(p,q): Y^{\frac{41}{180}}<p\le Y^{\frac{8}{33}},\ Y^{\frac{16}{135}}<q<p^{-\frac{29}{48}}Y^{\frac{89}{288}}\},\\
D_4 &= \{(p,q): Y^{\frac{41}{180}}<p\le Y^{\frac{8}{33}},\ p^{-1}Y^{\frac{41}{90}}<q<p\},\\
D_5 &= \{(p,q): Y^{\frac{8}{33}}<p\le Y^{\frac{37}{150}},\ Y^{\frac{16}{135}}<q<p^{-\frac{29}{48}}Y^{\frac{89}{288}}\},\\
D_6 &= \{(p,q): Y^{\frac{8}{33}}<p\le Y^{\frac{37}{150}},\ p^{-1}Y^{\frac{41}{90}}<q<p^{-1}Y^{\frac{16}{33}}\},\\
D_7 &= \{(p,q): Y^{\frac{37}{150}}<p\le Y^{\frac{17}{63}},\ Y^{\frac{16}{135}}<q<p^{-\frac{29}{48}}Y^{\frac{89}{288}}\},\\
D_8 &= \{(p,q): Y^{\frac{37}{150}}<p\le Y^{\frac{17}{63}},\ p^{-1}Y^{\frac{701}{1566}}<q<p^{-1}Y^{\frac{16}{33}}\},\\
D_9 &= \{(p,q): Y^{\frac{17}{63}}<p\le Y^{\frac{17}{57}},\ Y^{\frac{16}{135}}<q<p^{-\frac{29}{48}}Y^{\frac{89}{288}}\},\\
D_{10} &= \{(p,q): Y^{\frac{17}{63}}<p\le Y^{\frac{17}{57}},\ p^{-1}Y^{\frac{701}{1566}}<q<p^{-1}Y^{\frac{16}{33}}\},\\
D_{11} &= \{(p,q): Y^{\frac{17}{63}}<p\le Y^{\frac{17}{57}},\ p^{-\frac{6}{5}}Y^{\frac{17}{30}}<q<p^{-\frac{1}{8}}Y^{\frac{7}{24}}\},\\
D_{12} &= \{(p,q): Y^{\frac{17}{57}}<p\le Y^{\frac{31}{99}},\ Y^{\frac{16}{135}}<q<p^{-\frac{29}{48}}Y^{\frac{89}{288}}\},\\
D_{13} &= \{(p,q): Y^{\frac{17}{57}}<p\le Y^{\frac{31}{99}},\ p^{-1}Y^{\frac{701}{1566}}<q<p^{-1}Y^{\frac{16}{33}}\},\\
D_{14} &= \{(p,q): Y^{\frac{17}{57}}<p\le Y^{\frac{31}{99}},\ p^{-\frac{12}{11}}Y^{\frac{17}{33}}<q<p^{-\frac{1}{8}}Y^{\frac{7}{24}}\},
\end{aligned}$$

$$D_{15}=\{(p,q):Y^{\frac{31}{99}}<p\le Y^{\frac{17}{54}},\ Y^{\frac{16}{135}}<q<p^{-\frac{29}{48}}Y^{\frac{89}{288}}\},$$
$$D_{16}=\{(p,q):Y^{\frac{31}{99}}<p\le Y^{\frac{17}{54}},\ p^{-1}Y^{\frac{701}{1566}}<q<p^{-\frac{58}{67}}Y^{\frac{89}{201}}\},$$
$$D_{17}=\{(p,q):Y^{\frac{31}{99}}<p\le Y^{\frac{17}{54}},\ p^{-\frac{12}{11}}Y^{\frac{17}{33}}<q<p^{-\frac{1}{8}}Y^{\frac{7}{24}}\},$$
$$D_{18}=\{(p,q):Y^{\frac{17}{54}}<p\le Y^{\frac{221}{693}},\ p^{-1}Y^{\frac{701}{1566}}<q<p^{-\frac{58}{67}}Y^{\frac{89}{201}}\},$$
$$D_{19}=\{(p,q):Y^{\frac{17}{54}}<p\le Y^{\frac{221}{693}},\ Y^{\frac{17}{99}}<q<p^{-\frac{1}{8}}Y^{\frac{7}{24}}\},$$
$$D_{20}=\{(p,q):Y^{\frac{221}{693}}<p\le Y^{\frac{859}{2610}},\ p^{-1}Y^{\frac{701}{1566}}<q<p^{-\frac{58}{67}}Y^{\frac{89}{201}}\},$$
$$D_{21}=\{(p,q):Y^{\frac{221}{693}}<p\le Y^{\frac{859}{2610}},\ Y^{\frac{17}{99}}<q<p^{-1}Y^{\frac{113}{198}}\},$$
$$D_{22}=\{(p,q):Y^{\frac{859}{2610}}<p\le Y^{\frac{109}{330}},\ Y^{\frac{16}{135}}<q<p^{-\frac{58}{67}}Y^{\frac{89}{201}}\},$$
$$D_{23}=\{(p,q):Y^{\frac{859}{2610}}<p\le Y^{\frac{109}{330}},\ Y^{\frac{17}{99}}<q<p^{-1}Y^{\frac{113}{198}}\},$$
$$D_{24}=\{(p,q):Y^{\frac{109}{330}}<p\le Y^{\frac{151}{450}},\ Y^{\frac{16}{135}}<q<p^{-\frac{58}{67}}Y^{\frac{89}{201}}\},$$
$$D_{25}=\{(p,q):Y^{\frac{109}{330}}<p\le Y^{\frac{151}{450}},\ Y^{\frac{17}{99}}<q<p^{-6}Y^{\frac{20}{9}}\},$$
$$D_{26}=\{(p,q):Y^{\frac{151}{450}}<p\le Y^{\frac{34}{99}},\ Y^{\frac{16}{135}}<q<p^{-\frac{58}{67}}Y^{\frac{89}{201}}\},$$
$$D_{27}=\{(p,q):Y^{\frac{151}{450}}<p\le Y^{\frac{34}{99}},\ Y^{\frac{17}{99}}<q<p^{\frac{1}{4}}Y^{\frac{1}{8}}\},$$
$$D_{28}=\{(p,q):Y^{\frac{151}{450}}<p\le Y^{\frac{34}{99}},\ p^{-1}Y^{\frac{5}{9}}<q<p^{\frac{1}{3}}Y^{\frac{7}{54}}\},$$
$$D_{29}=\{(p,q):Y^{\frac{34}{99}}<p\le Y^{\frac{1649}{4752}},\ Y^{\frac{16}{135}}<q<p^{-\frac{58}{67}}Y^{\frac{89}{201}}\},$$
$$D_{30}=\{(p,q):Y^{\frac{34}{99}}<p\le Y^{\frac{1649}{4752}},\ p^{-2}Y^{\frac{85}{99}}<q<p^{-\frac{58}{49}}Y^{\frac{85}{147}}\},$$
$$D_{31}=\{(p,q):Y^{\frac{34}{99}}<p\le Y^{\frac{1649}{4752}},\ p^{-\frac{58}{49}}Y^{\frac{85}{147}}<q<p^{\frac{1}{4}}Y^{\frac{1}{8}}\},$$
$$D_{32}=\{(p,q):Y^{\frac{1649}{4752}}<p\le Y^{\frac{25}{72}},\ Y^{\frac{16}{135}}<q<p^{-\frac{58}{67}}Y^{\frac{89}{201}}\},$$
$$D_{33}=\{(p,q):Y^{\frac{1649}{4752}}<p\le Y^{\frac{25}{72}},\ p^{-\frac{70}{59}}Y^{\frac{34}{59}}<q<p^{-\frac{58}{49}}Y^{\frac{85}{147}}\},$$
$$D_{34}=\{(p,q):Y^{\frac{1649}{4752}}<p\le Y^{\frac{25}{72}},\ p^{-\frac{58}{49}}Y^{\frac{85}{147}}<q<p^{\frac{1}{4}}Y^{\frac{1}{8}}\},$$
$$D_{35}=\{(p,q):Y^{\frac{25}{72}}<p\le Y^{\frac{10}{27}},\ Y^{\frac{16}{135}}<q<p^{-\frac{58}{67}}Y^{\frac{89}{201}}\},$$
$$D_{36}=\{(p,q):Y^{\frac{25}{72}}<p\le Y^{\frac{10}{27}},\ p^{-\frac{70}{59}}Y^{\frac{34}{59}}<q<p^{-\frac{58}{49}}Y^{\frac{85}{147}}\},$$
$$D_{37}=\{(p,q):Y^{\frac{25}{72}}<p\le Y^{\frac{10}{27}},\ p^{-\frac{58}{49}}Y^{\frac{85}{147}}<q<p^{-\frac{23}{15}}Y^{\frac{7}{9}}\},$$
$$D_{38}=\{(p,q):Y^{\frac{10}{27}}<p\le Y^{\frac{617}{1620}},\ p^{-\frac{70}{59}}Y^{\frac{34}{59}}<q<p^{-\frac{58}{49}}Y^{\frac{85}{147}}\},$$
$$D_{39}=\{(p,q):Y^{\frac{10}{27}}<p\le Y^{\frac{617}{1620}},\ p^{-\frac{58}{49}}Y^{\frac{85}{147}}<q<p^{-1}Y^{\frac{47}{81}}\},$$
$$D_{40}=\{(p,q):Y^{\frac{617}{1620}}<p\le Y^{\frac{1823}{4725}},\ p^{-\frac{70}{59}}Y^{\frac{34}{59}}<q<p^{-\frac{58}{49}}Y^{\frac{85}{147}}\},$$
$$D_{41}=\{(p,q):Y^{\frac{617}{1620}}<p\le Y^{\frac{1823}{4725}},\ p^{-\frac{58}{49}}Y^{\frac{85}{147}}<q<p^{-\frac{29}{19}}Y^{\frac{89}{114}}\},$$
$$D_{42}=\{(p,q):Y^{\frac{1823}{4725}}<p\le Y^{\frac{3041}{7830}},\ Y^{\frac{16}{135}}<q<p^{-\frac{58}{49}}Y^{\frac{85}{147}}\},$$
$$D_{43}=\{(p,q):Y^{\frac{1823}{4725}}<p\le Y^{\frac{3041}{7830}},\ p^{-\frac{58}{49}}Y^{\frac{85}{147}}<q<p^{-\frac{29}{19}}Y^{\frac{89}{114}}\},$$
$$D_{44}=\{(p,q):Y^{\frac{3041}{7830}}<p\le Y^{\frac{11}{27}},\ Y^{\frac{16}{135}}<q<p^{-\frac{29}{19}}Y^{\frac{89}{114}}\},$$
$$D_{45}=\{(p,q):Y^{\frac{11}{27}}<p\le Y^{\frac{34}{81}},\ Y^{\frac{16}{135}}<q<p^{-\frac{29}{19}}Y^{\frac{89}{114}}\},$$

$$D_{46} = \{(p,q) : Y^{\frac{11}{27}} < p \le Y^{\frac{34}{81}},\ p^{-\frac{29}{19}} Y^{\frac{89}{114}} < q < p^{-\frac{35}{23}} Y^{\frac{18}{23}}\},$$
$$D_{47} = \{(p,q) : Y^{\frac{11}{27}} < p \le Y^{\frac{34}{81}},\ p^{-\frac{23}{8}} Y^{\frac{17}{12}} < q < p^{-\frac{1}{4}} Y^{\frac{25}{72}}\},$$
$$D_{48} = \{(p,q) : Y^{\frac{34}{81}} < p \le Y^{\frac{49}{114}},\ Y^{\frac{16}{135}} < q < p^{-\frac{29}{19}} Y^{\frac{89}{114}}\},$$
$$D_{49} = \{(p,q) : Y^{\frac{34}{81}} < p \le Y^{\frac{49}{114}},\ p^{-\frac{29}{19}} Y^{\frac{89}{114}} < q < p^{-\frac{35}{23}} Y^{\frac{18}{23}}\},$$
$$D_{50} = \{(p,q) : Y^{\frac{34}{81}} < p \le Y^{\frac{49}{114}},\ p^{-\frac{29}{10}} Y^{\frac{17}{12}} < q < p^{-\frac{1}{4}} Y^{\frac{25}{72}}\},$$
$$D_{51} = \{(p,q) : Y^{\frac{49}{114}} < p \le Y^{\frac{3397}{7830}},\ Y^{\frac{16}{135}} < q < p^{-\frac{29}{19}} Y^{\frac{89}{114}}\},$$
$$D_{52} = \{(p,q) : Y^{\frac{49}{114}} < p \le Y^{\frac{3397}{7830}},\ p^{-\frac{29}{10}} Y^{\frac{17}{12}} < q < p^{\frac{1}{7}} Y^{\frac{4}{21}}\},$$
$$D_{53} = \{(p,q) : Y^{\frac{3397}{7830}} < p \le Y^{\frac{4}{9}},\ p^{-\frac{35}{12}} Y^{\frac{17}{12}} < q < p^{-\frac{29}{10}} Y^{\frac{17}{12}}\},$$
$$D_{54} = \{(p,q) : Y^{\frac{3397}{7830}} < p \le Y^{\frac{4}{9}},\ p^{-\frac{29}{10}} Y^{\frac{17}{12}} < q < p^{\frac{1}{7}} Y^{\frac{4}{21}}\},$$
$$D_{55} = \{(p,q) : Y^{\frac{4}{9}} < p \le Y^{\frac{701}{1566}},\ p^{-\frac{29}{10}} Y^{\frac{17}{12}} < q < p^{-\frac{1}{5}} Y^{\frac{3}{10}}\},$$
$$D_{56} = \{(p,q) : Y^{\frac{4}{9}} < p \le Y^{\frac{701}{1566}},\ p^{-\frac{6}{5}} Y^{\frac{34}{45}} < q < p^{\frac{1}{7}} Y^{\frac{4}{21}}\},$$
$$D_{57} = \{(p,q) : Y^{\frac{701}{1566}} < p \le Y^{\frac{41}{90}},\ Y^{\frac{16}{135}} < q < p^{-\frac{1}{5}} Y^{\frac{3}{10}}\},$$
$$D_{58} = \{(p,q) : Y^{\frac{701}{1566}} < p \le Y^{\frac{41}{90}},\ p^{-\frac{6}{5}} Y^{\frac{34}{45}} < q < p^{\frac{1}{7}} Y^{\frac{4}{21}}\},$$
$$D_{59} = \{(p,q) : Y^{\frac{41}{90}} < p \le Y^{\frac{17}{36}},\ Y^{\frac{16}{135}} < q < p^{\frac{1}{7}} Y^{\frac{4}{21}}\},$$
$$D_{60} = \{(p,q) : Y^{\frac{17}{36}} < p \le Y^{\frac{16}{33}},\ Y^{\frac{16}{135}} < q < p^{-1} Y^{\frac{46}{63}}\},$$
$$D_{61} = \{(p,q) : Y^{\frac{16}{33}} < p \le Y^{\frac{307}{630}},\ Y^{\frac{16}{135}} < q < p^{-\frac{58}{9}} Y^{\frac{89}{27}}\},$$
$$D_{62} = \{(p,q) : Y^{\frac{16}{33}} < p \le Y^{\frac{307}{630}},\ Y^{\frac{17}{99}} < q < p^{-1} Y^{\frac{46}{63}}\},$$
$$D_{63} = \{(p,q) : Y^{\frac{307}{630}} < p \le Y^{\frac{211}{432}},\ Y^{\frac{16}{135}} < q < p^{-\frac{58}{9}} Y^{\frac{89}{27}}\},$$
$$D_{64} = \{(p,q) : Y^{\frac{307}{630}} < p \le Y^{\frac{211}{432}},\ Y^{\frac{17}{99}} < q < p^{-6} Y^{\frac{19}{6}}\},$$
$$D_{65} = \{(p,q) : Y^{\frac{211}{432}} < p \le Y^{\frac{281}{570}},\ Y^{\frac{16}{135}} < q < p^{-\frac{58}{9}} Y^{\frac{89}{27}}\},$$
$$D_{66} = \{(p,q) : Y^{\frac{211}{432}} < p \le Y^{\frac{281}{570}},\ p^{-\frac{58}{9}} Y^{\frac{89}{27}} < q < p^{-\frac{70}{11}} Y^{\frac{36}{11}}\},$$
$$D_{67} = \{(p,q) : Y^{\frac{211}{432}} < p \le Y^{\frac{281}{570}},\ Y^{\frac{17}{99}} < q < p^{-6} Y^{\frac{19}{6}}\},$$
$$D_{68} = \{(p,q) : Y^{\frac{281}{570}} < p \le Y^{\frac{429}{870}},\ Y^{\frac{16}{135}} < q < p^{-\frac{58}{9}} Y^{\frac{89}{27}}\},$$
$$D_{69} = \{(p,q) : Y^{\frac{429}{870}} < p \le Y^{\frac{1}{2}},\ Y^{\frac{17}{99}} < q < p^{-1} Y^{\frac{40}{57}}\}.$$

10. Bounds of the sieve functions

LEMMA 24. *We have*

$$\begin{aligned} &\Omega_{11} + \Omega_{14} + \Omega_{17} + \Omega_{19} + \Omega_{21} + \Omega_{23} + \Omega_{25} + \Omega_{27} + \Omega_{28} + \Omega_{31} \\ &\quad + \Omega_{34} + \Omega_{37} + \Omega_{39} + \Omega_{41} + \Omega_{43} + \Omega_{44} + \Omega_{45} + \Omega_{48} + \Omega_{51} \\ &\ge 0.422726 \cdot \frac{C(n)Y}{\log Y \log N}. \end{aligned}$$

Proof. We have

$$\Omega_{11}=\sum_{Y^{\frac{17}{63}}<p\le Y^{\frac{17}{57}}}\ \sum_{p^{-\frac{6}{5}}Y^{\frac{17}{30}}<q<p^{-\frac{1}{8}}Y^{\frac{7}{24}}} S(\mathcal{A}_{pq},q)=\sum_{pqr+p_1=n}1,$$

where $pqr\le 2Y$, $N-Y<p_1\le N$, $Y^{\frac{17}{63}}<p\le Y^{\frac{17}{57}}$, $p^{-\frac{6}{5}}Y^{\frac{17}{30}}<q<p^{-\frac{1}{8}}Y^{\frac{7}{24}}$, and the least prime factor of r is greater than q.

Let $h=q$ and $k=r$. By Lemma 8 with region (i), (8) holds. Then Lemmas 18 and 21 yield

$$\begin{aligned}
\Omega_{11}&=\frac{C(n)}{\log N}\sum_{Y^{\frac{17}{63}}<p\le Y^{\frac{17}{57}}}\ \sum_{p^{-\frac{6}{5}}Y^{\frac{17}{30}}<q<p^{-\frac{1}{8}}Y^{\frac{7}{24}}} S(\mathcal{B}_{pq},q)+O\Big(\frac{\varepsilon C(n)Y}{\log^2 N}\Big)\\
&=\frac{C(n)Y}{\log N}\sum_{Y^{\frac{17}{63}}<p\le Y^{\frac{17}{57}}}\ \sum_{p^{-\frac{6}{5}}Y^{\frac{17}{30}}<q<p^{-\frac{1}{8}}Y^{\frac{7}{24}}}\frac{1}{pq\log q}w\Big(\frac{\log(Y/(pq))}{\log q}\Big)\\
&\quad+O\Big(\frac{\varepsilon C(n)Y}{\log^2 N}\Big)\\
&=\frac{C(n)Y}{\log Y\log N}\int_{\frac{17}{63}}^{\frac{17}{57}}\frac{dt}{t}\int_{\frac{17}{30}-\frac{6}{5}t}^{\frac{7}{24}-\frac{t}{8}}\frac{1}{u^2}w\Big(\frac{1-t-u}{u}\Big)\,du+O\Big(\frac{\varepsilon C(n)Y}{\log^2 N}\Big)\\
&=\frac{C(n)Y}{\log Y\log N}\Bigg(\int_{\frac{17}{63}}^{\frac{17}{57}}\frac{dt}{t}\int_{\frac{17}{30}-\frac{6}{5}t}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\Big(1+\log\Big(\frac{1-t}{u}-2\Big)\Big)\,du\\
&\quad+\int_{\frac{17}{63}}^{\frac{17}{57}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{7}{24}-\frac{t}{8}}\frac{du}{u(1-t-u)}+O(\varepsilon)\Bigg)\\
&\ge(0.012655+0.015106)\cdot\frac{C(n)Y}{\log Y\log N}\\
&=0.027761\cdot\frac{C(n)Y}{\log Y\log N}.
\end{aligned}$$

Using the above discussion and Lemma 20, we have

$$\begin{aligned}
&\Omega_{14}+\Omega_{17}+\Omega_{19}+\Omega_{21}+\Omega_{23}+\Omega_{25}+\Omega_{27}+\Omega_{28}+\Omega_{31}\\
&\qquad+\Omega_{34}+\Omega_{37}+\Omega_{39}+\Omega_{41}+\Omega_{43}+\Omega_{44}+\Omega_{45}+\Omega_{48}+\Omega_{51}\\
&=\frac{C(n)Y}{\log Y\log N}\Bigg(\int_{\frac{17}{57}}^{\frac{17}{54}}\frac{dt}{t}\int_{\frac{17}{33}-\frac{12}{11}t}^{\frac{7}{24}-\frac{t}{8}}\frac{1}{u^2}w\Big(\frac{1-t-u}{u}\Big)\,du\\
&\quad+\int_{\frac{17}{54}}^{\frac{221}{693}}\frac{dt}{t}\int_{\frac{17}{99}}^{\frac{7}{24}-\frac{t}{8}}\frac{1}{u^2}w\Big(\frac{1-t-u}{u}\Big)\,du+\int_{\frac{221}{693}}^{\frac{109}{330}}\frac{dt}{t}\int_{\frac{17}{99}}^{\frac{113}{198}-t}\frac{1}{u^2}w\Big(\frac{1-t-u}{u}\Big)\,du
\end{aligned}$$

$$+\int_{\frac{109}{330}}^{\frac{151}{450}}\frac{dt}{t}\int_{\frac{17}{99}}^{\frac{20}{9}-6t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du+\int_{\frac{151}{450}}^{\frac{34}{99}}\frac{dt}{t}\int_{\frac{17}{99}}^{\frac{1}{8}+\frac{t}{4}}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du$$

$$+\int_{\frac{151}{450}}^{\frac{34}{99}}\frac{dt}{t}\int_{\frac{5}{9}-t}^{\frac{7}{54}+\frac{t}{3}}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du+\int_{\frac{34}{99}}^{\frac{25}{72}}\frac{dt}{t}\int_{\frac{85}{147}-\frac{58}{49}t}^{\frac{1}{8}+\frac{t}{4}}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du$$

$$+\int_{\frac{25}{72}}^{\frac{10}{27}}\frac{dt}{t}\int_{\frac{85}{147}-\frac{58}{49}t}^{\frac{7}{9}-\frac{23}{15}t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du+\int_{\frac{10}{27}}^{\frac{617}{1620}}\frac{dt}{t}\int_{\frac{85}{147}-\frac{58}{49}t}^{\frac{47}{81}-t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du$$

$$+\int_{\frac{617}{1620}}^{\frac{3041}{7830}}\frac{dt}{t}\int_{\frac{85}{147}-\frac{58}{49}t}^{\frac{89}{114}-\frac{29}{19}t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du$$

$$+\int_{\frac{3041}{7830}}^{\frac{3397}{7830}}\frac{dt}{t}\int_{\frac{16}{135}}^{\frac{89}{114}-\frac{29}{19}t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du+O(\varepsilon)\Bigg)$$

$$\geq\frac{C(n)Y}{\log Y\log N}\Bigg(\int_{\frac{17}{57}}^{\frac{17}{54}}\frac{dt}{t}\int_{\frac{17}{33}-\frac{12}{11}t}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{17}{57}}^{\frac{17}{54}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{7}{24}-\frac{t}{8}}\frac{du}{u(1-t-u)}$$

$$+\int_{\frac{17}{54}}^{\frac{221}{693}}\frac{dt}{t}\int_{\frac{17}{99}}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{17}{54}}^{\frac{221}{693}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{7}{24}-\frac{t}{8}}\frac{du}{u(1-t-u)}$$

$$+\int_{\frac{221}{693}}^{\frac{109}{330}}\frac{dt}{t}\int_{\frac{17}{99}}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{221}{693}}^{\frac{109}{330}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{113}{198}-t}\frac{du}{u(1-t-u)}$$

$$+\int_{\frac{109}{330}}^{\frac{1}{3}}\frac{dt}{t}\int_{\frac{17}{99}}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{109}{330}}^{\frac{1}{3}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{20}{9}-6t}\frac{du}{u(1-t-u)}$$

$$+\int_{\frac{1}{3}}^{\frac{151}{450}}\frac{dt}{t}\int_{\frac{17}{99}}^{\frac{20}{9}-6t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{151}{450}}^{\frac{34}{99}}\frac{dt}{t}\int_{\frac{17}{99}}^{\frac{1}{8}+\frac{t}{4}}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{151}{450}}^{\frac{34}{99}}\frac{dt}{t}\int_{\frac{5}{9}-t}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{151}{450}}^{\frac{34}{99}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{7}{54}+\frac{t}{3}}\frac{du}{u(1-t-u)}$$

$$+\int_{\frac{34}{99}}^{\frac{25}{72}}\frac{dt}{t}\int_{\frac{85}{147}-\frac{58}{49}t}^{\frac{1}{8}+\frac{t}{4}}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{25}{72}}^{\frac{193}{549}}\frac{dt}{t}\int_{\frac{85}{147}-\frac{58}{49}t}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{25}{72}}^{\frac{193}{549}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{7}{9}-\frac{23}{15}t}\frac{du}{u(1-t-u)}$$

$$+0.5607\int_{\frac{193}{549}}^{\frac{10}{27}}\frac{dt}{t}\int_{\frac{85}{147}-\frac{58}{49}t}^{\frac{1}{4}(1-t)}\frac{du}{u^2}$$

$$+\int_{\frac{193}{549}}^{\frac{10}{27}}\frac{dt}{t}\int_{\frac{1}{4}(1-t)}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{193}{549}}^{\frac{10}{27}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{7}{9}-\frac{23}{15}t}\frac{du}{u(1-t-u)}$$

$$+0.5607\int_{\frac{10}{27}}^{\frac{617}{1620}}\frac{dt}{t}\int_{\frac{85}{147}-\frac{58}{49}t}^{\frac{1}{4}(1-t)}\frac{du}{u^2}$$

$$+ \int_{\frac{10}{27}}^{\frac{617}{1620}} \frac{dt}{t} \int_{\frac{1}{4}(1-t)}^{\frac{47}{81}-t} \frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right) du$$

$$+ 0.5607 \int_{\frac{617}{1620}}^{\frac{3041}{7830}} \frac{dt}{t} \int_{\frac{85}{147}-\frac{58}{49}t}^{\frac{1}{4}(1-t)} \frac{du}{u^2}$$

$$+ \int_{\frac{617}{1620}}^{\frac{3041}{7830}} \frac{dt}{t} \int_{\frac{1}{4}(1-t)}^{\frac{89}{114}-\frac{29}{19}t} \frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right) du$$

$$+ 0.5607 \int_{\frac{3041}{7830}}^{\frac{121}{291}} \frac{dt}{t} \int_{\frac{16}{135}}^{\frac{1}{4}(1-t)} \frac{du}{u^2}$$

$$+ \int_{\frac{3041}{7830}}^{\frac{121}{291}} \frac{dt}{t} \int_{\frac{1}{4}(1-t)}^{\frac{89}{114}-\frac{29}{19}t} \frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right) du$$

$$+ 0.5607 \int_{\frac{121}{291}}^{\frac{3397}{7830}} \frac{dt}{t} \int_{\frac{16}{135}}^{\frac{89}{114}-\frac{29}{19}t} \frac{du}{u^2} + O(\varepsilon)\Bigg)$$

$$\begin{aligned}
&\geq (0.035949 + 0.010981 + 0.010207 + 0.002956 \\
&\quad + 0.026780 + 0.007110 + 0.006712 + 0.000780 \\
&\quad + 0.004363 + 0.013697 + 0.001020 + 0.005290 \\
&\quad + 0.007124 + 0.009992 + 0.003228 + 0.010892 \\
&\quad + 0.045105 + 0.006377 + 0.016938 + 0.023556 \\
&\quad + 0.017909 + 0.014602 + 0.066888 + 0.026127 \\
&\quad + 0.020382) \cdot \frac{C(n)Y}{\log Y \log N} \\
&= 0.394965 \cdot \frac{C(n)Y}{\log Y \log N}.
\end{aligned}$$

The proof of Lemma 24 is complete.

LEMMA 25. *We have*

$$\begin{aligned}
\Phi &= \Omega_{47} + \Omega_{50} + \Omega_{52} + \Omega_{54} + \Omega_{55} + \Omega_{56} + \Omega_{57} + \Omega_{58} + \Omega_{59} \\
&\quad + \Omega_{60} + \Omega_{61} + \Omega_{62} + \Omega_{63} + \Omega_{64} + \Omega_{65} + \Omega_{67} + \Omega_{68} + \Omega_{69} \\
&\geq 0.347181 \cdot \frac{C(n)Y}{\log Y \log N}.
\end{aligned}$$

Proof. On applying Lemmas 10, 18, 20 and 21, in the same way as in Lemma 24, we have

$$\Phi = \frac{C(n)Y}{\log Y \log N}\Bigg(\int_{\frac{11}{27}}^{\frac{34}{81}} \frac{dt}{t}\int_{\frac{17}{12}-\frac{23}{8}t}^{\frac{25}{72}-\frac{t}{4}} \frac{1}{u^2} w\Big(\frac{1-t-u}{u}\Big)\,du$$

$$+\int_{\frac{34}{81}}^{\frac{49}{114}} \frac{dt}{t}\int_{\frac{17}{12}-\frac{29}{10}t}^{\frac{25}{72}-\frac{t}{4}} \frac{1}{u^2} w\Big(\frac{1-t-u}{u}\Big)\,du$$

$$+\int_{\frac{49}{114}}^{\frac{4}{9}} \frac{dt}{t}\int_{\frac{17}{12}-\frac{29}{10}t}^{\frac{4}{21}+\frac{t}{7}} \frac{1}{u^2} w\Big(\frac{1-t-u}{u}\Big)\,du$$

$$+\int_{\frac{4}{9}}^{\frac{701}{1566}} \frac{dt}{t}\int_{\frac{17}{12}-\frac{29}{10}t}^{\frac{3}{10}-\frac{t}{5}} \frac{1}{u^2} w\Big(\frac{1-t-u}{u}\Big)\,du$$

$$+\int_{\frac{4}{9}}^{\frac{701}{1566}} \frac{dt}{t}\int_{\frac{34}{45}-\frac{6}{5}t}^{\frac{4}{21}+\frac{t}{7}} \frac{1}{u^2} w\Big(\frac{1-t-u}{u}\Big)\,du$$

$$+\int_{\frac{701}{1566}}^{\frac{41}{90}} \frac{dt}{t}\int_{\frac{16}{135}}^{\frac{3}{10}-\frac{t}{5}} \frac{1}{u^2} w\Big(\frac{1-t-u}{u}\Big)\,du$$

$$+\int_{\frac{701}{1566}}^{\frac{41}{90}} \frac{dt}{t}\int_{\frac{34}{45}-\frac{6}{5}t}^{\frac{4}{21}+\frac{t}{7}} \frac{1}{u^2} w\Big(\frac{1-t-u}{u}\Big)\,du$$

$$+\int_{\frac{41}{90}}^{\frac{17}{36}} \frac{dt}{t}\int_{\frac{16}{135}}^{\frac{4}{21}+\frac{t}{7}} \frac{1}{u^2} w\Big(\frac{1-t-u}{u}\Big)\,du$$

$$+\int_{\frac{17}{36}}^{\frac{16}{33}} \frac{dt}{t}\int_{\frac{16}{135}}^{\frac{46}{63}-t} \frac{1}{u^2} w\Big(\frac{1-t-u}{u}\Big)\,du + \int_{\frac{16}{33}}^{\frac{307}{630}} \frac{dt}{t}\int_{\frac{16}{135}}^{\frac{89}{27}-\frac{58}{9}t} \frac{1}{u^2} w\Big(\frac{1-t-u}{u}\Big)\,du$$

$$+\int_{\frac{16}{33}}^{\frac{307}{630}} \frac{dt}{t}\int_{\frac{17}{99}}^{\frac{46}{63}-t} \frac{1}{u^2} w\Big(\frac{1-t-u}{u}\Big)\,du + \int_{\frac{307}{630}}^{\frac{281}{570}} \frac{dt}{t}\int_{\frac{16}{135}}^{\frac{89}{27}-\frac{58}{9}t} \frac{1}{u^2} w\Big(\frac{1-t-u}{u}\Big)\,du$$

$$+\int_{\frac{307}{630}}^{\frac{281}{570}} \frac{dt}{t}\int_{\frac{17}{99}}^{\frac{19}{6}-6t} \frac{1}{u^2} w\Big(\frac{1-t-u}{u}\Big)\,du$$

$$
+\int_{\frac{281}{570}}^{\frac{429}{870}} \frac{dt}{t} \int_{\frac{16}{135}}^{\frac{89}{27}-\frac{58}{9}t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du
$$

$$
+\int_{\frac{429}{870}}^{\frac{1}{2}} \frac{dt}{t} \int_{\frac{17}{99}}^{\frac{40}{57}-t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du + O(\varepsilon)\Bigg)
$$

$$
\geq \frac{C(n)Y}{\log Y \log N}\Bigg(\int_{\frac{11}{27}}^{\frac{34}{81}} \frac{dt}{t} \int_{\frac{17}{12}-\frac{23}{8}t}^{\frac{25}{72}-\frac{t}{4}} \frac{du}{u(1-t-u)}
$$

$$
+\int_{\frac{34}{81}}^{\frac{65}{154}} \frac{dt}{t} \int_{\frac{17}{12}-\frac{29}{10}t}^{\frac{25}{72}-\frac{t}{4}} \frac{du}{u(1-t-u)}
$$

$$
+\int_{\frac{65}{154}}^{\frac{49}{114}} \frac{dt}{t} \int_{\frac{17}{12}-\frac{29}{10}t}^{\frac{1}{3}(1-t)} \frac{1}{u(1-t-u)} \left(1+\log\left(\frac{1-t}{u}-2\right)\right) du
$$

$$
+\int_{\frac{65}{154}}^{\frac{49}{114}} \frac{dt}{t} \int_{\frac{1}{3}(1-t)}^{\frac{25}{72}-\frac{t}{4}} \frac{du}{u(1-t-u)}
$$

$$
+\int_{\frac{49}{114}}^{\frac{70}{159}} \frac{dt}{t} \int_{\frac{17}{12}-\frac{29}{10}t}^{\frac{1}{3}(1-t)} \frac{1}{u(1-t-u)} \left(1+\log\left(\frac{1-t}{u}-2\right)\right) du
$$

$$
+\int_{\frac{49}{114}}^{\frac{70}{159}} \frac{dt}{t} \int_{\frac{1}{3}(1-t)}^{\frac{4}{21}+\frac{t}{7}} \frac{du}{u(1-t-u)}
$$

$$
+0.5607 \int_{\frac{70}{159}}^{\frac{4}{9}} \frac{dt}{t} \int_{\frac{17}{12}-\frac{29}{10}t}^{\frac{1}{4}(1-t)} \frac{du}{u^2}
$$

$$
+\int_{\frac{70}{159}}^{\frac{4}{9}} \frac{dt}{t} \int_{\frac{1}{4}(1-t)}^{\frac{1}{3}(1-t)} \frac{1}{u(1-t-u)} \left(1+\log\left(\frac{1-t}{u}-2\right)\right) du
$$

$$
+\int_{\frac{70}{159}}^{\frac{4}{9}} \frac{dt}{t} \int_{\frac{1}{3}(1-t)}^{\frac{4}{21}+\frac{t}{7}} \frac{du}{u(1-t-u)} + 0.5607 \int_{\frac{4}{9}}^{\frac{701}{1566}} \frac{dt}{t} \int_{\frac{17}{12}-\frac{29}{10}t}^{\frac{1}{4}(1-t)} \frac{du}{u^2}
$$

$$
+\int_{\frac{4}{9}}^{\frac{701}{1566}} \frac{dt}{t} \int_{\frac{1}{4}(1-t)}^{\frac{1}{3}(1-t)} \frac{1}{u(1-t-u)} \left(1+\log\left(\frac{1-t}{u}-2\right)\right) du
$$

$$+\int_{\frac{4}{9}}^{\frac{701}{1566}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{3}{10}-\frac{t}{5}}\frac{du}{u(1-t-u)}+\int_{\frac{4}{9}}^{\frac{701}{1566}}\frac{dt}{t}\int_{\frac{34}{45}-\frac{6}{5}t}^{\frac{4}{21}+\frac{t}{7}}\frac{du}{u(1-t-u)}$$

$$+0.5607\int_{\frac{701}{1566}}^{\frac{41}{90}}\frac{dt}{t}\int_{\frac{16}{135}}^{\frac{1}{4}(1-t)}\frac{du}{u^2}$$

$$+\int_{\frac{701}{1566}}^{\frac{41}{90}}\frac{dt}{t}\int_{\frac{1}{4}(1-t)}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{701}{1566}}^{\frac{41}{90}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{3}{10}-\frac{t}{5}}\frac{du}{u(1-t-u)}+\int_{\frac{701}{1566}}^{\frac{41}{90}}\frac{dt}{t}\int_{\frac{34}{45}-\frac{6}{5}t}^{\frac{4}{21}+\frac{t}{7}}\frac{du}{u(1-t-u)}$$

$$+0.5607\int_{\frac{41}{90}}^{\frac{17}{36}}\frac{dt}{t}\int_{\frac{16}{135}}^{\frac{1}{4}(1-t)}\frac{du}{u^2}$$

$$+\int_{\frac{41}{90}}^{\frac{17}{36}}\frac{dt}{t}\int_{\frac{1}{4}(1-t)}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{41}{90}}^{\frac{17}{36}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{4}{21}+\frac{t}{7}}\frac{du}{u(1-t-u)}+0.5607\int_{\frac{17}{36}}^{\frac{16}{33}}\frac{dt}{t}\int_{\frac{16}{135}}^{\frac{1}{4}(1-t)}\frac{du}{u^2}$$

$$+\int_{\frac{17}{36}}^{\frac{16}{33}}\frac{dt}{t}\int_{\frac{1}{4}(1-t)}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{17}{36}}^{\frac{16}{33}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{46}{63}-t}\frac{du}{u(1-t-u)}+0.5607\int_{\frac{16}{33}}^{\frac{307}{630}}\frac{dt}{t}\int_{\frac{16}{135}}^{\frac{1}{4}(1-t)}\frac{du}{u^2}$$

$$+\int_{\frac{16}{33}}^{\frac{307}{630}}\frac{dt}{t}\int_{\frac{1}{4}(1-t)}^{\frac{89}{27}-\frac{58}{9}t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{16}{33}}^{\frac{307}{630}}\frac{dt}{t}\int_{\frac{17}{99}}^{\frac{46}{63}-t}\frac{du}{u(1-t-u)}+0.5607\int_{\frac{307}{630}}^{\frac{329}{669}}\frac{dt}{t}\int_{\frac{16}{135}}^{\frac{1}{4}(1-t)}\frac{du}{u^2}$$

$$+\int_{\frac{307}{630}}^{\frac{329}{669}}\frac{dt}{t}\int_{\frac{1}{4}(1-t)}^{\frac{89}{27}-\frac{58}{9}t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+ 0.5607 \int_{\frac{329}{669}}^{\frac{281}{570}} \frac{dt}{t} \int_{\frac{16}{135}}^{\frac{89}{27}-\frac{58}{9}t} \frac{du}{u^2} + \int_{\frac{307}{630}}^{\frac{281}{570}} \frac{dt}{t} \int_{\frac{17}{99}}^{\frac{19}{6}-6t} \frac{du}{u(1-t-u)}$$

$$+ 0.5607 \int_{\frac{281}{570}}^{\frac{429}{870}} \frac{dt}{t} \int_{\frac{16}{135}}^{\frac{89}{27}-\frac{58}{9}t} \frac{du}{u^2} + \int_{\frac{429}{870}}^{\frac{1}{2}} \frac{dt}{t} \int_{\frac{17}{99}}^{\frac{40}{57}-t} \frac{du}{u(1-t-u)} + O(\varepsilon)\Big)$$

$$\begin{aligned}
&\geq (0.005888 + 0.003229 + 0.002778 + 0.011651 \\
&\quad + 0.015140 + 0.020407 + 0.001606 + 0.009394 \\
&\quad + 0.008706 + 0.003624 + 0.007141 + 0.002659 \\
&\quad + 0.003233 + 0.011243 + 0.017671 + 0.006828 \\
&\quad + 0.009875 + 0.019672 + 0.037042 + 0.040808 \\
&\quad + 0.011345 + 0.027966 + 0.031531 + 0.001851 \\
&\quad + 0.004715 + 0.005786 + 0.003084 + 0.003848 \\
&\quad + 0.000433 + 0.010176 + 0.000003 + 0.007848) \cdot \frac{C(n)Y}{\log Y \log N} \\
&= 0.347181 \cdot \frac{C(n)Y}{\log Y \log N}.
\end{aligned}$$

The proof of Lemma 25 is complete.

LEMMA 26. *We have*

$$\begin{aligned}
\Omega_4 + \Omega_6 + \Omega_8 + \Omega_{10} + \Omega_{13} + \Omega_{16} + \Omega_{18} + \Omega_{20} \\
+ \Omega_{22} + \Omega_{24} + \Omega_{26} + \Omega_{29} + \Omega_{32} + \Omega_{35} \\
\geq 0.298946 \cdot \frac{C(n)Y}{\log Y \log N}.
\end{aligned}$$

P r o o f. We have

$$\begin{aligned}
\Omega_4 &= \sum_{Y^{\frac{41}{180}} < p \leq Y^{\frac{8}{33}}} \sum_{p^{-1}Y^{\frac{41}{90}} < q < p} S(\mathcal{A}_{pq}, q) \\
&= \sum_{Y^{\frac{41}{180}} < p \leq Y^{\frac{8}{33}}} \sum_{p^{-1}Y^{\frac{41}{90}} < q < p} S(\mathcal{A}_{pq}, Y^{\delta}) \\
&\quad - \sum_{Y^{\frac{41}{180}} < p \leq Y^{\frac{8}{33}}} \sum_{p^{-1}Y^{\frac{41}{90}} < q < p} \sum_{Y^{\delta} < r < Y^{\frac{16}{135}}} S(\mathcal{A}_{pqr}, r) \\
&\quad - \sum_{Y^{\frac{41}{180}} < p \leq Y^{\frac{8}{33}}} \sum_{p^{-1}Y^{\frac{41}{90}} < q < p} \sum_{Y^{\frac{16}{135}} < r < \min(q, (\frac{2Y}{pq})^{\frac{1}{2}})} S(\mathcal{A}_{pqr}, r) \\
&= \Phi_1 - \Phi_2 - \Phi_3.
\end{aligned} \tag{53}$$

Let $z = Y^\delta$ and $D = D(p,q) = Y^{\frac{19}{36}}/(pq)$. Applying Iwaniec's sieve method, we have

$$\Phi_1 \le \frac{C(n)Y}{\log z \log N} \sum_{Y^{\frac{41}{180}}<p\le Y^{\frac{8}{33}}} \sum_{p^{-1}Y^{\frac{41}{90}}<q<p} \frac{1}{pq} F\left(\frac{\log D}{\log z}\right)$$
$$+ \sum_{Y^{\frac{41}{180}}<p\le Y^{\frac{8}{33}}} \sum_{p^{-1}Y^{\frac{41}{90}}<q<p} \sum_{r<Y^{\frac{19}{36}}/(pq)} a(r)\widetilde{r}(\mathcal{A}, pqr) + O\left(\frac{\delta C(n)Y}{\log^2 N}\right).$$

Let $m = pqr$. Then Lemma 17 yields

$$\sum_{Y^{\frac{41}{180}}<p\le Y^{\frac{8}{33}}} \sum_{p^{-1}Y^{\frac{41}{90}}<q<p} \sum_{r<Y^{\frac{19}{36}}/(pq)} a(r)\widetilde{r}(\mathcal{A}, pqr) = O(Y \log^{-5} N).$$

Hence,

$$\Phi_1 \le \frac{e^{-\gamma}}{\delta} \cdot \frac{C(n)Y}{\log Y \log N} \sum_{Y^{\frac{41}{180}}<p\le Y^{\frac{8}{33}}} \sum_{p^{-1}Y^{\frac{41}{90}}<q<p} \frac{1}{pq} + O\left(\frac{\delta C(n)Y}{\log^2 N}\right)$$
$$= \frac{C(n)}{\log N} \sum_{Y^{\frac{41}{180}}<p\le Y^{\frac{8}{33}}} \sum_{p^{-1}Y^{\frac{41}{90}}<q<p} S(\mathcal{B}_{pq}, Y^\delta) + O\left(\frac{\delta C(n)Y}{\log^2 N}\right).$$

In the same way, we can get the lower bound

$$\Phi_1 \ge \frac{C(n)}{\log N} \sum_{Y^{\frac{41}{180}}<p\le Y^{\frac{8}{33}}} \sum_{p^{-1}Y^{\frac{41}{90}}<q<p} S(\mathcal{B}_{pq}, Y^\delta) + O\left(\frac{\delta C(n)Y}{\log^2 N}\right).$$

Thus we have the asymptotic formula

$$\Phi_1 = \frac{C(n)}{\log N} \sum_{Y^{\frac{41}{180}}<p\le Y^{\frac{8}{33}}} \sum_{p^{-1}Y^{\frac{41}{90}}<q<p} S(\mathcal{B}_{pq}, Y^\delta) + O\left(\frac{\delta C(n)Y}{\log^2 N}\right). \tag{54}$$

Using Lemma 19, we have

$$\Phi_2 = \frac{C(n)}{\log N} \sum_{Y^{\frac{41}{180}}<p\le Y^{\frac{8}{33}}} \sum_{p^{-1}Y^{\frac{41}{90}}<q<p} \sum_{Y^\delta<r<Y^{\frac{16}{135}}} S(\mathcal{B}_{pqr}, r)$$
$$+ O\left(\frac{\delta C(n)Y}{\log^2 N}\right). \tag{55}$$

By Lemma 13 with region (i), (8) holds. Then Lemma 18 yields

$$\Phi_3 = \frac{C(n)}{\log N} \sum_{Y^{\frac{41}{180}}<p\le Y^{\frac{8}{33}}} \sum_{p^{-1}Y^{\frac{41}{90}}<q<p} \sum_{Y^{\frac{16}{135}}<r<\min(q,(\frac{2Y}{pq})^{\frac{1}{2}})} S(\mathcal{B}_{pqr}, r)$$
$$+ O\left(\frac{\delta C(n)Y}{\log^2 N}\right). \tag{56}$$

Gathering together (53)–(56), we have

$$
\begin{aligned}
\Omega_4 &= \frac{C(n)}{\log N} \sum_{Y^{\frac{41}{180}} < p \le Y^{\frac{8}{33}}} \sum_{p^{-1} Y^{\frac{41}{90}} < q < p} S(\mathcal{B}_{pq}, q) + O\bigg(\frac{\delta C(n) Y}{\log^2 N}\bigg) \\
&= \frac{C(n)Y}{\log Y \log N} \int_{\frac{41}{180}}^{\frac{8}{33}} \frac{dt}{t} \int_{\frac{41}{90}-t}^{t} \frac{1}{u^2} w\bigg(\frac{1-t-u}{u}\bigg)\, du + O\bigg(\frac{\delta C(n) Y}{\log^2 N}\bigg) \\
&= \frac{C(n)Y}{\log Y \log N} \int_{\frac{41}{180}}^{\frac{8}{33}} \frac{dt}{t} \int_{\frac{41}{90}-t}^{t} \frac{1}{u(1-t-u)} \bigg(1 + \log\bigg(\frac{1-t}{u} - 2\bigg)\bigg)\, du \\
&\quad + O\bigg(\frac{\delta C(n) Y}{\log^2 N}\bigg) \\
&\ge 0.009636 \cdot \frac{C(n)}{\log Y \log N}.
\end{aligned}
$$

In the same way, it can be shown that

$$
\begin{aligned}
&\Omega_6 + \Omega_8 + \Omega_{10} + \Omega_{13} + \Omega_{16} + \Omega_{18} + \Omega_{20} + \Omega_{22} + \Omega_{24} + \Omega_{26} + \Omega_{29} + \Omega_{32} + \Omega_{35} \\
&\quad = \frac{C(n)Y}{\log Y \log N} \bigg(\int_{\frac{8}{33}}^{\frac{37}{150}} \frac{dt}{t} \int_{\frac{41}{90}-t}^{\frac{16}{33}-t} \frac{1}{u^2} w\bigg(\frac{1-t-u}{u}\bigg)\, du \\
&\quad + \int_{\frac{37}{150}}^{\frac{31}{99}} \frac{dt}{t} \int_{\frac{701}{1566}-t}^{\frac{16}{33}-t} \frac{1}{u^2} w\bigg(\frac{1-t-u}{u}\bigg)\, du \\
&\quad + \int_{\frac{31}{99}}^{\frac{859}{2610}} \frac{dt}{t} \int_{\frac{701}{1566}-t}^{\frac{89}{201}-\frac{58}{67}t} \frac{1}{u^2} w\bigg(\frac{1-t-u}{u}\bigg)\, du \\
&\quad + \int_{\frac{859}{2610}}^{\frac{10}{27}} \frac{dt}{t} \int_{\frac{16}{135}}^{\frac{89}{201}-\frac{58}{67}t} \frac{1}{u^2} w\bigg(\frac{1-t-u}{u}\bigg)\, du + O(\delta)\bigg) \\
&\quad \ge \frac{C(n)Y}{\log Y \log N} \bigg(\int_{\frac{8}{33}}^{\frac{37}{150}} \frac{dt}{t} \int_{\frac{41}{90}-t}^{\frac{16}{33}-t} \frac{1}{u(1-t-u)} \bigg(1 + \log\bigg(\frac{1-t}{u} - 2\bigg)\bigg)\, du \\
&\quad + \int_{\frac{37}{150}}^{\frac{619}{2349}} \frac{dt}{t} \int_{\frac{701}{1566}-t}^{\frac{16}{33}-t} \frac{1}{u(1-t-u)} \bigg(1 + \log\bigg(\frac{1-t}{u} - 2\bigg)\bigg)\, du
\end{aligned}
$$

$$+ 0.5607 \int_{\frac{619}{2349}}^{\frac{31}{99}} \frac{dt}{t} \int_{\frac{701}{1566}-t}^{\frac{1}{4}(1-t)} \frac{du}{u^2}$$

$$+ \int_{\frac{619}{2349}}^{\frac{31}{99}} \frac{dt}{t} \int_{\frac{1}{4}(1-t)}^{\frac{16}{33}-t} \frac{1}{u(1-t-u)} \left(1 + \log\left(\frac{1-t}{u} - 2\right)\right) du$$

$$+ 0.5607 \int_{\frac{31}{99}}^{\frac{859}{2610}} \frac{dt}{t} \int_{\frac{701}{1566}-t}^{\frac{89}{201}-\frac{58}{67}t} \frac{du}{u^2}$$

$$+ 0.5607 \int_{\frac{859}{2610}}^{\frac{10}{27}} \frac{dt}{t} \int_{\frac{16}{135}}^{\frac{89}{201}-\frac{58}{67}t} \frac{du}{u^2} + O(\delta)\Bigg)$$

$$\geq (0.005530 + 0.031194 + 0.066046 + 0.050732 + 0.051408 + 0.084400) \cdot \frac{C(n)Y}{\log Y \log N}$$

$$= 0.289310 \cdot \frac{C(n)Y}{\log Y \log N}.$$

The proof of Lemma 26 is complete.

Lemma 27. *We have*

$$\Phi = \Omega_{30} + \Omega_{33} + \Omega_{36} + \Omega_{38} + \Omega_{40} + \Omega_{42} + \Omega_{46} + \Omega_{49} + \Omega_{53} + \Omega_{66} \geq 0.024582 \cdot \frac{C(n)Y}{\log Y \log N}.$$

Proof. On applying Lemmas 11, 12, 18, 20 and 21, in the same way as in Lemma 24, we have

$$\Phi = \frac{C(n)Y}{\log Y \log N}\Bigg(\int_{\frac{34}{99}}^{\frac{1649}{4752}} \frac{dt}{t} \int_{\frac{85}{99}-2t}^{\frac{85}{147}-\frac{58}{49}t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du$$

$$+ \int_{\frac{1649}{4752}}^{\frac{1823}{4725}} \frac{dt}{t} \int_{\frac{34}{59}-\frac{70}{59}t}^{\frac{85}{147}-\frac{58}{49}t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du$$

$$+ \int_{\frac{1823}{4725}}^{\frac{3041}{7830}} \frac{dt}{t} \int_{\frac{16}{135}}^{\frac{85}{147}-\frac{58}{49}t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du$$

$$+ \int_{\frac{11}{27}}^{\frac{49}{114}} \frac{dt}{t} \int_{\frac{89}{114}-\frac{29}{19}t}^{\frac{18}{23}-\frac{35}{23}t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du$$

$$+\int_{\frac{3397}{7830}}^{\frac{4}{9}}\frac{dt}{t}\int_{\frac{17}{12}-\frac{35}{12}t}^{\frac{17}{12}-\frac{29}{10}t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du$$

$$+\int_{\frac{211}{432}}^{\frac{281}{570}}\frac{dt}{t}\int_{\frac{89}{27}-\frac{58}{9}t}^{\frac{36}{11}-\frac{70}{11}t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du+O(\varepsilon)\Bigg)$$

$$\geq\frac{C(n)Y}{\log Y\log N}\left(\int_{\frac{34}{99}}^{\frac{1649}{4752}}\frac{dt}{t}\int_{\frac{85}{99}-2t}^{\frac{85}{147}-\frac{58}{49}t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du\right.$$

$$+\int_{\frac{1649}{4752}}^{\frac{77}{221}}\frac{dt}{t}\int_{\frac{34}{59}-\frac{70}{59}t}^{\frac{85}{147}-\frac{58}{49}t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+0.5607\int_{\frac{77}{221}}^{\frac{193}{549}}\frac{dt}{t}\int_{\frac{34}{59}-\frac{70}{59}t}^{\frac{1}{4}(1-t)}\frac{du}{u^2}$$

$$+\int_{\frac{77}{221}}^{\frac{193}{549}}\frac{dt}{t}\int_{\frac{1}{4}(1-t)}^{\frac{85}{147}-\frac{58}{49}t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+0.5607\int_{\frac{193}{549}}^{\frac{1823}{4725}}\frac{dt}{t}\int_{\frac{34}{59}-\frac{70}{59}t}^{\frac{85}{147}-\frac{58}{49}t}\frac{du}{u^2}+0.5607\int_{\frac{1823}{4725}}^{\frac{3041}{7830}}\frac{dt}{t}\int_{\frac{16}{135}}^{\frac{85}{147}-\frac{58}{49}t}\frac{du}{u^2}$$

$$+\int_{\frac{11}{27}}^{\frac{121}{291}}\frac{dt}{t}\int_{\frac{89}{114}-\frac{29}{19}t}^{\frac{18}{23}-\frac{35}{23}t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+0.5607\int_{\frac{121}{291}}^{\frac{49}{117}}\frac{dt}{t}\int_{\frac{89}{114}-\frac{29}{19}t}^{\frac{1}{4}(1-t)}\frac{du}{u^2}$$

$$+\int_{\frac{121}{291}}^{\frac{49}{117}}\frac{dt}{t}\int_{\frac{1}{4}(1-t)}^{\frac{18}{23}-\frac{35}{23}t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+0.5607\int_{\frac{49}{117}}^{\frac{49}{114}}\frac{dt}{t}\int_{\frac{89}{114}-\frac{29}{19}t}^{\frac{18}{23}-\frac{35}{23}t}\frac{du}{u^2}$$

$$+\int_{\frac{3397}{7830}}^{\frac{7}{16}}\frac{dt}{t}\int_{\frac{17}{12}-\frac{35}{12}t}^{\frac{17}{12}-\frac{29}{10}t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+ 0.5607 \int_{\frac{7}{16}}^{\frac{70}{159}} \frac{dt}{t} \int_{\frac{17}{12}-\frac{35}{12}t}^{\frac{1}{4}(1-t)} \frac{du}{u^2}$$

$$+ \int_{\frac{7}{16}}^{\frac{70}{159}} \frac{dt}{t} \int_{\frac{1}{4}(1-t)}^{\frac{17}{12}-\frac{29}{10}t} \frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right) du$$

$$+ 0.5607 \int_{\frac{70}{159}}^{\frac{4}{9}} \frac{dt}{t} \int_{\frac{17}{12}-\frac{35}{12}t}^{\frac{17}{12}-\frac{29}{10}t} \frac{du}{u^2}$$

$$+ \int_{\frac{211}{432}}^{\frac{329}{669}} \frac{dt}{t} \int_{\frac{89}{27}-\frac{58}{9}t}^{\frac{36}{11}-\frac{70}{11}t} \frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right) du$$

$$+ 0.5607 \int_{\frac{329}{669}}^{\frac{281}{570}} \frac{dt}{t} \int_{\frac{89}{27}-\frac{58}{9}t}^{\frac{1}{4}(1-t)} \frac{du}{u^2}$$

$$+ \int_{\frac{329}{669}}^{\frac{281}{570}} \frac{dt}{t} \int_{\frac{1}{4}(1-t)}^{\frac{36}{11}-\frac{70}{11}t} \frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right) du + O(\varepsilon)\Bigg)$$

$$\geq (0.000303 + 0.000244 + 0.000282 + 0.000276 + 0.008036$$
$$+ 0.000393 + 0.001843 + 0.000369 + 0.000357 + 0.003087$$
$$+ 0.001545 + 0.000679 + 0.000634 + 0.002319 + 0.002910$$
$$+ 0.000330 + 0.000975) \cdot \frac{C(n)Y}{\log Y \log N}$$
$$= 0.024582 \cdot \frac{C(n)Y}{\log Y \log N}.$$

The proof of Lemma 27 is complete.

LEMMA 28. *We have*

$$\Phi = \Omega_1 + \Omega_2 + \Omega_3 + \Omega_5 + \Omega_7 + \Omega_9 + \Omega_{12} + \Omega_{15}$$
$$\geq 1.010623 \cdot \frac{C(n)Y}{\log Y \log N}.$$

Proof. The main idea in this lemma is adopted from [21]. We have

$$(57) \quad \Omega_1 \geq \sum_{Y^{\frac{16}{135}} < p \leq Y^{\frac{89}{462}-10^{-8}}} \sum_{Y^{\frac{16}{135}} < q < p} S(\mathcal{A}_{pq}, q)$$
$$= \sum_{Y^{\frac{16}{135}} < p \leq Y^{\frac{89}{462}-10^{-8}}} \sum_{Y^{\frac{16}{135}} < q < p} S(\mathcal{A}_{pq}, Y^{\delta})$$

$$- \sum_{Y^{\frac{16}{135}} < p \le Y^{\frac{89}{462} - 10^{-8}}} \sum_{Y^{\frac{16}{135}} < q < p} \sum_{Y^{\delta} < r < Y^{\frac{16}{135}}} S(\mathcal{A}_{pqr}, r)$$
$$- \sum_{Y^{\frac{16}{135}} < p \le Y^{\frac{89}{462} - 10^{-8}}} \sum_{Y^{\frac{16}{135}} < q < p} \sum_{Y^{\frac{16}{135}} < r < \min(q, (\frac{2Y}{pq})^{\frac{1}{2}})} S(\mathcal{A}_{pqr}, r)$$
$$= \Phi_1 - \Phi_2 - \Phi_3.$$

Let $z = Y^{\delta}$ and $D(p,q) = Y^{\frac{19}{36}}/(pq)$. Using Iwaniec's sieve method, in the same way as in Lemma 26, we have

$$(58)\quad \Phi_1 = \frac{C(n)}{\log N} \sum_{Y^{\frac{16}{135}} < p \le Y^{\frac{89}{462} - 10^{-8}}} \sum_{Y^{\frac{16}{135}} < q < p} S(\mathcal{B}_{pq}, Y^{\delta}) + O\left(\frac{\delta C(n) Y}{\log^2 N}\right).$$

Lemma 19 yields

$$(59)\quad \Phi_2 = \frac{C(n)}{\log N} \sum_{Y^{\frac{16}{135}} < p \le Y^{\frac{89}{462} - 10^{-8}}} \sum_{Y^{\frac{16}{135}} < q < p} \sum_{Y^{\delta} < r < Y^{\frac{16}{135}}} S(\mathcal{B}_{pqr}, r) + O\left(\frac{\delta C(n) Y}{\log^2 N}\right).$$

Hence,

$$(60)\quad \Phi_1 - \Phi_2 = \frac{C(n)}{\log N} \sum_{Y^{\frac{16}{135}} < p \le Y^{\frac{89}{462} - 10^{-8}}} \sum_{Y^{\frac{16}{135}} < q < p} S(\mathcal{B}_{pq}, Y^{\frac{16}{135}}) + O\left(\frac{\delta C(n) Y}{\log^2 N}\right)$$
$$= \frac{C(n) Y}{\log Y \log N} \cdot \frac{135}{16} \int_{\frac{16}{135}}^{\frac{89}{462} - 10^{-8}} \frac{dt}{t} \int_{\frac{16}{135}}^{t} \frac{1}{u} w\left(\frac{135}{16}(1 - t - u)\right) du + O\left(\frac{\delta C(n) Y}{\log^2 N}\right)$$
$$\ge \frac{C(n) Y}{\log Y \log N} \left(\frac{135}{16} \int_{\frac{16}{135}}^{\frac{89}{462}} \frac{dt}{t} \int_{\frac{16}{135}}^{t} \frac{1}{u} w\left(\frac{135}{16}(1 - t - u)\right) du - 10^{-6}\right)$$
$$\ge \frac{C(n) Y}{\log Y \log N} \left(0.5612 \cdot \frac{135}{16} \int_{\frac{16}{135}}^{\frac{89}{462}} \frac{dt}{t} \int_{\frac{16}{135}}^{t} \frac{du}{u} - 10^{-6}\right)$$
$$\ge 0.558649 \cdot \frac{C(n) Y}{\log Y \log N}.$$

Note that if $q < (2Y/p)^{\frac{1}{3}}$, then $q < (2Y/(pq))^{\frac{1}{2}}$. We have

$$(61)\quad \begin{aligned} \Phi_3 &\le \sum_{Y^{\frac{16}{135}} < p \le Y^{\frac{89}{462} - 10^{-8}}} \sum_{Y^{\frac{16}{135}} < q < p} \sum_{Y^{\frac{16}{135}} < r < q} S(\mathcal{A}_{pqr}, Y^{\frac{16}{135}}) \\ &= \sum_{Y^{\frac{16}{135}} < p \le Y^{\frac{89}{462} - 10^{-8}}} \sum_{Y^{\frac{16}{135}} < q < p} \sum_{Y^{\frac{16}{135}} < r < q} S(\mathcal{A}_{pqr}, Y^{\delta}) \\ &\quad - \sum_{Y^{\frac{16}{135}} < p \le Y^{\frac{89}{462} - 10^{-8}}} \sum_{Y^{\frac{16}{135}} < q < p} \sum_{Y^{\frac{16}{135}} < r < q} \sum_{Y^{\delta} < s < Y^{\frac{16}{135}}} S(\mathcal{A}_{pqrs}, s) \\ &= \Phi_4 - \Phi_5. \end{aligned}$$

Let $z = Y^{\delta}$ and $D(p, q, r) = Y^{10^{-8}}$. Using Iwaniec's sieve method and Lemma 16, in the same way as in Lemma 26, we have

$$(62)\quad \Phi_4 = \frac{C(n)}{\log N} \sum_{Y^{\frac{16}{135}} < p \le Y^{\frac{89}{462} - 10^{-8}}} \sum_{Y^{\frac{16}{135}} < q < p} \sum_{Y^{\frac{16}{135}} < r < q} S(\mathcal{B}_{pqr}, Y^{\delta}) + O\left(\frac{\delta C(n) Y}{\log^2 N}\right).$$

Lemma 19 yields

$$(63)\quad \Phi_5 = \frac{C(n)}{\log N} \sum_{Y^{\frac{16}{135}} < p \le Y^{\frac{89}{462} - 10^{-8}}} \sum_{Y^{\frac{16}{135}} < q < p} \sum_{Y^{\frac{16}{135}} < r < q} \sum_{Y^{\delta} < s < Y^{\frac{16}{135}}} S(\mathcal{B}_{pqrs}, s) + O\left(\frac{\delta C(n) Y}{\log^2 N}\right).$$

We therefore have

$$\begin{aligned} \Phi_3 &\le \frac{C(n)}{\log N} \sum_{Y^{\frac{16}{135}} < p \le Y^{\frac{89}{462} - 10^{-8}}} \sum_{Y^{\frac{16}{135}} < q < p} \sum_{Y^{\frac{16}{135}} < r < q} S(\mathcal{B}_{pqr}, Y^{\frac{16}{135}}) + O\left(\frac{\delta C(n) Y}{\log^2 N}\right) \\ &\le \frac{C(n) Y}{\log Y \log N} \cdot \frac{135}{16} \int_{\frac{16}{135}}^{\frac{89}{462}} \frac{dt}{t} \int_{\frac{16}{135}}^{t} \frac{du}{u} \int_{\frac{16}{135}}^{u} \frac{1}{v} w\left(\frac{135}{16}(1 - t - u - v)\right) dv \\ &\le \frac{C(n) Y}{\log Y \log N} \left(0.5617 \cdot \frac{135}{16} \int_{\frac{16}{135}}^{\frac{71}{405}} \frac{dt}{t} \int_{\frac{16}{135}}^{t} \frac{du}{u} \int_{\frac{16}{135}}^{u} \frac{dv}{v} \right. \end{aligned}$$

$$+ 0.5617 \cdot \frac{135}{16} \int_{\frac{71}{405}}^{\frac{89}{462}} \frac{dt}{t} \int_{\frac{16}{135}}^{\frac{1}{2}(\frac{71}{135}-t)} \frac{du}{u} \int_{\frac{16}{135}}^{u} \frac{dv}{v}$$

$$+ 0.5644 \cdot \frac{135}{16} \int_{\frac{71}{405}}^{\frac{89}{462}} \frac{dt}{t} \int_{\frac{1}{2}(\frac{71}{135}-t)}^{t} \frac{du}{u} \int_{\frac{16}{135}}^{u} \frac{dv}{v}\Bigg)$$

$$\le (0.047392 + 0.030099 + 0.013113) \cdot \frac{C(n)Y}{\log Y \log N}$$

$$= 0.090604 \cdot \frac{C(n)Y}{\log Y \log N}.$$

Hence,

$$\Omega_1 \ge 0.468045 \cdot \frac{C(n)Y}{\log Y \log N}. \tag{64}$$

In the same way, it can be shown that

$$\Omega_2 + \Omega_3 + \Omega_5 + \Omega_7 + \Omega_9 + \Omega_{12} + \Omega_{15}$$

$$\ge \frac{C(n)Y}{\log Y \log N}\Bigg(\frac{135}{16} \int_{\frac{89}{462}}^{\frac{17}{54}} \frac{dt}{t} \int_{\frac{16}{135}}^{\frac{89}{288}-\frac{29}{48}t} \frac{1}{u} w\Big(\frac{135}{16}(1-t-u)\Big)\, du$$

$$- \frac{135}{16} \int_{\frac{89}{462}}^{\frac{17}{54}} \frac{dt}{t} \int_{\frac{16}{135}}^{\frac{89}{288}-\frac{29}{48}t} \frac{du}{u} \int_{\frac{16}{135}}^{u} \frac{1}{v} w\Big(\frac{135}{16}(1-t-u-v)\Big)\, dv - 10^{-6}\Bigg)$$

$$\ge \frac{C(n)Y}{\log Y \log N}\Bigg(0.5612 \cdot \frac{135}{16} \int_{\frac{89}{462}}^{\frac{17}{54}} \frac{dt}{t} \int_{\frac{16}{135}}^{\frac{89}{288}-\frac{29}{48}t} \frac{du}{u}$$

$$- 0.5617 \cdot \frac{135}{16} \int_{\frac{89}{462}}^{\frac{13}{45}} \frac{dt}{t} \int_{\frac{16}{135}}^{\frac{1}{2}(\frac{71}{135}-t)} \frac{du}{u} \int_{\frac{16}{135}}^{u} \frac{dv}{v}$$

$$- 0.5644 \cdot \frac{135}{16} \int_{\frac{89}{462}}^{\frac{13}{45}} \frac{dt}{t} \int_{\frac{1}{2}(\frac{71}{135}-t)}^{\frac{89}{288}-\frac{29}{48}t} \frac{du}{u} \int_{\frac{16}{135}}^{u} \frac{dv}{v}$$

$$- 0.5644 \cdot \frac{135}{16} \int_{\frac{13}{45}}^{\frac{17}{54}} \frac{dt}{t} \int_{\frac{16}{135}}^{\frac{89}{288}-\frac{29}{48}t} \frac{du}{u} \int_{\frac{16}{135}}^{u} \frac{dv}{v} - 10^{-6}\Bigg)$$

$$\ge (0.658048 - 0.044320 - 0.069979 - 0.001171) \cdot \frac{C(n)Y}{\log Y \log N}$$

$$= 0.542578 \cdot \frac{C(n)Y}{\log Y \log N}.$$

Hence,

$$\Phi \geq 1.010623 \cdot \frac{C(n)Y}{\log Y \log N}.$$

The proof of Lemma 28 is complete.

11. Proof of the Theorem. From Lemmas 24–28, it follows that

$$\Omega \geq 2.104057 \cdot \frac{C(n)Y}{\log Y \log N}. \tag{65}$$

Then using Lemmas 22 and 23, we obtain

$$T(n) = S(\mathcal{A}, (2Y)^{\frac{1}{2}}) + O(Y^{\frac{1}{2}}) \geq 0.011 \cdot \frac{C(n)Y}{\log Y \log N}.$$

Now (44) holds and the proof of the Theorem is complete.

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