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On the distribution of primitive abundant numbers

by

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A number m is *primitive abundant* if it is abundant ($\sigma(m) \geq 2m$), and all its proper divisors d are deficient ($\sigma(d) < 2d$), where $\sigma(m)$ is the sum of the divisors of m. Let $P(n)$ represent the number of primitive abundant numbers (p.a.n.) $\leq n$. In 1935, Erdős [2] proved the following result. For n sufficiently large,

$$n \cdot \exp[-c_1\sqrt{\log n \cdot \log\log n}] \leq P(n) \leq n \cdot \exp[-c_2\sqrt{\log n \cdot \log\log n}]$$

with $c_1 = 8$ and $c_2 = 1/25$. In 1985, Ivić [4] improved this, proving the inequalities with $c_1 = \sqrt{6} + \varepsilon$ and $c_2 = 1/\sqrt{12} - \varepsilon$. In this paper, we improve it to the following.

THEOREM. *For $n \geq n_0(\varepsilon)$*

$$n \cdot \exp[-(\sqrt{2}+\varepsilon)\sqrt{\log n \cdot \log\log n}] \leq P(n) \leq n \cdot \exp[-(1-\varepsilon)\sqrt{\log n \cdot \log\log n}].$$

I would like to thank Professor Carl Pomerance for suggesting the problem and for his abundant assistance.

The following notation will be standard throughout:

- $h(m) = \sigma(m)/m$,
- $E = e^{\sqrt{\log n \cdot \log\log n}}$, $L = \sqrt{\log n/\log\log n}$,
- $p_1 = p_1(m) =$ largest prime divisor of m,
- $q =$ largest squarefull divisor of m
  (n is squarefull means $p \mid n \Rightarrow p^2 \mid n$ for all primes p),
- $f =$ squarefree part of m; i.e. $f = m/q$,
- $p, p_j =$ prime numbers,
- $\varepsilon =$ arbitrarily small, positive numbers, not necessarily the same at each occurrence.

The method of proof, for both bounds, is a refinement of the one in [2] and [4]. For the upper bound, rather than divide into 3 cases (small p_1, large q, or large p_1 and small q), we divide into many cases, where both p_1 and q are restricted to short intervals. We are consequently able to combine either 2 or 3 bounds from the 3 original cases (Lemmas 7 and 9). Further, in the main case (large p_1 and small q), we are able to get an improvement by removing the restriction that the map constructed is 1-1 (Lemmas 10 and 11).

As for the lower bound, we still consider only numbers of the form $2^l \cdot p_k \dots p_1$. Rather than choosing all the primes from one small interval, the smallest is chosen to essentially be as small as possible, which has the effect of allowing the others to be chosen larger; hence there are more choices. At the same time, one must still restrict each to a short interval, to get sharp approximations of $h(p_j)$.

The upper bound. We first state some results from other papers that are used.

Lemma 1. *Let $F(x)$ be the number of squarefull numbers $n \le x$. Then*

$$F(x) \sim \frac{\zeta(3/2)}{\zeta(3)} x^{1/2}.$$

This result is proved in [3].

Lemma 2. *Let $\psi(x,y) = \sum_{n \le x,\, p_1(n) \le y} 1$ and $u = \log x / \log y$. Suppose $\log x < y < x^{o(1)}$. Then*

$$\psi(x,y) \le x/u^{(1+o(1))u}.$$

This result is proved in [1].

Lemma 3. *Suppose $\eta \ge 1$, $m = p_1 \dots p_t$, $p_1 \ge \dots \ge p_t$ and*

$$p_i \le \eta \cdot (p_{i+1} \dots p_t), \quad 1 \le i \le t-1.$$

Then for any D with $1 \le D < m$, there exists $d \mid m$ with

$$D/(\eta p_t) < d \le D.$$

This is equivalent to Lemma 4 of [5], with $a_i = \log p_i$.

We now prove some preliminary results.

Lemma 4. *If m is a p.a.n., $m \le n$, and $m = p_1 \dots p_t$, where $p_1 \ge \dots \ge p_t$, then*

$$p_i \le 2 \log n \cdot p_{i+1} \dots p_t + 1, \quad i = 1, \dots, t.$$

Proof. Let $v = p_1 \dots p_i$ and $u = p_{i+1} \dots p_t$ (if $i = t$, then $u = 1$). If the lemma is false, there exists i such that $p_i - 1 > 2 \log n \cdot u$. Note that u

is deficient, and since $v \le n$, the number of distinct prime divisors v has is $\le \log n$. Thus

$$\begin{aligned} h(m) = h(u)h(v) &\le \Big(2 - \frac{1}{u}\Big) \prod_{p\mid v} \Big(1 + \frac{1}{p-1}\Big) \\ &< \Big(2 - \frac{1}{u}\Big) \prod_{p\mid v} \Big(1 + \frac{1}{2\log n \cdot u}\Big) < \Big(2 - \frac{1}{u}\Big)\Big(1 + \frac{1}{2\log n \cdot u}\Big)^{\log n} \\ &< \Big(2 - \frac{1}{u}\Big)\Big(1 + \frac{1}{2u} + \frac{1}{(2u)^2} + \ldots\Big) = 2. \end{aligned}$$

This contradicts the abundance of m.

Corollary. *If m is a p.a.n., $m \le n$, and $1 \le D < m$, then there exists $d \mid m$ with*

$$D/(2\log n + 1)^2 < d \le D.$$

Proof. This follows immediately from Lemmas 3 and 4, since $p_t \le 2\log n + 1$.

Lemma 5. *Let S be the set of m that satisfy* (i) $m \le n$ *and* (ii) $q \ge E^{\zeta}$. *Then*

$$|S| \ll n/E^{\zeta/2}.$$

Proof. Using Lemma 1 and partial summation we obtain

$$\begin{aligned} \sum_{\substack{m\le n\\ q\ge E^{\zeta}}} 1 \le \sum_{E^{\zeta}\le q\le n} \frac{n}{q} &= \sum_{E^{\zeta}\le q\le n} 1 + n \int_{E^{\zeta}}^{n} \Big(\sum_{E^{\zeta}\le q\le t} 1\Big)\frac{dt}{t^2} \\ &\ll n^{1/2} + n \int_{E^{\zeta}}^{n} t^{-3/2}\,dt \ll n \cdot E^{-\zeta/2}. \end{aligned}$$

Lemma 6. *Let S be the set of m that satisfy* (i) $m \le n$ *and* (ii) $p_1 \le E^{\beta}$. *Then for each $\varepsilon > 0$ there is a number $n_0(\varepsilon)$ such that if $n \ge n_0(\varepsilon)$ then*

$$|S| \le n/E^{1/(2\beta)-\varepsilon}.$$

Proof. This follows from Lemma 2 with $x = n, y = E^{\beta}$, since

$$\begin{aligned} u^u &= \exp[u \cdot \log u] \\ &\ge \exp\Big[\frac{\log n}{\beta\sqrt{\log n \cdot \log\log n}} \log([\log n]^{1/2-\varepsilon})\Big] \\ &= \exp\Big[\frac{1/2-\varepsilon}{\beta}\sqrt{\log n \cdot \log\log n}\Big]. \end{aligned}$$

Lemma 7. *Let S be the set of m that satisfy* (i) $m \le n$, (ii) $p_1 \le E^{\beta}$, *and* (iii) $E^{\zeta} \le q$. *Then for each $\varepsilon > 0$ there is a number $n_0(\varepsilon)$ such that if $n \ge n_0(\varepsilon)$ then*

$$|S| \le n/E^{1/(2\beta)+\zeta/2-\varepsilon}.$$

Proof. By Lemma 5, we may assume that $q < E^{1/\beta+\zeta}$. Then we may apply Lemma 6 to obtain

$$\psi(n/q; E^{\beta}) \le \frac{n/q}{E^{1/(2\beta)-\varepsilon}}.$$

Then, as in Lemma 5,

$$\sum_{\substack{m\le n,\, p_1\le E^{\beta}\\ E^{\zeta}\le q<E^{1/\beta+\zeta}}} 1 \le \sum_{E^{\zeta}\le q<E^{1/\beta+\zeta}} \psi(n/q; E^{\beta}) \le \frac{1}{E^{1/(2\beta)-\varepsilon}} \sum_{E^{\zeta}\le q<E^{1/\beta+\zeta}} \frac{n}{q} \le \frac{n}{E^{1/(2\beta)+\zeta/2-\varepsilon}}.$$

Lemma 8. *Let m be a p.a.n. with $p_1 \ge E^{\beta}$ and $p_1^2 \nmid m$. Then*

$$2 \le h(m) < 2 + 2/E^{\beta}.$$

Proof. Since $p_1^2 \nmid m$, we have $(m/p_1, p_1) = 1$. Also m/p_1 is deficient. Therefore

$$h(m) = h(m/p_1)h(p_1) < 2(1 + 1/p_1) \le 2(1 + 1/E^{\beta}).$$

Lemma 9. *Let S be the set of p.a.n.s m that satisfy* (i) $m \le n$, (ii) $E^{\beta} \le p_1 \le E^{\alpha}$, $0 < \beta \le \alpha$, (iii) $E^{\zeta} \le q \le E^{\gamma}$, $0 < \zeta \le \gamma$, *and $\beta > \gamma/2$,* (iv) *there exists $d \mid f$ such that $E^{c} \le d \le \frac{1}{2}E^{\beta/2}$. Then for each $\varepsilon > 0$ there is a number $n_0(\varepsilon)$ such that if $n \ge n_0(\varepsilon)$ then*

$$|S| \le n/E^{c+1/(2\alpha)+\zeta/2-\varepsilon}.$$

Proof. Define a map from S to $[1, n/E^{c}]$ by $m \mapsto m/d$. We claim that this map is 1-1.

If $m_1 \ne m_2$ and $d_1 = d_2$ then $m_1/d_1 \ne m_2/d_2$. So consider $m_1 \ne m_2$ and $d_1 \ne d_2$ and suppose that $m_1/d_1 = m_2/d_2$. Then

$$h(m_1/d_1) = h(m_2/d_2).$$

Since $(m_i/d_i, d_i) = 1$ we have

$$h(m_i) = h(m_i/d_i)h(d_i), \quad i = 1, 2.$$

It follows that

$$\frac{h(d_1)}{h(d_2)} = \frac{h(m_1)}{h(m_2)} < \frac{2 + 2/E^{\beta}}{2} = 1 + \frac{1}{E^{\beta}}$$

using Lemma 8 (which is valid by (ii) and (iii)).

On the other hand, since d_1 and d_2 are squarefree, $h(d_1) \neq h(d_2)$. Therefore, we may assume that $h(d_1)/h(d_2) > 1$. Thus, since d_2 is deficient,

$$\frac{h(d_1)}{h(d_2)} = \frac{\sigma(d_1)d_2}{d_1\sigma(d_2)} \geq 1 + \frac{1}{d_1\sigma(d_2)} > 1 + \frac{1}{2d_1d_2} \geq 1 + \frac{2}{E^\beta}$$

by (iv), which is a contradiction. Hence, the map is 1-1.

This gives us a 1-1 correspondence between S and a subset T of $[1, n/E^c]$. Since $d \mid f$, the squarefull part of m/d is the same as that of m, and since $d \leq \frac{1}{2}E^{\beta/2}$, p_1 is the same for both. Thus m/d satisfies (ii) and (iii). Applying Lemma 7 to T yields the result.

LEMMA 10. *Let S be the set of p.a.n.s m that satisfy* (i) $m \leq n$, (ii) $E^\beta \leq p_1 \leq E^\alpha$, (iii) $E^\zeta \leq q \leq E^\gamma$, *where* $\beta > \gamma/2$, (iv) *there exists* $d \mid f$ *such that* $E^\lambda \leq d \leq E^\eta$, *where* $\eta \geq \beta/2$. *Then*

$$|S| \leq n/E^{2\lambda - 3\eta + \beta + 1/(2\alpha) + \zeta/2 - \varepsilon}.$$

Proof. We follow the proof of Lemma 9, except that the map is not necessarily 1-1. Suppose it is at worst N to 1. If d_1 and d_2 are divisors as in (iv), we know that repeats satisfy

$$1 + \frac{1}{E^\beta} > \frac{\sigma(d_1)d_2}{d_1\sigma(d_2)} > 1$$

with $\sigma(d_i) < 2d_i$. It follows that

$$\frac{2d_1d_2}{E^\beta} > \sigma(d_1)d_2 - d_1\sigma(d_2) > 0$$

and thus, by (iv),

$$\frac{2E^{2\eta-\beta}}{\tau} > D > 0,$$

where

$$D = \frac{\sigma(d_1)}{\tau}d_2 - \frac{d_1}{\tau}\sigma(d_2) \quad \text{and} \quad \tau = (d_1, \sigma(d_1)).$$

For given values of D and d_1, d_2 is fixed $\bmod d_1/\tau$, so the number of possibilities for d_2 is

$$< \frac{E^\eta}{d_1/\tau} \leq E^{\eta-\lambda}\tau.$$

Thus, given d_1, the total number of possibilities for d_2 is

$$< \frac{2E^{2\eta-\beta}}{\tau}E^{\eta-\lambda}\tau = 2E^{3\eta-\lambda-\beta}.$$

This is a bound for N. It follows, as in Lemma 9, that

$$|S| \leq 2E^{3\eta-\lambda-\beta}\frac{n}{E^{\lambda+1/(2\alpha)+\zeta/2-\varepsilon}},$$

and the result follows.

LEMMA 11. *Let S be the set of p.a.n.s m that satisfy* (i) $m \le n$, (ii) $E^\beta \le p_1 \le E^{\beta+\varepsilon}$, (iii) $E^\zeta \le q \le E^{\zeta+\varepsilon}$, *where* $\beta > (\zeta+\varepsilon)/2$, (iv) *there exists* $d \mid f$ *with* $E^{\beta/2-\zeta/4} \le d \le E^{\beta/2+\zeta/2-\delta}$, *where* $\delta = (\zeta/2)\cdot(2/3)^{J-1}$ *and* $J = E^{o(1)}$. *Then*

$$|S| \le n/E^{1-\varepsilon}.$$

Proof. Let $\eta_j = \beta/2 + \zeta/2 - (3/2)^{j-1}\delta$ and $\lambda_j = \eta_{j+1}$ for $j = 1, \ldots, J$. Note that $\eta_J = \beta/2$. Let S_j be the set of p.a.n.s that satisfy (i)–(iii) and for which there exists $d \mid f$ with $E^{\lambda_j} \le d \le E^{\eta_j}$. Then $S = \bigcup_{j=1}^J S_j$ and therefore $|S| \le \sum_{j=1}^J |S_j|$. Now Lemma 10 applies to S_j and

$$2\lambda_j - 3\eta_j + \beta + 1/(2\beta) + \zeta/2 - \varepsilon = \beta/2 + 1/(2\beta) - \varepsilon$$

so it follows that

$$|S_j| \le n/E^{\beta/2+1/(2\beta)-\varepsilon} \le n/E^{1-\varepsilon}.$$

Since $J = E^{o(1)}$, the result follows.

LEMMA 12. *Let S be the set of p.a.n.s m that satisfy* (i) $m \le n$, (ii) $E^\beta \le p_1 \le E^{\beta+\varepsilon}$, (iii) $E^\zeta \le q \le E^{\zeta+\varepsilon}$, *where* $\beta > (\zeta+\varepsilon)/2$, (iv) *there exists* $d \mid f$ *with* $E^{\beta/2-\zeta/2-\delta-\varepsilon} \le d \le E^{\beta/2-\zeta/4}$, *where* $\delta = o(1)$. *Then*

$$|S| \le n/E^{1-\varepsilon}.$$

Proof. This follows immediately from Lemma 9, since $\min(\beta/2+1/(2\beta)) = 1$.

We now proceed to establish

$$P(n) \le n/E^{1-\varepsilon}.$$

First, by Lemma 5, those m with $q \ge E^2$ can be ignored. Next, it follows from the Corollary to Lemma 4 that if $q < E^2$, there exists $d \mid f$ with $E^{1-\varepsilon} < d \le \frac{1}{2}E^3$. Thus, by Lemma 9, those m with $p_1 \ge E^6$ and $q < E^2$ also can be ignored.

We take the remaining set of p.a.n.s $m \le n$, with $p_1 < E^6$ and $q < E^2$, divide it into $\ll (\log n)^2 = E^{o(1)}$ subsets, and establish the desired bound on each of these.

Specifically, let $S_{j,k}$ be the set of p.a.n.s $m \le n$, with

$$e^{j/L} \le p_1 \le e^{(j+1)/L} \quad \text{and} \quad e^{k/L} \le q \le e^{(k+1)/L}$$

for $0 \le j \le 6\log n$ and $0 \le k \le 2\log n$.

First, we will handle those $S_{j,k}$ with $j < k$. Noting that, with $\beta = (j+1)/\log n$ and $\zeta = k/\log n$, we have $\beta - \varepsilon < \zeta$, Lemma 7 yields

$$|S_{j,k}| \le n/E^{1/(2\beta)+\beta/2-\varepsilon} \le n/E^{1-\varepsilon}.$$

Now, for those with $j \ge k$, we will use Lemmas 11 and 12. First note that, by the Corollary to Lemma 4, if $q \le E^{\zeta+\varepsilon}$ then there exists a $d \mid f$

with $E^{\beta/2-\zeta/2-\delta-\varepsilon} \le d \le E^{\beta/2+\zeta/2-\delta}$. With J and δ as in Lemma 11, let $J = \log n = E^{o(1)}$, so that $\delta = o(1)$. Since $j \ge k$, setting $\beta = j/\log n$ and $\zeta = k/\log n$, we have $\beta \ge \zeta$. Thus the conditions of Lemmas 11 or 12 are satisfied by the p.a.n.s in $S_{j,k}$, and hence we have the desired bound.

The lower bound. Here we construct a set of numbers, show them to be primitive abundant, and underestimate the cardinality of the set. We first need to define many parameters.

$$\alpha \in (\sqrt{2} - 1/L, \sqrt{2}] \tag{1}$$

is chosen so that

$$k = \alpha L \in \mathbb{N}. \tag{2}$$

Now define

$$\delta = \frac{1}{8ke^k}, \tag{3}$$

$$t = [8k^2 e^k]. \tag{4}$$

We consider sequences (k_j) with

$$k_j \in \{0, 1\}, \quad \sum_{j=1}^{t} k_j = k - 2, \quad \text{and} \quad k_{t+1} = 1. \tag{5}$$

For each sequence we will define a set of numbers which will be shown to be primitive abundant. These sets will be disjoint, so $P(n)$ is at least the sum of their cardinalities. Now define β by

$$n = 8E^{\beta(k+1)}(4k)^{k-1}(1+\delta)^{\Sigma_{j=1}^{t+1} jk_j}. \tag{6}$$

We can now define

$$l = \left[\frac{\beta \log E}{\log 2}\right], \tag{7}$$

$$S = \sum_{j=1}^{t} \frac{k_j}{2k(1+\delta)^{j-1}}. \tag{8}$$

Note that, by (5),

$$0 < S < 1/2. \tag{9}$$

We claim that it follows from (6) that

$$\beta = 1/\sqrt{2} + o(1). \tag{10}$$

Therefore, from (2) and (7),

$$k = o(l). \tag{11}$$

Indeed, since we have $n = E^L$ from the definitions of E and L, (6) implies

$$E^{\beta k} < E^L < E^{\beta k+\beta} e^{k\log 4k} e^{tk\delta} < E^{\beta k+\beta} e^{k^2(1+\varepsilon)} = E^{\beta k+o(k)}.$$

Thus $\beta\alpha L < L < \beta\alpha L + o(L)$ by (2), which implies $1 - o(1) < \beta\alpha < 1$.

Our set of numbers will be those of the form $m = 2^l p_k \ldots p_1$, where $p_k < p_{k-1} < \ldots < p_1$ are primes chosen as follows:

$$p_k \in \left[\frac{1}{1-S}2^{l+1}, \frac{1+\delta}{1-S}2^{l+1}\right]. \tag{12}$$

We define intervals I_j by

$$I_j = [k(1+\delta)^{j-1}2^{l+2},\ k(1+\delta)^j 2^{l+2}), \quad j = 1, \ldots, t, \tag{13}$$

and we choose k_j primes from I_j to give us $p_{k-1}, \ldots, p_2$. Finally, we choose p_1 with

$$p_1 \in [k(1+\delta)^t 2^{l+2},\ k(1+\delta)^t 2^{l+3}]. \tag{14}$$

Note that, for any number m with $p_1 \ge 2q$, which holds for the numbers m of the above form (here $q = 2^l$), if we have $h(m/p_1) < 2$ then it follows that $h(d) < 2$ for all proper divisors d of m. Thus, we need to establish:

(A) $h(m/p_1) < 2$,
(B) $h(m) \ge 2$,
(C) $m \le n$,
(D) $\#\{m\} \ge n/E^{\sqrt{2}+\varepsilon}$.

Proof of (A). Since $q = 2^l$,

$$h(q) = 2 - \frac{1}{2^l} = 2\left(1 - \frac{1}{2^{l+1}}\right). \tag{15}$$

From (12) we have

$$h(p_k) = 1 + \frac{1}{p_k} \le 1 + \frac{1-S}{2^{l+1}}. \tag{16}$$

From (13) we have

$$h(p_{k-1}\ldots p_2) \le \prod_{j=1}^{t}\left(1 + \frac{k_j}{k(1+\delta)^{j-1}\cdot 2^{l+2}}\right).$$

Note that $\prod(1+\varepsilon_i) < 1 + \sum\varepsilon_i + (\sum\varepsilon_i)^2$ when each $\varepsilon_i > 0$ and $\sum\varepsilon_i < 1/2$. Using this fact with (8) and the above, we have

$$h(p_{k-1}\ldots p_2) < 1 + \frac{S}{2^{l+1}} + \frac{S^2}{2^{2l+2}}. \tag{17}$$

Combining (15)–(17) with (9), we have

$$\begin{aligned} h(m/p_1) &= h(q)h(p_k)h(p_{k-1}\dots p_2) \\ &\le 2\Big(1-\frac{1}{2^{l+1}}\Big)\Big(1+\frac{1-S}{2^{l+1}}\Big)\Big(1+\frac{S}{2^{l+1}}+\frac{S^2}{2^{2l+2}}\Big) \\ &= 2\Big(1-\frac{1}{2^{l+1}}\Big)\Big[1+\frac{1}{2^{l+1}}+\frac{S}{2^{2l+2}}+O\Big(\frac{1}{2^{3l}}\Big)\Big] \\ &< 2\Big[1-\frac{1}{2^{2l+2}}+\frac{1}{2^{2l+3}}+O\Big(\frac{1}{2^{3l}}\Big)\Big] < 2. \end{aligned}$$

P r o o f o f (B). From (12) and (14) we have

$$h(p_k) \ge 1 + \frac{1-S}{(1+\delta)\cdot 2^{l+1}}, \tag{18}$$

$$h(p_1) \ge 1 + \frac{1}{k(1+\delta)^t\cdot 2^{l+3}}. \tag{19}$$

From (13) and (8) we have

$$\begin{aligned} h(p_{k-1}\dots p_2) &> \prod_{j=1}^{t}\Big(1+\frac{k_j}{k(1+\delta)^j\cdot 2^{l+2}}\Big) \\ &> 1+\sum_{j=1}^{t}\frac{k_j}{2k(1+\delta)^j\cdot 2^{l+1}} = 1+\frac{S}{(1+\delta)\cdot 2^{l+1}}. \end{aligned} \tag{20}$$

Combining (15), (18)–(20) and (9) gives

$$\begin{aligned} h(m) &> 2\Big(1-\frac{1}{2^{l+1}}\Big)\Big(1+\frac{1-S}{(1+\delta)\cdot 2^{l+1}}\Big)\Big(1+\frac{S}{(1+\delta)\cdot 2^{l+1}}\Big) \\ &\quad\times\Big(1+\frac{1}{k(1+\delta)^t\cdot 2^{l+3}}\Big) \\ &= 2\Big[1+\Big(-1+\frac{1-S}{1+\delta}+\frac{S}{1+\delta}+\frac{1}{4k(1+\delta)^t}\Big)\frac{1}{2^{l+1}}+O\Big(\frac{1}{2^{2l}}\Big)\Big] \\ &= 2\Big[1+\Big(\frac{-\delta}{1+\delta}+\frac{1}{4k(1+\delta)^t}\Big)\frac{1}{2^{l+1}}+O\Big(\frac{1}{2^{2l}}\Big)\Big]. \end{aligned}$$

By (3) and (4), $(1+\delta)^t < (1+\delta)^{k/\delta} < e^k$ and therefore

$$\frac{1}{4k(1+\delta)^t}-\frac{\delta}{1+\delta} > \frac{1}{4ke^k}-\frac{1}{8ke^k} = \frac{1}{8ke^k}.$$

It follows from (11) that $h(m) > 2$.

Proof of (C). Using (7), (12)–(14), and (9) we have

$$\begin{aligned} m &= 2^l p_k(p_{k-1}\dots p_2)p_1 \\ &\le E^\beta \cdot 4(1+\delta)E^\beta(4kE^\beta)^{k-2}(1+\delta)^{\Sigma_{j=1}^t jk_j}\cdot 8kE^\beta(1+\delta)^t \\ &= 8E^{\beta(k+1)}(4k)^{k-1}(1+\delta)^{\Sigma_{j=1}^{t+1} jk_j}. \end{aligned}$$

Thus, by (6), $m \le n$.

Proof of (D). By (12)–(14), (7) and the prime number theorem,

$$P(n) \gg \sum_{(k_j)} \frac{\delta E^\beta}{\log E}\prod_{j=1}^t \left[\frac{4\delta k(1+\delta)^{j-1}E^\beta}{\log(4k(1+\delta)^j E^\beta)}\right]^{k_j}\frac{k(1+\delta)^t E^\beta}{\log(8k(1+\delta)^t E^\beta)}.$$

Note that (3), (4), (11), and (7) tell us

$$4k(1+\delta)^j < 4k(1+\delta)^{k/\delta} < 4ke^k = e^{o(l)} < E^\beta.$$

Likewise, for any constant c, $c^k = E^{o(1)}$. Thus, using (5),

$$P(n) \gg \sum_{(k_j)} \frac{\delta^{k-1}\cdot k^{k-1}\cdot E^{\beta k}(1+\delta)^{\Sigma_{j=1}^{t+1} jk_j}}{E^{o(1)}\cdot(\log E)^k}.$$

Note that, by (2), $(\log E)^k = e^{\alpha L\log\log E} = e^{(\alpha/2)L\log\log n(1+o(1))} = E^{\alpha/2+o(1)}$. Therefore, using (6) we find

$$P(n) \gg \sum_{(k_j)} \frac{\delta^{k-1}\cdot n}{E^{\alpha/2+\beta+o(1)}}.$$

Applying (1), (10), (5), (3), and (4) shows that

$$\begin{aligned} P(n) &\gg \sum_{(k_j)} \frac{\delta^{k-1}n}{E^{\sqrt{2}+o(1)}} = \binom{t}{k-2}\delta^{k-1}\frac{n}{E^{\sqrt{2}+o(1)}} \\ &\ge \left(\frac{t}{k}\right)^{k-2}\delta^{k-1}\frac{n}{E^{\sqrt{2}+o(1)}} = \frac{n}{E^{\sqrt{2}+o(1)}}. \end{aligned}$$

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