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On values of L-functions of totally real algebraic number fields at integers

by

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Dedicated to Professor H. Shimizu on the occasion of his 60th birthday

0. Let K be a totally real algebraic number field. In his paper [20], Siegel obtained explicit arithmetic expressions of the values of a zeta function of K at negative integers by using the method of restricting Hilbert–Eisenstein series for $\mathrm{SL}_2(\mathcal{O})$ to a diagonal, $\mathcal{O}$ denoting the ring of integers of K. Let us consider Hilbert–Eisenstein series of higher level whose 0th Fourier coefficients are special values of L-functions. Then a modified method of Siegel's gives formulas for the values of L-functions at integers, which is one of the purposes of the present paper. Such Eisenstein series have been considered for example in Shimura [18] and Deligne–Ribet [7]. However, for our purpose it is desirable that the Eisenstein series have many 0 as their 0th coefficients at cusps except for a specific cusp. After constructing such Eisenstein series, we give formulas for values of L-functions of K at integers. As a particular case, they turn out to be formulas for relative class numbers of totally imaginary quadratic extensions of K, where the exact form of fundamental units is not necessary. We also give several numerical examples of special values of L-functions and relative class numbers.

Our result is twofold. After Section 5, we take as K a real quadratic field. Under some condition on a character we obtain an elliptic modular form whose 0th coefficient is a product of two L-functions over $\mathbb{Q}$ and whose higher coefficients are elementary arithmetic. These modular forms can be applied to the investigation of numbers of representations of a natural number by a positive quadratic form with odd number of variables. We obtain a relation between special values of L-functions and numbers of representations by some such quadratic forms. For example, Gauss' three-square theorem is an easy consequence of our theorem.

1. Let $\mathfrak{H}$ denote the upper half plane $\{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$. For $N \in \mathbb{N}$, we put

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1,\ c \equiv 0 \pmod{N} \right\}$$

and

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

Let χ_0 be a Dirichlet character modulo N. Let $k \in \mathbb{N}$, and let Γ be $\Gamma_0(N)$ or $\Gamma_1(N)$. A holomorphic function f on $\mathfrak{H}$ is called a *modular form for Γ of weight k* if it satisfies (i) $f|A = f$ for $A \in \Gamma$, where $(f|A)(z) = (cz+d)^{-k} f(Az)$ with $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $Az = \frac{az+b}{cz+d}$, and (ii) f is holomorphic also at cusps. Let $\mathbf{M}_{k,\chi_0}(N)$ denote the space of modular forms f for $\Gamma_0(N)$ of weight k with character χ_0, that is, modular forms f for $\Gamma_1(N)$ which satisfy $f|A = \chi_0(d) f$ for any $A \in \Gamma_0(N)$. If χ_0 is trivial, we denote it by $\mathbf{M}_k(N)$, which is the space of modular forms for $\Gamma_0(N)$.

We set $\mathsf{e}(z) = \exp(2\pi\sqrt{-1}\, z)$. A modular form f for $\Gamma_1(N)$ has the Fourier expansion $f(z) = \sum_{n=0}^{\infty} a_n \mathsf{e}(nz)$ at the cusp $\sqrt{-1}\,\infty$. An operator U_l $(l \in \mathbb{N})$ on Fourier series is defined by

$$U_l(f)(z) = \sum_{n=0}^{\infty} a_{ln} \mathsf{e}(nz);$$

it maps $\mathbf{M}_k(N)$ to itself if any prime divisor of l is a factor of N (Atkin–Lehner [2]). We also consider a function for which the holomorphy condition in (ii) is replaced by meromorphy. Such a function is called a *meromorphic modular form*; its weight is not necessarily positive.

Let $\mathbf{M}^{\infty}_{k,\chi_0}(N)$ (resp. $\mathbf{M}^{0}_{k,\chi_0}(N)$, resp. $\mathbf{M}^{\infty,0}_{k,\chi_0}(N)$) denote the subspace of $\mathbf{M}_{k,\chi_0}(N)$ consisting of modular forms which vanish at all cusps but $\sqrt{-1}\,\infty$ (resp. 0, resp. $\sqrt{-1}\,\infty$ and 0). All of them coincide if $N = 1$, and the spaces $\mathbf{M}_{k,\chi_0}(N)$ and $\mathbf{M}^{\infty,0}_{k,\chi_0}(N)$ coincide if N is prime.

Since $\mathbf{M}^{\infty,0}_{k,\chi_0}(N)$ is of finite dimension, there are nontrivial linear relations satisfied by the 0th Fourier coefficient at 0 and first several coefficients at $\sqrt{-1}\,\infty$, of arbitrary modular forms in $\mathbf{M}^{\infty,0}_{k,\chi_0}(N)$. Let $N > 1$. We define $\mathrm{LR}_{k,\chi_0}(N)$ to be the set consisting of ordered sets $\{c_0, c'_0, c_{-1}, \ldots, c_{-n_0}\}$ where c_i's and c'_0 are constants such that the equality $c'_0 a^{(0)}_0 + \sum_{n=0}^{n_0} c_{-n} a_n = 0$ holds for the 0th Fourier coefficient $a^{(0)}_0$ at 0 and first $n_0 + 1$ coefficients $a_0, \ldots, a_{n_0}$ at $\sqrt{-1}\,\infty$ of any modular form f in $\mathbf{M}^{\infty,0}_{k,\chi_0}(N)$. Here we note that $a^{(0)}_0$ is a complex number so that $\lim_{z\to\infty} z^{-k} f(-1/z) = a^{(0)}_0$. If the modular form is in $\mathbf{M}^{\infty}_{k,\chi_0}(N)$ (resp. $\mathbf{M}^{0}_{k,\chi_0}(N)$), then the equality $\sum_{n=0}^{n_0} c_{-n} a_n = 0$ (resp. $c'_0 a^{(0)}_0 + \sum_{n=1}^{n_0} c_{-n} a_n = 0$) holds. Similarly for $N \geq 1$, $\mathrm{LR}'_{k,\chi_0}(N)$ is defined to be the set consisting of $\{c_0, c_{-1}, \ldots, c_{-n_0}\}$ for which the equality $\sum_{n=0}^{n_0} c_{-n}a_n = 0$ holds for any modular form in $\mathbf{M}_{k,\chi_0}^{\infty,0}(N)$. If χ_0 is trivial, then we omit χ_0 from $\mathbf{M}_{k,\chi_0}^{\infty,0}(N)$, $\mathrm{LR}_{k,\chi_0}(N)$ etc., for example $\mathrm{LR}_k(N) := \mathrm{LR}_{k,\chi_0}(N)$.

Elements of $\mathrm{LR}_{k,\chi_0}(N)$, $\mathrm{LR}'_{k,\chi_0}(N)$ can be obtained by the following method initially employed by Siegel [20] in the case $N = 1$. Cusps of $\Gamma_0(N)$ are represented as i/M $(i, M \in \mathbb{N},\ (i, M) = 1,\ M \mid N)$, and two such cusps i/M, i'/M' are equivalent if and only if M equals M', and i' is congruent to i modulo M or modulo N/M. The cusp $\sqrt{-1}\,\infty$ (resp. 0) is equivalent to $1/N$ (resp. $1/1$). A local parameter at a cusp i/M is $\mathsf{e}((M^2, N)/N \times Az)$, where $A \in \mathrm{SL}_2(\mathbb{Z})$ maps i/M to $\sqrt{-1}\,\infty$.

Lemma 1. *Let $k \in \mathbb{N}$. Let $h(z) = \sum_{n=-n_0}^{\infty} c_n \mathsf{e}(-nz)$ be a meromorphic modular form for $\Gamma_0(N)$ of weight $-k+2$ with character χ_0^{-1} having the only pole at $\sqrt{-1}\,\infty$. Let $c_0^{(i/M)}$ be the 0th Fourier coefficient at the cusp i/M. Let $f(z) \in \mathbf{M}_{k,\chi_0}(N)$, $f(z) = \sum_{n=0}^{\infty} a_n \mathsf{e}(nz)$, and let $a_0^{(i/M)}$ be its 0th coefficient at i/M. Then*

$$\sum_{M,i} (N/(M^2, N)) c_0^{(i/M)} a_0^{(i/M)} + \sum_{n=0}^{n_0} c_{-n} a_n = 0,$$

where the first summation is taken over a complete set of representatives of cusps of $\Gamma_0(N)$.

Proof. By the assumption, $f(z)h(z)\,dz$ is a meromorphic differential form on the compactified modular curve for $\Gamma_0(N)$ with poles only at cusps. Then by the residue theorem, the residue of the differential form, which is $(2\sqrt{-1}\,\pi)^{-1}$ times the left hand side of the equality in the lemma, is equal to 0. This shows our assertion. ∎

Corollary. *Let h and c_n be as in the lemma. Let $c_0^{(0)}$ denote the 0th Fourier coefficient of h at the cusp 0. Then $\{c_0, Nc_0^{(0)}, c_{-1}, \ldots, c_{-n_0}\} \in \mathrm{LR}_{k,\chi_0}(N)$. If $c_0^{(0)} = 0$, then $\{c_0, c_{-1}, \ldots, c_{-n_0}\} \in \mathrm{LR}'_{k,\chi_0}(N)$.*

For a prime p, denote by v_p the p-adic valuation. For a proper divisor M of N, $\mathrm{LR}_k(N)$ is not a subset of $\mathrm{LR}_k(M)$ in general since $\mathbf{M}_k^{\infty,0}(M) \not\subset \mathbf{M}_k^{\infty,0}(N)$ in general. Suppose that $v_p(N) \geq 2$. Then by Atkin–Lehner [2], $U_p(f)$ is in $\mathbf{M}_k(N/p)$ for $f \in \mathbf{M}_k(N)$. It is easy to show that $U_p(f) \subset \mathbf{M}_k^{\infty,0}(N/p)$ if $f \in \mathbf{M}_k^{\infty,0}(N)$, and that $U_p(f)$ has $p^{k-1}a_0^{(0)}$ as its 0th coefficient at the cusp 0, $a_0^{(0)}$ being the 0th coefficient of f at 0. We also have $U_p(\mathbf{M}_k^{\infty}(N)) \subset \mathbf{M}_k^{\infty}(N/p)$ and $U_p(\mathbf{M}_k^{0}(N)) \subset \mathbf{M}_k^{0}(N/p)$. If $\{c_0, c'_0, c_{-1}, \ldots, c_{-n_0}\} \in \mathrm{LR}_k(N/p)$, then $\{c_0, p^{k-1}c'_0, (p-1 \text{ times } 0), c_{-1}, (p-1 \text{ times } 0), \ldots, c_{-n_0}\}$ is in $\mathrm{LR}_k(N)$. This implies that some elements in

$\mathrm{LR}_k(N)$ are obtainable from $\mathrm{LR}_k(\prod_{p|N} p)$. Similarly, if $\{c_0, c_{-1}, \ldots, c_{-n_0}\} \in \mathrm{LR}'_k(N/p)$, then $\{c_0, (p-1 \text{ times } 0), c_{-1}, (p-1 \text{ times } 0), \ldots, c_{-n_0}\}$ is in $\mathrm{LR}'_k(N)$. We note that the inclusion $\mathbf{M}^0_k(M) \subset \mathbf{M}^0_k(N)$ holds for $M \mid N$ if $v_p(M) \geq 1$ for any prime factor p of N.

Hecke [11] investigated Eisenstein series of higher level (see also [22]). If N and k are sufficiently small, the spaces of modular forms are spanned by their linear combinations. In that case, elements of $\mathrm{LR}_{k,\chi_0}(N)$, etc., can be obtained from their Fourier coefficients through simple calculation. In the present paper we need several elements of $\mathrm{LR}_{k,\chi_0}(N)$, etc. However, we omit the detail of getting them.

2. Let K be a totally real algebraic number field of degree g. We denote by $\mathcal{O}$, $\mathfrak{d}_K$ and D_K the ring of integers, the different and the discriminant respectively. Let $\mathfrak{N}$ be an integral ideal. Let $\mathcal{E}_{\mathfrak{N}}$ denote the group of units $\varepsilon \succ 0$ congruent $1 \bmod \mathfrak{N}$, where $\varepsilon \succ 0$ means that ε is totally positive. We denote by $\mathbf{C}_{\mathfrak{N}}$ the narrow ray class group modulo $\mathfrak{N}$, and by $\mathbf{C}^*_{\mathfrak{N}}$ the character group. Although $\mathbf{C}_{\mathfrak{N}}$ denotes an integral ideal class group, we evaluate its character also at fractional ideals by the obvious extension. We call a character $\psi \in \mathbf{C}^*_{\mathfrak{N}}$ *even* (resp. *odd*) if $\psi(\mu) = 1$ (resp. $\psi(\mu) = \mathrm{sgn}(\mathrm{Nm}(\mu))$) for all $\mu \neq 0$, $\mu \equiv 1 \pmod{\mathfrak{N}}$. The conductor of ψ is denoted by $\mathfrak{f}_\psi$. For an ideal $\mathfrak{M}$ such that $\mathfrak{N} \subset \mathfrak{M} \subset \mathfrak{f}_\psi$, we denote by $\psi_{\mathfrak{M}}$ the character in $\mathbf{C}^*_{\mathfrak{M}}$ satisfying $\psi(\mathfrak{A}) = \psi_{\mathfrak{M}}(\mathfrak{A})$ for any $\mathfrak{A}$ relatively prime to $\mathfrak{N}$.

Let $\mathfrak{H}^g$ denote the product of g copies of $\mathfrak{H}$. For $\mathfrak{z} = (z_1, \ldots, z_g) \in \mathfrak{H}^g$, $\mathrm{Nm}(\gamma\mathfrak{z} + \delta)$ stands for $\prod_{i=1}^g (\gamma^{(i)} z_i + \delta^{(i)})$, where $\gamma^{(1)}, \ldots, \gamma^{(g)}$ denote conjugates of γ. Let $\mathfrak{N}$, $\mathfrak{N}'$ be integral ideals. Let $\mathfrak{A}$ be an ideal relatively prime to $\mathfrak{N}\mathfrak{N}'$. Let $k \in \mathbb{N}$. For $\gamma_0 \in \mathfrak{A}\mathfrak{d}_K^{-1}$, $\delta_0 \in \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}$, an *Eisenstein series* on $\mathfrak{H}^g$ is defined by setting

$$E_{k,\mathfrak{A}}(\mathfrak{z}, \gamma_0, \delta_0; \mathfrak{N}', \mathfrak{N}) := \mathrm{Nm}(\mathfrak{A})^k \sideset{}{'}\sum_{\gamma,\delta} \mathrm{Nm}(\gamma\mathfrak{z} + \delta)^{-k} |\mathrm{Nm}(\gamma\mathfrak{z} + \delta)|^{-s}|_{s=0},$$

where the summation is taken over all $(\gamma, \delta) \neq (0,0)$, $\gamma \equiv \gamma_0 \pmod{\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}}$, $\delta \equiv \delta_0 \pmod{\mathfrak{A}\mathfrak{d}_K^{-1}}$ which are not associated under the action of $\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}$: $(\gamma, \delta) \to (\varepsilon\gamma, \varepsilon\delta)$, $\varepsilon \in \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}$.

Let $\psi \in \mathbf{C}^*_{\mathfrak{N}}$ and $\psi' = \mathbf{C}^*_{\mathfrak{N}'}$. Suppose that $\psi\psi' \in \mathbf{C}^*_{\mathfrak{N}\mathfrak{N}'}$ has the same parity as k. Then we put

$$\begin{aligned}\widetilde{\lambda}^{\psi'}_{k,\psi}(\mathfrak{z}) := \left(\frac{(k-1)!}{(2\sqrt{-1}\,\pi)^k}\right)^g D_K^{-1/2} \mathrm{Nm}(\mathfrak{N})^{-1} [\mathcal{E}_{\mathfrak{N}} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1} \sum_{\mathfrak{A} \in \mathbf{C}_{\mathfrak{N}}} \psi(\mathfrak{A}) \\ \times \sum_{\gamma_0 \in \mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1},\, \gamma_0 \succ 0} \psi'(\gamma_0 \mathfrak{A}^{-1}\mathfrak{d}_K) \sum_{\delta_0 \in \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1}} \mathsf{e}(\mathrm{tr}(\delta_0)) \\ \times E_{k,\mathfrak{A}}(\mathfrak{z}, -\gamma_0, \delta_0; \mathfrak{N}', \mathfrak{N}),\end{aligned}$$

where $\mathfrak{A}$ is a representative relatively prime to $\mathfrak{N}'$. This is a modular form for

$$\Gamma_0(\mathfrak{N}\mathfrak{N}')_K := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}) : \gamma \equiv 0 \pmod{\mathfrak{N}\mathfrak{N}'} \right\}$$

of weight k with a character. In case $K = \mathbb{Q}$ and $k = 2$ we assume that either $\mathfrak{N} \neq \mathcal{O}$ or at least one of ψ, ψ' is nontrivial. The Fourier expansion of $\widetilde{\lambda}^{\psi'}_{k,\psi}(\mathfrak{z})$ at the cusp $\sqrt{-1}\,\infty$ is given as

$$\widetilde{\lambda}^{\psi'}_{k,\psi}(\mathfrak{z}) = C + 2^g \sum_{\nu \in \mathfrak{d}_K^{-1},\, \nu \succ 0} \Big(\sum_{\mathcal{O} \supset \mathfrak{B} \supset \nu\mathfrak{d}_K} \psi'(\nu\mathfrak{B}^{-1}\mathfrak{d}_K)\psi(\mathfrak{B})\,\mathrm{Nm}(\mathfrak{B})^{k-1} \Big) \mathsf{e}(\mathrm{tr}(\nu\mathfrak{z}))$$

with a constant C, where $\mathfrak{B}$ runs over integral ideals containing $\nu\mathfrak{d}_K$. If $\mathfrak{N}' = \mathcal{O}$ and ψ' is trivial, we denote the modular form by $\widetilde{\lambda}_{k,\psi}(\mathfrak{z})$. Similarly $\widetilde{\lambda}^{\psi'}_k(\mathfrak{z})$ is also defined. We can obtain C and the 0th Fourier coefficients of $\widetilde{\lambda}_{k,\psi}(\mathfrak{z})$ and $\widetilde{\lambda}^{\psi'}_k(\mathfrak{z})$ at other cusps by a similar computation to that in Shimura [18].

Proposition 1. *Let $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O})$. Let $k \in \mathbb{N}$ and let $\psi \in \mathbf{C}^*_{\mathfrak{N}}$ and k have the same parity.*

(1) *In case $K = \mathbb{Q}$ and $k = 2$, assume that $\mathfrak{N} \neq \mathcal{O}$ or ψ is nontrivial. Then the 0th Fourier coefficient of $\widetilde{\lambda}_{k,\psi}(\mathfrak{z})|A$ is equal to*

$$\mathrm{sgn}(\mathrm{Nm}(\delta))^{k-1}\psi(\delta) \prod_{\substack{\mathfrak{P}|\mathfrak{N} \\ \mathfrak{P} \nmid (\gamma,\mathfrak{N})}} (1 - \mathrm{Nm}(\mathfrak{P})^{-1}) L_K(1-k, \psi_{(\gamma,\mathfrak{N})}) \qquad ((\gamma,\mathfrak{N}) \subset \mathfrak{f}_\psi)$$

$$+ (\sqrt{-1}\,\pi)^{-g} D_K^{-1/2} \psi(\gamma) L_K(1,\psi) \qquad (k = 1 \text{ and } (\gamma,\mathfrak{N}) = \mathcal{O}),$$

where $\psi(0) = 1$ in case $\mathfrak{N} = \mathcal{O}$.

(2) *In case $K = \mathbb{Q}$ and $k = 2$, assume that ψ is nontrivial. Then the 0th Fourier coefficient of $\widetilde{\lambda}^{\psi}_k(\mathfrak{z})|A$ is equal to*

$$\left(\frac{2(k-1)!}{(2\sqrt{-1}\,\pi)^k} \right)^g D_K^{k-1/2} \psi(\gamma) L_K(k,\psi) \qquad ((\gamma,\mathfrak{N}) = \mathcal{O})$$

$$+ \psi(\alpha)^{-1} \prod_{\substack{\mathfrak{P}|\mathfrak{N} \\ \mathfrak{P} \nmid (\gamma,\mathfrak{N})}} (1 - \mathrm{Nm}(\mathfrak{P})^{-1}) L_K(0, \psi_{(\gamma,\mathfrak{N})}) \qquad (k = 1 \text{ and } (\gamma,\mathfrak{N}) = \mathfrak{f}_\psi).$$

3. We put $\lambda^{\psi'}_{gk,\psi}(z) := \widetilde{\lambda}^{\psi'}_{k,\psi}(z,\ldots,z)$. Let $N \in \mathbb{N} \cap \mathfrak{N}\mathfrak{N}'$, and let χ_0 be an element of the group $(\mathbb{Z}/N)^*$ of characters mod N such that $\chi_0(i) = \psi(i)\psi'(i)$. Then $\lambda^{\psi'}_{gk,\psi}(z)$ is in $\mathbf{M}_{gk,\chi_0}(N)$. We have the Fourier expansion

$$\lambda^{\psi'}_{gk,\psi}(z) = C + 2^g \sum_{n=1}^{\infty} \mathfrak{f}^{\psi'}_{k-1,\psi}(n)\mathsf{e}(nz)$$

with

$$\mathfrak{f}_{k-1,\psi}^{\psi'}(n) := \sum_{\substack{\nu\in\mathfrak{d}_K^{-1},\,\nu\succ 0\\ \mathrm{tr}(\nu)=n}} \sum_{\mathcal{O}\supset\mathfrak{A}\supset\nu\mathfrak{d}_K} \psi'(\nu\mathfrak{A}^{-1}\mathfrak{d}_K)\psi(\mathfrak{A})\,\mathrm{Nm}(\mathfrak{A})^{k-1}.$$

If ψ' (resp. ψ) is trivial, then we write $\mathfrak{f}_{k-1,\psi}^{\psi'}$ as $\mathfrak{f}_{k-1,\psi}$ (resp. $\mathfrak{f}_{k-1}^{\psi'}$). Further, we put $\lambda_{gk,\psi}(z) := \widetilde{\lambda}_{k,\psi}(z,\ldots,z)$ and $\lambda_{gk}^{\psi}(z) := \widetilde{\lambda}_{k}^{\psi}(z,\ldots,z)$. By Proposition 1, we have the following:

Proposition 2. *Let ψ be as in Proposition* 1. *Let $N\in\mathbb{N}\cap\mathfrak{N}$, and let $\chi_0\in(\mathbb{Z}/N)^*$ be such that $\chi_0(i)=\psi(i)$. Let $M\in\mathbb{N}$ be a divisor of N. The modular forms $\lambda_{gk,\psi}$ and λ_{gk}^{ψ} are in $\mathbf{M}_{gk,\chi_0}(N)$. The* 0*th Fourier coefficient of $\lambda_{gk,\psi}$ at a cusp i/M $(i\in\mathbb{N},\ (i,M)=1)$ is*

$$\chi_0(i)^{-1}\prod_{\substack{\mathfrak{P}\mid\mathfrak{N}\\ \mathfrak{P}\nmid(M,\mathfrak{N})}}(1-\mathrm{Nm}(\mathfrak{P})^{-1})L_K(1-k,\psi_{(M,\mathfrak{N})})\quad((M,\mathfrak{N})\subset\mathfrak{f}_\psi)$$

or 0 (*otherwise*), *and there is an additional term $(\sqrt{-1}\,\pi)^{-g}D_K^{1/2}\chi_0(M)\times L_K(1,\psi)$ if $k=1$ and $(M,\mathfrak{N})=\mathcal{O}$. Let $k>1$. Then the* 0*th Fourier coefficient of λ_{gk}^{ψ} at i/M is*

$$\left(\frac{2(k-1)!}{(2\sqrt{-1}\,\pi)^k}\right)^g D_K^{k-1/2}\chi_0(M)L_K(k,\psi)\quad((M,\mathfrak{N})=\mathcal{O})$$

or 0 (*otherwise*).

Corollary. *Suppose that ψ is a primitive character with $\mathfrak{f}_\psi=\mathfrak{N}$. Let N be the least element in $\mathbb{N}\cap\mathfrak{N}$. Then $\lambda_{gk,\psi}\in\mathbf{M}_{gk,\chi_0}^{\infty}(N)$, $\lambda_{gk}^{\psi}\in\mathbf{M}_{gk,\chi_0}^{0}(N)$ for $k>1$, and $\lambda_{g,\psi}\in\mathbf{M}_{g,\chi_0}^{\infty,0}(N)$ for $k=1$.*

Let $W(\psi)$ be the root of unity appearing in the functional equation of the L-function $L_K(s,\psi)$ in Hecke [12]. It is written as a Gauss sum, in the form

$$W(\psi)=w\,\mathrm{Nm}(\mathfrak{N})^{-1/2}\psi(\varrho\mathfrak{N}\mathfrak{d}_K)\sum_{\mu\in\mathcal{O}/\mathfrak{N},\,\mu\succ 0}\psi(\mu)\mathsf{e}(\mathrm{tr}(\varrho\mu)),$$

where w equals 1 or $\sqrt{-1}^{\,-g}$ according as ψ is even or odd and where $\varrho\in K$, $\varrho\succ 0$, is such that $\varrho\mathfrak{N}\mathfrak{d}_K$ is an integral ideal relatively prime to $\mathfrak{N}$. Then the additional term in the above proposition is written as $\sqrt{-1}^{\,-g}\psi(M)W(\psi)\,\mathrm{Nm}(\mathfrak{N})^{-1/2}L(0,\overline{\psi})$, $\overline{\psi}$ being the complex conjugate of ψ.

By the Corollary to Lemma 1 and Proposition 2 we obtain the following:

Theorem 1. *Let $k\in\mathbb{N}$. Let ψ be a primitive character with conductor $\mathfrak{N}$ and with the same parity as k, and let N be the least element in $\mathbb{N}\cap\mathfrak{N}$. Let $\chi_0\in(\mathbb{Z}/N)^*$ be such that $\chi_0(i)=\psi(i)$. Assume that $\mathfrak{N}\neq\mathcal{O}$ if $k=1$.*

(1) *We have the identity*

$$c_0 L_K(1-k,\psi) = -2^g \sum_{n=1}^{n_0} c_{-n}\mathfrak{f}_{k-1,\psi}(n)$$

where $\{c_0, *, c_{-1}, \ldots, c_{-n_0}\} \in \mathrm{LR}_{gk,\chi_0}(N)$ $(N > 1,\ k > 1)$, *and* $\{c_0, c_{-1}, \ldots, c_{-n_0}\} \in \mathrm{LR}'_{g,\chi_0}(N)$ $(N = 1$ *or* $k = 1)$. *Let* $k = 1$ *and suppose that* $L_K(0,\psi) \in \mathbb{R}$. *Then*

$$\{c_0 + \sqrt{-1}^{\,-g} W(\psi)\,\mathrm{Nm}(\mathfrak{N})^{-1/2} c_0'\} L_K(0,\psi) = -2^g \sum_{n=1}^{n_0} c_{-n}\mathfrak{f}_{0,\psi}(n)$$

with $\{c_0, c_0', c_{-1}, \ldots, c_{-n_0}\} \in \mathrm{LR}_{g,\chi_0}(N)$.

(2) *Let* $k > 1$. *Then*

$$c_0' L_K(k,\psi) = -\left(\frac{(2\sqrt{-1}\,\pi)^k}{(k-1)!}\right)^g D_K^{-k+1/2} \sum_{n=1}^{n_0} c_{-n}\mathfrak{f}^{\psi}_{k-1}(n)$$

with $\{*, c_0', c_{-1}, \ldots, c_{-n_0}\} \in \mathrm{LR}_{gk,\chi_0}(N)$ $(N > 1)$, *and* $\{c_0', c_{-1}, \ldots, c_{-n_0}\} \in \mathrm{LR}'_{g,\chi_0}(1)$ $(N = 1)$.

Consider the case $k = 1$ and $\mathfrak{N} = \mathcal{O}$. The existence of an odd character ψ of $\mathbf{C}_{\mathcal{O}}$ implies that g is even. Then $W(\psi)$ is equal to $(-1)^{g/2}\psi(\mathfrak{d}_K)$. Let $\mathfrak{P}$ be a prime ideal of K with $\psi(\mathfrak{P}) \neq 1$, and let ψ' be a character mod $\mathfrak{P}$ such that $\psi'_{\mathfrak{P}} = \psi$. Then by Proposition 2,

$$\lambda_{g,\psi'}(z) = (1-\psi(\mathfrak{P}))L_K(0,\psi) + 2^g \sum_{n=1}^{\infty} \mathfrak{f}_{0,\psi,\mathfrak{P}}(n)\mathsf{e}(nz)$$

with

$$\mathfrak{f}_{0,\psi,\mathfrak{P}}(n) := \sum_{\substack{\nu\in\mathfrak{d}_K^{-1},\,\nu\succ 0\\ \mathrm{tr}(\nu)=n}} \sum_{\substack{\mathcal{O}\supset\mathfrak{A}\supset\nu\mathfrak{d}_K\\ (\mathfrak{A},\mathfrak{P})=\mathcal{O}}} \psi(\mathfrak{A})$$

is in $\mathbf{M}_g(p)$, where p is a rational prime in $\mathfrak{P}$. Hence for $\{c_0, c_{-1}, \ldots, c_{-n_0}\} \in \mathrm{LR}'_g(p)$, we have

$$c_0 L_K(0,\psi) = -2^g(1-\psi(\mathfrak{P}))^{-1} \sum_{n=1}^{n_0} c_{-n}\mathfrak{f}_{0,\psi,\mathfrak{P}}(n).$$

However, in the next proposition we obtain a formula which may be better in the sense that n_0 is possibly smaller.

PROPOSITION 3. *Let* $\mathfrak{P}$ *be a prime ideal of* K *with* $\psi(\mathfrak{P}) \neq 1$ *and let* $p \in \mathbb{N}$ *be a prime in* $\mathfrak{P}$.

(1) *Suppose that* $L_K(0,\psi)\in\mathbb{R}$ *and* $\psi(\mathfrak{d}_K)\neq -1$. *Then*

$$c_0L_K(0,\psi) = -2^g(1+\psi(\mathfrak{d}_K))^{-1}\sum_{n=1}^{n_0} c_{-n}\mathfrak{f}_{0,\psi}(n)$$

for $\{c_0,\dots,c_{-n_0}\}\in \mathrm{LR}'_g(1)$.

(2) *Suppose that* $L_K(0,\psi)\in\mathbb{R}$ *and* $\psi(\mathfrak{d}_K) = -1$. *Then*

$$\{c_0-\mathrm{Nm}(\mathfrak{P})^{-1}c'_0\}L_K(0,\psi) = -2^g(1-\psi(\mathfrak{P}))^{-1}\sum_{n=1}^{n_0} c_{-n}\mathfrak{f}_{0,\psi,\mathfrak{P}}(n)$$

for $\{c_0,c'_0,c_{-1},\dots,c_{-n_0}\}\in \mathrm{LR}_g(p)$, *where* p *is the rational prime in* $\mathfrak{P}$.

(3) *We have the identity*

$$\begin{aligned}\{c_0-&\mathrm{Nm}(\mathfrak{P})^{-1}c'_0\}L_K(0,\psi)\\ &= 2^g(1-\psi(\mathfrak{P}))^{-1}\\ &\quad\times\Big\{(1-\psi(\mathfrak{P})\,\mathrm{Nm}(\mathfrak{P})^{-1})c'_0d_0^{-1}\sum_{n=1}^{m_0} d_{-n}\mathfrak{f}_{0,\psi}(n)-\sum_{n=1}^{n_0}c_{-n}\mathfrak{f}_{0,\psi,\mathfrak{P}}(n)\Big\}\end{aligned}$$

for $\{d_0,\dots,d_{m_0}\}\in \mathrm{LR}'_g(1)$ *with* $d_0\neq 0$, *and for* $\{c_0,c'_0,c_{-1},\dots,c_{-n_0}\}\in \mathrm{LR}_g(p)$.

Proof. Since $\lambda_{g,\psi}(z)=C+2^g\sum_{n=1}^{\infty}\mathfrak{f}_{k-1,\psi}(n)\mathsf{e}(nz)$ with $C=L_K(0,\psi)+\psi(\mathfrak{d}_K)L_K(0,\overline{\psi})$, is in $\mathbf{M}_g(1)$, the assertion (1) follows immediately. The 0th Fourier coefficient of $\lambda_{g,\psi'}\in\mathbf{M}_g(p)$ at 0 is $(1-\psi(\mathfrak{P})\,\mathrm{Nm}(\mathfrak{P})^{-1})\times\psi(\mathfrak{d}_K)L_K(0,\overline{\psi})+(1-\mathrm{Nm}(\mathfrak{P})^{-1})L_K(0,\psi)$, which is equal to $-(1-\psi(\mathfrak{P}))\times\mathrm{Nm}(\mathfrak{P})^{-1}L_K(0,\psi)$ under the assumption of (2). Then the equality in (2) follows.

Consider the case (3). By Proposition 2 the 0th coefficient of $\lambda_{g,\psi'}$ at 0 is calculated to be $(1-\psi(\mathfrak{P}))\,\mathrm{Nm}(\mathfrak{P})^{-1}L_K(0,\psi)+(1-\psi(\mathfrak{P})\,\mathrm{Nm}(\mathfrak{P})^{-1})C$, and C is equal to $-2^gd_0^{-1}\sum_{n=1}^{m_0}d_{-n}f_{0,\psi}(n)$. Since

$$\begin{aligned}c_0(1-&\psi(\mathfrak{P}))L_K(0,\psi)\\ &+c'_0\{-(1-\psi(\mathfrak{P}))\,\mathrm{Nm}(\mathfrak{P})^{-1}L_K(0,\psi)+(1-\psi(\mathfrak{P})\,\mathrm{Nm}(\mathfrak{P})^{-1})C\}\\ =&\ -2^g\sum_{n=1}^{n_0}c_{-n}\mathfrak{f}_{0,\psi,\mathfrak{P}}(n),\end{aligned}$$

our assertion follows. ■

Let F be a totally imaginary quadratic extension of a totally real field K. Let H and h denote the class numbers of F and K respectively. Let $\mathfrak{D}$ be the relative discriminant and let $\psi\in\mathbf{C}^*_{\mathfrak{D}}$ be the character associated with the extension in the sense of class field theory. Then the relative class number is given by

$$H/h = \frac{w(F)R_K}{2R_F} L_K(0, \psi),$$

where $w(F)$ denotes the number of roots of unity in F and R_F, R_K denote the regulators of F, K respectively. Since $W(\psi)$ is trivial in this case, we have the following formulas for the relative class numbers as a corollary of Theorem 1 and of Proposition 3, where the exact form of fundamental units is not necessary.

COROLLARY. *Let N be the minimum of $\mathfrak{D} \cap \mathbb{N}$, and let $\chi_0 \in (\mathbb{Z}/N)^*$ be such that $\chi_0(i) = \psi(i)$. If $\mathfrak{D} \neq \mathcal{O}$, then*

$$\{c_0 + \sqrt{-1}^{\,-g}\, \mathrm{Nm}(\mathfrak{D})^{-1/2} c_0'\} H/h = -2^{g-1} w(F) R_K R_F^{-1} \sum_{n=1}^{n_0} c_{-n} \mathfrak{f}_{0,\psi}(n)$$

with $\{c_0, c_0', c_{-1}, \ldots, c_{-n_0}\} \in \mathrm{LR}_{g,\chi_0}(N)$. Suppose that $\mathfrak{D} = \mathcal{O}$. If $g \equiv 0 \pmod 4$, then

$$c_0 H/h = -2^{g-2} w(F) R_K R_F^{-1} \sum_{n=1}^{n_0} c_{-n} \mathfrak{f}_{0,\psi}(n)$$

with $\{c_0, \ldots, c_{-n_0}\} \in \mathrm{LR}'_g(1)$. Let $\mathfrak{P}$ and p be as in Proposition 3. Then if $g \equiv 2 \pmod 4$, then

$$\{c_0 - \mathrm{Nm}(\mathfrak{P})^{-1} c_0'\} H/h = -2^{g-1} w(F) R_K R_F^{-1} (1 - \psi(\mathfrak{P}))^{-1} \sum_{n=1}^{n_0} c_{-n} \mathfrak{f}_{0,\psi,\mathfrak{P}}(n)$$

with $\{c_0, c_0', c_{-1}, \ldots, c_{-n_0}\} \in \mathrm{LR}_g(p)$.

4. We give some examples to illustrate the results of Section 3. First we show the following:

LEMMA 2. *Let K be a real quadratic field of discriminant D_K. If $\psi'\psi$ has the same parity as k, then*

$$\mathfrak{f}^{\psi'}_{k-1,\psi}(n) = \sum_{\substack{|m| < n\sqrt{D_K} \\ m \equiv nD_K \ (\mathrm{mod}\ 2)}} \ \sum_{\mathcal{O} \supset \mathfrak{A} \supset ((m+n\sqrt{D_K})/2)} \psi'\left(\frac{m+n\sqrt{D_K}}{2}\mathfrak{A}^{-1}\right) \times \psi(\mathfrak{A})\, \mathrm{Nm}(\mathfrak{A})^{k-1}.$$

Let $\mathfrak{P}$ be a prime ideal and let $\psi \in \mathbf{C}^*_{\mathcal{O}}$ be odd. Then

$$\mathfrak{f}_{0,\psi,\mathfrak{P}}(n) = -\psi(\mathfrak{P}) \sum_{\substack{|m| < n\sqrt{D_K} \\ m \equiv nD_K \ (\mathrm{mod}\ 2)}} \ \sum_{\mathcal{O} \supset \mathfrak{A} \supset \mathfrak{P}^{-1}((m+n\sqrt{D_K})/2)} \psi(\mathfrak{A}).$$

Proof. A totally positive number in $\mathfrak{d}_K^{-1}$ with trace $n \in \mathbb{N}$ is of the form $(m + n\sqrt{D_K})/2\sqrt{D_K}$ with $m \equiv nD_K \pmod 2$ and $|m| < n\sqrt{D_K}$.

Then the first equality follows immediately. Consider the second one. Since $\lambda_{2,\psi} \in \mathbf{M}_2(1) = \{0\}$, its nth Fourier coefficient $\mathfrak{f}_{0,\psi}(n)$ is equal to 0. Then

$$\mathfrak{f}_{0,\psi,\mathfrak{P}}(n) = -(\mathfrak{f}_{0,\psi}(n) - \mathfrak{f}_{0,\psi,\mathfrak{P}}(n)) = -\sum_{\substack{\nu\in\mathfrak{d}_K^{-1},\,\nu\succ 0\\ \mathrm{tr}(\nu)=n}}\ \sum_{\mathfrak{P}\supset\mathfrak{A}\supset\nu\mathfrak{d}_K} \psi(\mathfrak{A}).$$

This shows our assertion. ∎

Example 1. Let $K = \mathbb{Q}(\sqrt{79}\,)$. The class number h is 3, and the narrow ideal class group $\mathbf{C}_{\mathcal{O}}$ is a cyclic group of order six. There are six characters of $\mathbf{C}_{\mathcal{O}}$, three odd ones and three even ones. Let $\mathfrak{P}_7 = (7, 3+\sqrt{79}\,)$. It is a prime ideal with norm 7 and the class containing $\mathfrak{P}_7$ generates $\mathbf{C}_{\mathcal{O}}$. Let ψ_i $(0 \le i \le 6)$ be a character such that $\psi_i(\mathfrak{P}_7) = \mathsf{e}(i/6)$, where the parity of ψ_i is the same as i. Since $\{-1, 4\} \in L_2'(7)$, by the formula before Proposition 3 and by Lemma 2 we have

$$\begin{aligned} 4L_K(0,\psi_i) = {} & -4\mathsf{e}\left(\frac{i}{6}\right)\left(1-\mathsf{e}\left(\frac{i}{6}\right)\right)^{-1} \\ & \times \sum_{|m|<\sqrt{79}}\ \sum_{\mathcal{O}\supset\mathfrak{A}\supset\mathfrak{P}_7^{-1}(m+\sqrt{79}\,)} \psi_i(\mathfrak{A}) \qquad (i=1,3,5). \end{aligned}$$

The inclusion $\mathfrak{P}_7 \supset (m+\sqrt{79}\,)$ $(|m| < \sqrt{79}\,)$ holds only for $m = 3, -4$, and decompositions of $m+\sqrt{79}$ into products of primes are $3+\sqrt{79} = (9+\sqrt{79}\,)(5, 3+\sqrt{79}\,)\mathfrak{P}_7$ and $4+\sqrt{79} = (3, 2+\sqrt{79}\,)^2\mathfrak{P}_7$. Hence if we put $\omega = \psi_i(\mathfrak{P}_7)$, then

$$L_K(0,\psi_i) = -(1-\omega)^{-1}\omega\{(1+1+\omega^2+\omega^2) + (1+\omega+\omega^2)\}.$$

By substituting $\mathsf{e}(1/6), -1, \mathsf{e}(5/6)$ for ω, we obtain $L_K(0,\psi_1) = L_K(0,\psi_5)$ $= 4$ and $L_K(0,\psi_3) = 5/2$.

Let $\psi \in \mathbf{C}_{\mathcal{O}}^*$ and let $\omega = \psi(\mathfrak{P}_7)$. Considering the prime decompositions of $(m+\sqrt{79}\,)$ $(|m| \le 8)$, we obtain

$$\begin{aligned} \mathfrak{f}_{k-1,\psi}(1) = {} & 17 + 8\cdot 2^{k-1} \\ & + (6\cdot 3^{k-1} + 3\cdot 6^{k-1} + 2\cdot 7^{k-1} + 14^{k-1} + 15^{k-1})(\omega+\omega^5) \\ & + (4\cdot 5^{k-1} + 2\cdot 9^{k-1} + 2\cdot 10^{k-1} + 13^{k-1} + 18^{k-1} + 21^{k-1} \\ & + 25^{k-1} + 26^{k-1})(\omega^2+\omega^4) \\ & + \{4\cdot 15^{k-1} + 2(27^{k-1} + 30^{k-1} + 35^{k-1} + 39^{k-1} + 43^{k-1} \\ & + 54^{k-1} + 63^{k-1} + 70^{k-1} + 75^{k-1} + 78^{k-1}) + 79^{k-1}\}\omega^3. \end{aligned}$$

From this and the fact that $\{240, -1\} \in \mathrm{LR}_4(1)$, $\{504, 1\} \in \mathrm{LR}_6(1)$, $\{480, -1\} \in \mathrm{LR}_8(1)$ and $\{264, 1\} \in \mathrm{LR}_{10}(1)$ (Siegel [20]), we obtain $L_K(-1,\psi_2) = L_K(-1,\psi_4) = 16$, $L_K(-1,\psi_0) = \zeta_K(-1) = 28$; $L_K(-2,\psi_1)$ $= L_K(-2,\psi_5) = 544$, $L_K(-2,\psi_0) = \zeta_K(-2) = 496$; $L_K(-3,\psi_2) =$

$L_K(-3,\psi_4) = 34960$, $L_K(-3,\psi_0) = \zeta_K(-3) = 182558/5$; $L_K(-4,\psi_1) = L_K(-4,\psi_5) = 4412992$, $L_K(-4,\psi_3) = 4362400$.

Let F be a totally imaginary extension of a totally real field of K. Let $Q_{F/K}$ denote the unit index of Hasse, that is, $Q_{F/K} = [\widetilde{\mathcal{E}}_F : \Omega_F\widetilde{\mathcal{E}}_K]$, where $\widetilde{\mathcal{E}}_F$ and $\widetilde{\mathcal{E}}_K$ denote the groups of all units in F and K respectively and Ω_F denotes the group of roots of unity in F. Then R_K/R_F is equal to $2^{-g+1}Q_{F/K}$. The index is 1 or 2, and is readily obtained (Hasse [10], Okazaki [16]). Let $F = K(\sqrt{-\nu}\,)$ with a totally positive integer ν in K. Let $\mathfrak{D}$ be the relative discriminant of the extension, and let $\psi \in \mathbf{C}^*_{\mathfrak{D}}$ be the associated character. Let $\mathfrak{A}$ be an ideal with $(\mathfrak{A},\mathfrak{D}) = \mathcal{O}$. If $\mathfrak{A}$ is relatively prime to 2, then $\psi(\mathfrak{A})$ is equal to $\left(\frac{-\nu}{\mathfrak{A}}\right)_K$ where $\left(\frac{\cdot}{\cdot}\right)_K$ is the quadratic residue symbol in K. If $(\mathfrak{A},2) \neq \mathcal{O}$, then we take another integral ideal $\mathfrak{B}$ relatively prime to $2\mathfrak{D}$ which is of the form $\mathfrak{B} = \varrho\mathfrak{C}^2\mathfrak{A}$ for some $\varrho \in K$, $\varrho \succ 0$ multiplicatively congruent 1 mod $\mathfrak{D}$ and for a fractional ideal $\mathfrak{C}$. The computation of $\psi(\mathfrak{A})$ is reduced to that of $\psi(\mathfrak{B})$. Let χ_0 be the character on $\mathbb{Z}$ defined by $\chi_0(i) = \psi(i)$. Obviously $\chi_0(-1) = 1$, that is, χ_0 is even.

Suppose that K is real quadratic. Then if $\mathfrak{P}$ is of degree one, then $\left(\frac{-\nu}{\mathfrak{P}}\right)_K$ is written as $\left(\frac{n}{p}\right)$, where $\left(\frac{\cdot}{\cdot}\right)$ denotes the usual Jacobi–Legendre symbol and $p = \mathrm{Nm}(\mathfrak{P})$, $n \in \mathbb{Z}$, $n \equiv -\nu \pmod{\mathfrak{P}}$. If $\mathfrak{P}$ is of degree two, then it is written as $\left(\frac{\mathrm{Nm}(\nu)}{p}\right)$, where p is a prime in $\mathfrak{P}$.

For D a discriminant of a quadratic field, we denote by χ_D the Kronecker–Jacobi–Legendre symbol.

Example 2. Let K be a real quadratic field where 2 is not inert and its prime factor $\mathfrak{P}_2$ is a principal ideal (ν) with $\nu \succ 0$. A necessary condition for this is that D_K is free from a prime factor congruent to 3 or 5 mod 8. Let $F = K(\sqrt{-\nu}\,)$. We show that the relative class number of F over K is given by

$$H/h = c \sum_{\substack{|m|<\sqrt{D_K} \\ m\equiv D_K \ (\mathrm{mod}\ 2)}} \ \sum_{\mathcal{O}\supset\mathfrak{A}\supset((m+\sqrt{D_K}\,)/2)} \psi(\mathfrak{A})$$

where $c = 1/7$ ($D_K \equiv 1 \pmod 8$ and $\mathrm{tr}(\nu) \equiv 1 \pmod 4$), and $c = 1/3$ (otherwise).

The conductor $\mathfrak{D}$ of the extension is $\mathfrak{P}_2^3$ or $4\mathfrak{P}_2$, where the former is the case when $c = 1/7$. The character χ_0 is in $(\mathbb{Z}/8)^*$. For p prime, $\chi_0(p) = \left(\frac{2}{p}\right)$ or 1 according as p is decomposed in K or not, and hence $\chi_0 = \chi_8$. Since $\{2, 32\sqrt{2}, 1\} \in \mathrm{LR}_{2,\chi_8}(8)$, and since $w(F) = 2$ and $R_K/R_F = 1/2$, we have $H/h = \{16\sqrt{2}\,\mathrm{Nm}(\mathfrak{D})^{-1/2} - 1\}^{-1}\mathfrak{f}_{0,\psi}(1)$ by the last corollary in Section 3, which shows our formula.

There are nine real quadratic fields K with $D_K < 100$ having ν satisfying the condition, to which we apply the formula.

Let $K=\mathbb{Q}(\sqrt{2}\,)$ and $F=\mathbb{Q}(\sqrt{-2-\sqrt{2}}\,)$. Then

$$H/h=\frac{1}{3}\sum_{|m|<\sqrt{2}}\ \sum_{\mathfrak{A}\supset(m+\sqrt{2}\,)}\psi(\mathfrak{A})=\frac{1}{3}(1+1+1)=1.$$

Thus the class number of F is 1.

Let $K=\mathbb{Q}(\sqrt{17}\,)$ and $F=K(\sqrt{-\nu}\,)$ with $\nu=(5+\sqrt{17}\,)/2$. Put $\mathfrak{P}_2=(\nu)$. In this case the conductor is $\mathfrak{P}_2^3$. We note that $\psi(\overline{\mathfrak{P}}_2)=\psi(7)=1$ because $\overline{\nu}\equiv 7 \pmod{\mathfrak{P}_2^3}$. Then

$$H/h=\frac{1}{7}\sum_{\substack{|m|<\sqrt{17}\\ m\,\mathrm{odd}}}\ \sum_{\mathfrak{A}\supset((m+\sqrt{17}\,)/2)}\psi(\mathfrak{A})=\frac{1}{7}(5+2\psi(\overline{\mathfrak{P}}_2))=1.$$

Let $K=\mathbb{Q}(\sqrt{7}\,)$ and $F=K(\sqrt{-3-\sqrt{7}}\,)$. Then

$$\begin{aligned}H/h&=\frac{1}{3}\sum_{|m|<\sqrt{7}}\ \sum_{\mathfrak{A}\supset(m+\sqrt{7}\,)}\psi(\mathfrak{A})\\ &=\frac{1}{3}\left\{5+\left(\frac{-3-\sqrt{7}}{\sqrt{7}}\right)_K+2\left(\frac{-3-\sqrt{7}}{-2+\sqrt{7}}\right)_K+2\left(\frac{-3-\sqrt{7}}{2+\sqrt{7}}\right)_K\right\}\\ &=\frac{1}{3}\left\{5+\left(\frac{-3}{7}\right)+2\left(\frac{2}{3}\right)+2\left(\frac{1}{3}\right)\right\}=2.\end{aligned}$$

Let $\varepsilon=8+3\sqrt{7}$ a fundamental unit of K, let $F'=K(\sqrt{(-3-\sqrt{7}\,)\varepsilon}\,)$, and let H' be the class number. Then $H'=2$.

By similar computations we get the following class numbers:

$$\begin{array}{llll}2 & (F=\mathbb{Q}(\sqrt{(-7-\sqrt{41}\,)/2}\,)), & 2 & (F=\mathbb{Q}(\sqrt{-4-\sqrt{14}}\,)),\\ 1 & (F=\mathbb{Q}(\sqrt{(-9-\sqrt{73}\,)/2}\,)), & 3 & (F=\mathbb{Q}(\sqrt{(-217-23\sqrt{89}\,)/2}\,)),\\ 2 & (F=\mathbb{Q}(\sqrt{-5-\sqrt{23}}\,)), & 3 & (F=\mathbb{Q}(\sqrt{(-69-7\sqrt{97}\,)/2}\,)).\end{array}$$

Example 3. Let K be a real quadratic field where $13=\mathfrak{P}_{13}\overline{\mathfrak{P}}_{13}$ in K and $\mathfrak{P}_{13}$ is a principal ideal (ν) with $\nu\succ 0$. Here $\overline{\mathfrak{P}}_{13}$ is the conjugate of $\mathfrak{P}_{13}$. Let $F=K(\sqrt{-\nu}\,)$. Assume that the relative discriminant of F over K is $\mathfrak{P}_{13}$. The character χ_0 is equal to χ_{13}. Since $\{1,13\sqrt{13},1\}\in \mathrm{LR}_{2,\chi_0}(13)$, we have

$$H/h=\frac{1}{6}\sum_{\substack{|m|<\sqrt{D_K}\\ m\equiv D_K \ (\mathrm{mod}\ 2)}}\ \sum_{\mathcal{O}\supset\mathfrak{A}\supset((m+\sqrt{D_K}\,)/2)}\psi(\mathfrak{A}).$$

If $K = \mathbb{Q}(\sqrt{13})$, then our conditions are satisfied, and

$$F = K(\sqrt{-\sqrt{13}\,\varepsilon}) \quad \text{with } \varepsilon = \frac{3+\sqrt{13}}{2},$$

and

$$\begin{aligned} H/h &= \frac{1}{6} \sum_{\substack{|m|\le 3 \\ m \text{ odd}}} \sum_{\mathfrak{A}\supset((m+\sqrt{13})/2)} \psi(\mathfrak{A}) \\ &= \frac{1}{6}\left\{4 + \left(\frac{-\sqrt{13}\,\varepsilon}{(1+\sqrt{13})/2}\right)_K + \left(\frac{-\sqrt{13}\,\varepsilon}{(-1+\sqrt{13})/2}\right)_K\right\} \\ &= \frac{1}{6}\left\{4 + \left(\frac{-5}{3}\right) + \left(\frac{-8}{3}\right)\right\} = 1. \end{aligned}$$

Let $K = \mathbb{Q}(\sqrt{17})$. Then $13 = (9+2\sqrt{17})(9-2\sqrt{17})$, and if we put $F = K(\sqrt{-9-2\sqrt{17}})$, then our conditions are satisfied. We have a decomposition $2 = \mathfrak{P}_2\overline{\mathfrak{P}}_2$ in K. Since

$$\psi(2) = \psi(14)\psi(7) = 1 \cdot \left(\frac{-9-2\sqrt{17}}{7}\right)_K = \left(\frac{13}{7}\right) = -1,$$

we have $\{\psi(\mathfrak{P}_2), \psi(\overline{\mathfrak{P}}_2)\} = \{\pm 1\}$. Then

$$\begin{aligned} H/h &= \frac{1}{6} \sum_{\substack{|m|\le 3 \\ m \text{ odd}}} \sum_{\mathfrak{A}\supset((m+\sqrt{17})/2)} \psi(\mathfrak{A}) \\ &= \frac{1}{6}\{4 + 2\psi(\mathfrak{P}_2) + 2\psi(\overline{\mathfrak{P}}_2) + \psi(\mathfrak{P}_2)^2 + \psi(\overline{\mathfrak{P}}_2)^2\} = 1. \end{aligned}$$

Let $K = \mathbb{Q}(\sqrt{29})$. Then we have $13 = \left(\frac{9+\sqrt{29}}{2}\right)\left(\frac{9-\sqrt{29}}{2}\right)$. Let $F = K(\sqrt{(-9-\sqrt{29})/2})$. Then a similar calculation gives $H/h = \frac{1}{6}\cdot 6 = 1$.

Let $K = \mathbb{Q}(\sqrt{69})$. Then $13 = (17+2\sqrt{69})(17-2\sqrt{69})$. Let $F = K(\sqrt{-17-2\sqrt{69}})$. Then $H/h = \frac{1}{6}\cdot 12 = 2$.

The class numbers of some of the fields in Examples 2 and 3 have already been computed in Okazaki [16], where Shintani's formula [19] is employed. Our results are compatible with his. Grundman [9] obtained numerical examples of values of zeta functions of totally real cubic fields also by adapting Shintani's method.

Example 4. Let K be a totally real cubic field, and let $\varepsilon \succ 0$ be a unit. Let $F = K(\sqrt{-\varepsilon})$. Then the conductor $\mathfrak{D}$ of the extension is a factor of 4, and $w(F) = 4$, $Q_{F/K} = 1$ for $\varepsilon = 1$ or $w(F) = 2$, $Q_{F/K} = 2$ for $\varepsilon \notin (K^\times)^2$ (see for example Okazaki [16], Sect. 3). The character χ_0 is equal to χ_{-4}, namely $\chi_{-4}(n) = (-1)^{(n-1)/2}$ for n odd. Since $\{1, 32\sqrt{-1}, 1/4\} \in \mathrm{LR}_{3,\chi_{-4}}(4)$, by the last corollary of Section 3 we have a formula for the relative class number $H/h = (32\,\mathrm{Nm}(\mathfrak{D})^{-1/2} - 1)^{-1}\mathfrak{f}_{0,\psi}(1)$. If the absolute discriminant of K is odd, then $\mathfrak{D} = (4)$ and we have

$$H/h = \frac{1}{3} \sum_{\substack{\nu\in\mathfrak{d}_K^{-1},\,\nu\succ 0\\ \mathrm{tr}(\nu)=1}} \ \sum_{\mathcal{O}\supset\mathfrak{A}\supset\nu\mathfrak{d}_K} \psi(\mathfrak{A}).$$

Here we take as K a totally real nonabelian cubic field of discriminant 257, whose class number h is 1. We have $K = \mathbb{Q}(\theta)$, where θ is a root of $x^3 - x^2 - 4x + 3 = 0$. Because the above polynomial is equal to $x(x^2 - x - 1) \bmod 3$, $(x+1)(x^2 - 2x - 2) \bmod 5$, $(x+3)(x^2+x+1) \bmod 7$, there are decompositions of 3, 5 and 7 into primes as $3 = \mathfrak{P}_3\mathfrak{P}_3'$, $5 = \mathfrak{P}_5\mathfrak{P}_5'$ and $7 = \mathfrak{P}_7\mathfrak{P}_7'$, where $\mathfrak{P}_i$'s are of degree 1 and $\mathfrak{P}_i'$'s are of degree 2. There are seven $\mu \in \mathfrak{d}_K^{-1}$ with $\mu \succ 0$ and $\mathrm{tr}(\mu) = 1$, and the ideals $\mu\mathfrak{d}_K$ are equal to $\mathfrak{P}_3$ for three of them, to $\mathfrak{P}_5$ for two μ's, to $\mathfrak{P}_7$ for one μ and to $\mathfrak{P}_3'$ for one μ. This computation was made by Cohen [5], Sect. 7. Let $F = K(\sqrt{-1}\,)$. Then

$$\begin{aligned} H/h &= \frac{1}{3}\left\{7 + 3\left(\frac{-1}{\mathfrak{P}_3}\right)_K + 2\left(\frac{-1}{\mathfrak{P}_5}\right)_K + \left(\frac{-1}{\mathfrak{P}_7}\right)_K + \left(\frac{-1}{\mathfrak{P}_3'}\right)_K\right\} \\ &= \frac{1}{3}\left\{7 + 3\left(\frac{-1}{3}\right) + 2\left(\frac{-1}{5}\right) + \left(\frac{-1}{7}\right) + 1\right\} = 2 \end{aligned}$$

where $\left(\frac{-1}{\mathfrak{P}_3'}\right)_K = 1$ since -1 is a square in $\mathbb{F}_9$. Thus the class number of F is 2. Let $F' = K(\sqrt{-\varepsilon}\,)$ with $\varepsilon = 2 + \theta \succ 0$. Then if H' is the class number of F', then

$$H'/h = \frac{1}{3}\left\{7 + 3\left(\frac{-\varepsilon}{\mathfrak{P}_3}\right)_K + 2\left(\frac{-\varepsilon}{\mathfrak{P}_5}\right)_K + \left(\frac{-\varepsilon}{\mathfrak{P}_7}\right)_K + \left(\frac{-\varepsilon}{\mathfrak{P}_3'}\right)_K\right\}.$$

From the above factorizations of $x^3 - x^2 - 4x + 3$ modulo 3, 5, 7, it follows that $-\varepsilon \equiv 1 \pmod{\mathfrak{P}_3}$, $-\varepsilon \equiv 4 \pmod{\mathfrak{P}_5}$, $-\varepsilon \equiv 1 \pmod{\mathfrak{P}_7}$ and that $-\varepsilon \pmod{\mathfrak{P}_3'}$ is not a square in $\mathbb{F}_9$. Therefore

$$H'/h = \frac{1}{3}\left\{7 + 3\left(\frac{1}{3}\right) + 2\left(\frac{4}{5}\right) + \left(\frac{1}{7}\right) - 1\right\} = 4.$$

Example 5. Let K be a totally real quartic field, and let F be its totally imaginary quadratic unramified extension. Since $\{-240, 1\} \in \mathrm{LR}_4'(1)$ (Siegel [20]), by the last corollary in Section 3 we have

$$H/h = \frac{1}{480} w(F) Q_{F/K} \mathfrak{f}_{0,\psi}(1).$$

Let $K = \mathbb{Q}(\sqrt{5}, \sqrt{6}\,)$ and let $F = \mathbb{Q}(\sqrt{-2}, \sqrt{-3}, \sqrt{5}\,)$, where F is an unramified extension of K. Then $h = 2$, $\mathfrak{d}_K = (2\sqrt{30}\,)$, $w(F) = 6$, and $Q_{F/K} = 2$. There are 22 numbers $\mu \in \mathfrak{d}_K^{-1}$ with $\mu \succ 0$ and $\mathrm{tr}(\mu) = 1$, and $\mu\mathfrak{d}_K$'s are the ideals generated by $(\pm1+\sqrt{5})(\pm\sqrt{5}+\sqrt{6})/2$ (norm 1), $(\pm1+\sqrt{5})(\pm2+\sqrt{6})/2$ (norm 4), $\{\pm(3+\sqrt{5})+\sqrt{6}+\sqrt{30}\}/2$, $\{\pm(3-\sqrt{5})-\sqrt{6}+\sqrt{30}\}/2$, $\{\pm2\pm\sqrt{6}+\sqrt{30}\}/2$ (norm 19), $(\pm1+\sqrt{5})(\pm1+\sqrt{6})/2$ (norm 25), $(\pm\sqrt{6}+\sqrt{30})/2$ (norm 36). In K we have the prime decompositions $2=\mathfrak{P}_2^2$, $3=\mathfrak{P}_3^2$, $5=\mathfrak{P}_5^2\mathfrak{P}_5'^2$ and $19=\mathfrak{P}_{19}\mathfrak{P}_{19}'\mathfrak{P}_{19}''\mathfrak{P}_{19}'''$, where $\mathfrak{P}_2=(2+\sqrt{6})$, $\mathfrak{P}_5^2=(1+\sqrt{6})$ and $\mathfrak{P}_5'^2=(1-\sqrt{6})$. Since $\mathfrak{P}_2$ and $\mathfrak{P}_5^2$ are in the same class of $\mathbf{C}_{\mathcal{O}}$, we have $\psi(\mathfrak{P}_2)=1$. Therefore

$$\begin{aligned}H/h=\frac{1}{40}\bigg\{&4+4(1+\psi(\mathfrak{P}_2))+8\bigg(1+\Big(\frac{-2}{19}\Big)\bigg)\\&+4\bigg(1+\Big(\frac{-2}{5}\Big)+\psi(\mathfrak{P}_5^2)\bigg)+2(1+\psi(\mathfrak{P}_2))\bigg(1+\Big(\frac{-2}{\mathfrak{P}_3}\Big)\bigg)\bigg\}\\=1.&\end{aligned}$$

EXAMPLE 6. Let K be a totally real quartic field, and let F be a totally imaginary quadratic extension of K with conductor $\mathfrak{D}$. Let $\psi=\mathbf{C}_{\mathfrak{D}}^*$ be the character associated with the extension. Suppose that $\mathfrak{D}=(4)$. Then $\chi_0=(\mathbb{Z}/4)^*$ is trivial. Since $\{0,-256,1\}\in\mathrm{LR}_4(4)$, we have

$$H/h=\frac{1}{16}w(F)Q_{F/K}\mathfrak{f}_{0,\psi}(1).$$

Next, suppose that 7 is the least element in $\mathbb{N}\cap\mathfrak{D}$ and that $\chi_0\in(\mathbb{Z}/7)^*$ is trivial. Since $\{1,-7^4,1,1\}\in\mathrm{LR}_4(7)$, we have

$$H/h=w(F)Q_{F/K}(7^4\,\mathrm{Nm}(\mathfrak{D})^{-1/2}-1)^{-1}\{\mathfrak{f}_{0,\psi}(1)+\mathfrak{f}_{0,\psi}(2)\}.$$

Let $K=\mathbb{Q}(\theta)$ with θ a zero of $f(x):=x^4-8x^3+20x^2-17x+3$. It is a nonabelian totally real quartic field of discriminant 1957 ($=19\cdot103$) and its $\mathbb{Z}$-basis is provided by 1, θ, θ^2, θ^3 (Godwin [8]). The ideal (2) remains prime at K. There are decompositions $3=\mathfrak{P}_3\mathfrak{P}_3'$ and $7=\mathfrak{P}_7\mathfrak{P}_7'$, where $\mathfrak{P}_3$, $\mathfrak{P}_7$ are primes of degree 1 and $\mathfrak{P}_3'$, $\mathfrak{P}_7'$ are of degree 3. The inverse different $\mathfrak{d}_K^{-1}=(1/f'(\theta))$ has 1, θ, θ^2, $\frac{1}{1957}(\theta^3+691\theta^2-350\theta-42)$ as its $\mathbb{Z}$-basis. With the aid of a computer, we can show that there are seven totally positive elements μ in $\mathfrak{d}_K^{-1}$ with trace 1. The ideals $\mu\mathfrak{d}_K$'s are equal to $\mathcal{O}$ for four elements and to $\mathfrak{P}_3$ for two and to $\mathfrak{P}_7$ for one. Let $F=K(\sqrt{-1})$. Then $\mathfrak{D}=(4)$, $w(F)=4$, $Q_{F/K}=1$, and $H/h=\frac14\{7+2(\frac{-1}{3})+(\frac{-1}{7})\}=1$. Let $\varepsilon=-\theta^3+5\theta^2-7\theta+2$, which is a totally positive unit. Let $F=K(\sqrt{-\varepsilon})$. Then $\mathfrak{D}=(4)$, $w(F)=2$, $Q_{F/K}=2$ and

$$H/h=\frac14\bigg\{7+2\Big(\frac{-\varepsilon}{\mathfrak{P}_3}\Big)_K+\Big(\frac{-\varepsilon}{\mathfrak{P}_7}\Big)_K\bigg\}=\frac14\bigg\{7+2\Big(\frac{1}{3}\Big)+\Big(\frac{-1}{7}\Big)\bigg\}=2.$$

Let $F=K(\sqrt{-7})$. Then $\mathfrak{D}=(7)$, $w(F)=2$, $Q_{F/K}=1$. We have

$$\chi_0(3)=\psi(3)=\Big(\frac{-7}{\mathfrak{P}_3}\Big)_K\Big(\frac{-7}{\mathfrak{P}_3'}\Big)_K=(-1)\cdot(-1)=1$$

since -7 is not a square in $\mathbb{F}_3$ and in $\mathbb{F}_{3^3}$. Since 3 is a generator of $(\mathbb{Z}/7)^\times$, χ_0 is trivial. Then $H/h = \frac{1}{24}\{\mathfrak{f}_{0,\psi}(1) + \mathfrak{f}_{0,\psi}(2)\}$. It can be shown that there are 58 totally positive elements in $\mathfrak{d}_K^{-1}$ with trace 2. By a similar computation to the above, we obtain $H/h = \frac{1}{24} \cdot 48 = 2$.

5. Hereafter we consider exclusively the case where K is a real quadratic field. Let χ_K denote the Kronecker–Jacobi–Legendre symbol of K. For an ideal $\mathfrak{A}$, $\overline{\mathfrak{A}}$ denotes its conjugate in K. If $\psi \in \mathbf{C}^*_{\mathfrak{N}}$ is invariant under conjugation, that is, $\psi(\mathfrak{A}) = \psi(\overline{\mathfrak{A}})$ for any $\mathfrak{A}$, then there is a completely multiplicative function χ on $\mathbb{N}$ such that $\psi(\mathfrak{A}) = \chi(\mathrm{Nm}(\mathfrak{A}))$ for any ideal $\mathfrak{A}$. Indeed, ψ obviously gives a completely multiplicative function χ on the subset of $\mathbb{N}$ consisting of norms of ideals. The desired χ is constructed by assigning to $\chi(p)$ any square root of $\chi(p^2)$, for each prime p which is inert. In particular, χ is not uniquely determined.

For completely multiplicative functions χ, χ', we define $\sigma^{\chi'}_{k-1,\chi}$ by setting

$$\sigma^{\chi'}_{k-1,\chi}(m) := \sum_{0<d|m} \chi'(m/d)\chi(d)d^{k-1}$$

for $m \in \mathbb{N}$, and $\sigma^{\chi'}_{k-1,\chi}(m) := 0$ for $m \notin \mathbb{N} \cup \{0\}$. In the sequel we denote it by $\sigma_{k-1,\chi}$ (resp. $\sigma^{\chi'}_{k-1}$) if χ' (resp. χ) is trivial. The value $\sigma_{k-1,\chi}(0)$ is defined to be $\frac{1}{2}L(1-k,\chi)$. The value $\sigma^{\chi'}_{k-1}(0)$ is defined to be 0 if $\chi' \neq 1$. For later use we present the following lemma. The proof is parallel to that of Theorem 3.4 in Cohen [5].

Lemma 3. (1) *Let $m, n \in \mathbb{N}$. Then*

$$\sigma^{\chi'}_{k-1,\chi}(m)\sigma^{\chi'}_{k-1,\chi}(n) = \sum_{d|(m,n)} \chi'(d)\chi(d)d^{k-1}\sigma^{\chi'}_{k-1,\chi}\left(\frac{mn}{d^2}\right).$$

(2) *Let $n \in \mathbb{N}$. Then*

$$\sum_{m=0}^{n} \sigma_{k-1,\chi}(m)\sigma_{k-1,\chi}(n-m) = \sum_{d|n} \chi(d)d^{k-1} \sum_{m\in\mathbb{Z}} \sigma_{k-1,\chi}\left(\frac{(n/d)^2 - m^2}{4}\right).$$

(3) *Suppose that $\chi' \neq 1$. Then*

$$\sum_{m=0}^{n} \sigma^{\chi'}_{k-1}(m)\sigma^{\chi'}_{k-1}(n-m) = \sum_{d|n} \chi'(d)d^{k-1} \sum_{m\in\mathbb{Z}} \sigma^{\chi'}_{k-1}\left(\frac{(n/d)^2 - m^2}{4}\right).$$

Proposition 4. *Let K be a real quadratic field.*

(1) *Let ψ, ψ' be as in Section 2 and let k be a natural number with the same parity as $\psi'\psi$. Suppose that there are completely multiplicative*

functions χ, χ' *with* $\psi = \chi \circ \mathrm{Nm}$, $\psi' = \chi' \circ \mathrm{Nm}$. *Then*

$$\mathfrak{f}^{\psi'}_{k-1,\psi}(n) = \sum_{0<d\mid n} \chi_K(d)\chi'(d)\chi(d)d^{k-1} \sum_{m\in\mathbb{Z}} \sigma^{\chi'}_{k-1,\chi}\left(\frac{(n/d)^2 D_K - m^2}{4}\right).$$

(2) *Let* $\psi = \chi \circ \mathrm{Nm} \in \mathbf{C}^*_{\mathcal{O}}$ *be odd. Let* p *be a rational prime which is not inert. Suppose that* $\chi(p) = -1$ *if* $\chi_K(p) = 1$. *If* $\mathfrak{P}$ *is a prime factor of* p *in* K, *then*

$$\mathfrak{f}_{0,\psi,\mathfrak{P}}(n) = -\frac{\chi(p)}{1+\chi_K(p)} \sum_{\substack{0<d\mid n\\ (d,p)=1}} \chi_K(d)\chi(d) \sum_{m\in\mathbb{Z}} \sigma_{0,\chi}\left(\frac{(n/d)^2 D_K - m^2}{4p}\right).$$

Proof. (1) Let $N(d,\mathfrak{A},K)$ denote the number of integral ideals of K dividing $\mathfrak{A}$ whose norms are d. By Lemma 2, $\mathfrak{f}^{\psi'}_{k-1,\psi}(n)$ is equal to

$$\sum_{\substack{|m|<n\sqrt{D_K}\\ m\equiv nD_K \pmod 2}} \sum_{0<d\mid (n^2D_K-m^2)/4} \chi'\left(\frac{n^2D_K - m^2}{4d}\right) \times \chi(d)d^{k-1} N\left(d, \frac{m+n\sqrt{D_K}}{2}, K\right).$$

It has been shown in Cohen [5] that

$$N\left(d, \frac{m+n\sqrt{D_K}}{2}, K\right) = \sum_{0<e\mid \gcd(m,n,d,(n^2D_K-m^2)/4)} \chi_K(e).$$

Then

$$\begin{aligned}\mathfrak{f}^{\psi'}_{k-1,\psi}(n) &= \sum_{\substack{|m|<n\sqrt{D_K}\\ m\equiv nD_K \pmod 2}} \sum_{0<e\mid \gcd(m,n,(n^2D_K-m^2)/4)} \\ &\quad\times \sum_{0<d_1\mid ((n/e)^2D_K-(m/e)^2)/4} \chi_K(e)\chi'(e) \\ &\quad\times \chi'\left(\frac{(n/e)^2D_K-(m/e)^2}{4d_1}\right)\chi(ed_1)e^{k-1}d^{k-1} \\ &= \sum_{0<d\mid n} \chi_K(d)\chi'(d)\chi(d)d^{k-1} \sum_{m\in\mathbb{Z}} \sigma^{\chi'}_{k-1,\chi}\left(\frac{(n/d)^2D_K-m^2}{4}\right).\end{aligned}$$

(2) First suppose $\chi_K(p) = 0$, that is, p is ramified at K. If $d \mid ((n/d)^2D_K - m^2)/(4p)$ and if $\mathfrak{P}^{-1}\left(\frac{m+n\sqrt{D_K}}{2}\right)$ is integral, then

$$N\left(d, \mathfrak{P}^{-1}\left(\frac{m+n\sqrt{D_K}}{2}\right), K\right) = N\left(d, \frac{m+n\sqrt{D_K}}{2}, K\right).$$

By Lemma 2 and by the same argument as in (1),

$$\mathfrak{f}_{0,\psi,\mathfrak{P}}(n) = -\chi(p) \sum_{0<d|n} \chi_K(d)\chi(d) \sum_{m\in\mathbb{Z}} \sigma_{0,\chi}\Big(\frac{(n/d)^2 D_K - m^2}{4p}\Big).$$

Now suppose that $\chi_K(p) = 1$, that is, p is decomposed at K, and that $\chi(p) = -1$. Let $v(m,n)$ (resp. $\overline{v}(m,n)$) denote the $\mathfrak{P}$-adic (resp. $\overline{\mathfrak{P}}$-adic) valuation of $(m + n\sqrt{D_K}\,)/2$, and let $v_p(m)$ denote the p-adic valuation of $m \in \mathbb{Z}$. Then by Lemma 2,

$$\begin{aligned}
&\mathfrak{f}_{0,\psi,\mathfrak{P}}(n) \\
&\quad = \sum_{\substack{|m|<n\sqrt{D_K} \\ m\equiv nD_K \ (\mathrm{mod}\ 2)}} (1+\chi(p)+\ldots+\chi(p)^{v(m,n)-1}) \\
&\qquad \times (1+\chi(p)+\ldots+\chi(p)^{\overline{v}(m,n)}) \\
&\qquad \times \sum_{\substack{0<d|(n^2D_K-m^2)/(4p) \\ (d,p)=1}} \chi(d) N\Big(d, \frac{m+n\sqrt{D_K}}{2}, K\Big) \\
&\quad = \sum_{\substack{|m|<n\sqrt{D_K} \\ m\equiv nD_K \ (\mathrm{mod}\ 2) \\ v(m,n)\ \mathrm{odd},\, \overline{v}(m,n)\ \mathrm{even}}} \ \sum_{\substack{0<d|(n^2D_K-m^2)/(4p) \\ (d,p)=1}} \chi(d) N\Big(d, \frac{m+n\sqrt{D_K}}{2}, K\Big).
\end{aligned}$$

A necessary condition that $v(m,n)$ be odd and $\overline{v}(m,n)$ be even is that $v_p((n^2D_K - m^2)/4)$ be odd. Under this condition, $v(m,n)$ and $\overline{v}(m,n)$ have the above properties only for one of $\pm m$ ($\neq 0$) for a fixed n. Hence since

$$N\Big(d, \frac{m+n\sqrt{D_K}}{2}, K\Big) = N\Big(d, \frac{-m+n\sqrt{D_K}}{2}, K\Big),$$

it follows that

$$\begin{aligned}
&\mathfrak{f}_{0,\psi,\mathfrak{P}}(n) \\
&\quad = \sum_{\substack{0<m<n\sqrt{D_K} \\ m\equiv nD_K \ (\mathrm{mod}\ 2) \\ v_p((n^2D_K-m^2)/4)\ \mathrm{odd}}} \ \sum_{\substack{0<d|(n^2D_K-m^2)/(4p) \\ (d,p)=1}} \chi(d) N\Big(d, \frac{m+n\sqrt{D_K}}{2}, K\Big).
\end{aligned}$$

If $v_p((n^2D_K - m^2)/4)$ is even, then $\sum_{m\in\mathbb{Z}} \sigma_{0,\chi}\big(\frac{(n/d)^2D_K-m^2}{4p}\big)$ vanishes. Then by the same argument as in (1) it is shown that

$$\mathfrak{f}_{0,\psi,\mathfrak{P}}(n) = \frac{1}{2} \sum_{\substack{0<d|n \\ (d,p)=1}} \chi_K(d)\chi(d) \sum_{m\in\mathbb{Z}} \sigma_{0,\chi}\Big(\frac{(n/d)^2 D_K - m^2}{4p}\Big).$$

This shows our assertion. ∎

Let χ be a completely multiplicative function on $\mathbb{N}$ and suppose that $\psi := \chi \circ \mathrm{Nm} \in \mathbf{C}^*_{(N)}$. Let M be a divisor of N contained in the conductor $\mathfrak{f}_\psi$. Then $\psi_{(M)} \in \mathbf{C}^*_{(M)}$ (see Section 2 for the notation) is also invariant under conjugation, and in particular there is a completely multiplicative function $\chi_{(M)}$ on $\mathbb{N}$ such that $\psi_{(M)} = \chi_{(M)} \circ \mathrm{Nm}$.

Since there is an identity

$$L_K(s,\psi) = L(s,\chi)L(s,\chi\chi_K),$$

by Propositions 2 and 4 we have the following:

THEOREM 2. *Let $k, N \in \mathbb{N}$ with $kN \neq 1$. Let K be a real quadratic field and let $\mathbf{C}_{(N)}$ be its narrow ideal class group modulo N. Let χ be a completely multiplicative function on $\mathbb{N}$ such that $\psi := \chi \circ \mathrm{Nm} \in \mathbf{C}^*_{(N)}$ has the same parity as k. Let χ_0 be such that $\chi_0(i) = \chi(i^2)$.*

(1) *We have the identity*

$$\begin{aligned}\lambda_{2k,\psi}(z) = {} & L(1-k,\chi)L(1-k,\chi\chi_K) \\ & + 4\sum_{n=1}^{\infty}\sum_{0<d\mid n}\chi_K(d)\chi(d)d^{k-1} \\ & \times \sum_{m\in\mathbb{Z}}\sigma_{k-1,\chi}\left(\frac{(n/d)^2D_K - m^2}{4}\right)\mathsf{e}(nz),\end{aligned}$$

which is in $\mathbf{M}_{2k,\chi_0}(N)$. For M with $M \mid N$, the 0th Fourier coefficient at a cusp i/M, $(i,M) = 1$, is equal to 0 if $M \notin \mathfrak{f}_\psi$, and to

$$\chi_0(i)^{-1}\prod_{p\mid(N/M)}(1-p^{-1})(1-\chi_K(p)p^{-1})L(1-k,\chi_{(M)})L(1-k,\chi_{(M)}\chi_K)$$

otherwise, and there is an additional term $-\pi^{-2}D_K^{1/2}L(1,\chi)L(1,\chi\chi_K)$ at a cusp 0 if $k = 1$. Suppose that N is the least element in $\mathbb{N}\cap\mathfrak{f}_\psi$. Then the modular form is in $\mathbf{M}^{\infty}_{2k,\chi_0}(N)$ $(k > 1)$ or in $\mathbf{M}^{\infty,0}_{2,\chi_0}(N)$ $(k = 1)$.

(2) *Let $k > 1$ and $N > 1$. Then*

$$\lambda^{\psi}_{2k}(z) = 4\sum_{n=1}^{\infty}\chi_K(d)\chi(d)d^{k-1}\sum_{m\in\mathbb{Z}}\sigma^{\chi}_{k-1}\left(\frac{(n/d)^2D_K - m^2}{4}\right)\mathsf{e}(nz)$$

is in $\mathbf{M}^0_{2k,\chi_0}(N)$. The 0th Fourier coefficient at the cusp 0 is equal to

$$4(-1)^k\left(\frac{(k-1)!}{(2\pi)^k}\right)^2 D_K^{k-1/2}L(k,\chi)L(k,\chi\chi_K).$$

Let $N \in \mathbb{N}$, $N > 1$, and let $\chi \in (\mathbb{Z}/N)^*$. Then χ is said to be *even* or *odd* according as $\chi(-1) = 1$ or -1. Let $\mathfrak{N}$ be an integral ideal of K

containing N so that $N \mid \mathrm{Nm}(\mathfrak{N})$ and $\mathrm{tr}(\mathfrak{N}) \subset N\mathbb{Z}$. Put $\psi := \chi \circ \mathrm{Nm}$. We show that $\psi \in \mathbf{C}^*_{\mathfrak{N}}$. If $\alpha \equiv \beta \pmod{\mathfrak{N}}$ with $\alpha, \beta \in \mathcal{O}$ relatively prime to $\mathfrak{N}$, then $\mathrm{Nm}(\alpha) \equiv \mathrm{Nm}(\beta) \pmod{N}$. Indeed, putting $\alpha/\beta = 1 + \xi/\beta$, $\xi \in \mathfrak{N}$, we have

$$\frac{\mathrm{Nm}(\alpha)}{\mathrm{Nm}(\beta)} = 1 + \frac{1}{\mathrm{Nm}(\beta)}(\mathrm{tr}(\beta\overline{\xi}) + \mathrm{Nm}(\xi)) \in 1 + \frac{N}{\mathrm{Nm}(\beta)}\mathbb{Z},$$

where we note that $(\mathrm{Nm}(\beta), N) = 1$. Then $\psi(\alpha\mathfrak{A}) = \psi(\mathfrak{A})$ for $\alpha \succ 0$, $\alpha \equiv 1 \pmod{\mathfrak{N}}$, which implies that $\psi \in \mathbf{C}^*_{\mathfrak{N}}$. For $\alpha \equiv \beta \pmod{\mathfrak{N}}$, we have $|\mathrm{Nm}(\alpha)| \equiv \mathrm{sgn}(\mathrm{Nm}(\alpha/\beta))|\mathrm{Nm}(\beta)| \pmod{N}$ and so ψ is even or odd according as χ is even or odd.

Now let $\mathfrak{N} = (N)$. The above argument shows that for $\chi \in (\mathbb{Z}/N)^*$, $\psi := \chi \circ \mathrm{Nm}$ is a character in $\mathbf{C}^*_{(N)}$. However, it is sometimes possible that even if χ is in $(\mathbb{Z}/N')^*$ with $N \mid N'$, $N' > N$, ψ is still a character in $\mathbf{C}^*_{(N)}$. For example, suppose that $4 \mid D_K$ and $2 \mid N$. Then $2N \mid \mathrm{Nm}(\mathfrak{N})$ and $2N\mathbb{Z} \subset \mathrm{tr}(\mathfrak{N})$, that is, $2N$ plays the same role as N in the above argument. Hence $\chi \in (\mathbb{Z}/2N)^*$ gives a character ψ of the group $\mathbf{C}_{(N)}$. Later for a Dirichlet character χ we obtain the minimal $N \in \mathbb{N}$ for which $\psi \in \mathbf{C}^*_{(N)}$.

Let χ be a Dirichlet character in $(\mathbb{Z}/N)^*$ with the same parity as k. Consider the case $K = \mathbb{Q}$ in Section 2, where we have constructed a modular form $\widetilde{\lambda}^{\psi'}_{k,\psi}$. Put $G_{k,\chi} := \widetilde{\lambda}_{k,\chi} \in \mathbf{M}_{k,\chi}(N)$ ($k \neq 2$ or $N \neq 1$), and $G^{\chi}_k := \widetilde{\lambda}^{\chi}_k \in \mathbf{M}_{k,\chi}(N)$ ($k \neq 2$ or χ is nontrivial). For $k \geq 2$, we have the expansions

$$G_{k,\chi}(z) = L(1-k,\chi) + 2\sum_{n=1}^{\infty} \sigma_{k-1,\chi}(n)\mathsf{e}(nz)$$

and

$$G^{\chi}_k(z) = 2\sum_{n=1}^{\infty} \sigma^{\chi}_{k-1}(n)\mathsf{e}(nz).$$

This holds also for $k = 1$, except possibly for the constant term. Let $\theta(z) := \sum_{n=1}^{\infty} \mathsf{e}\big(\frac{1}{2}n^2 z\big)$ be a thetanullwerte. Then

$$\theta(2z)G_{k,\chi}(4z) = L(1-k,\chi) + 2\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} \sigma_{k-1,\chi}\bigg(\frac{n-m^2}{4}\bigg)\mathsf{e}(nz)$$

and

$$\theta(2z)G^{\chi}_k(4z) = 2\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} \sigma^{\chi}_{k-1}\bigg(\frac{n-m^2}{4}\bigg)\mathsf{e}(nz)$$

are modular forms for $\Gamma_1(4N)$ of weight $k + 1/2$ with character χ. Then $\theta(2z)G_{k,\chi}(4z)$ and $\lambda_{2k,\psi}(z)$, or $\theta(2z)G^{\chi}_k(4z)$ and $\lambda^{\psi}_{2k}(z)$ give an example of Shimura correspondence between noncusp forms of half-integral and integral weight. In a later paper we shall investigate a Shimura correspondence by using this fact.

The following lemma is easily verified. Here we denote by $\bar{i}$ ($i \in \mathbb{Z}$) the class of $\mathbb{Z}/8\mathbb{Z}$ containing i.

LEMMA 4. (1) *Let p be an odd prime. Then the map of $\mathcal{O}$ ($\subset K$) to $\mathbb{Z}/p\mathbb{Z}$ defined by $\alpha \to \mathrm{Nm}(\alpha) \pmod p$, $\alpha \in \mathcal{O}$, is surjective if $p \nmid D_K$. If $p \mid D_K$, then the image is the set of squares in $\mathbb{Z}/p\mathbb{Z}$.*

(2) *The image of the map $\alpha \to \mathrm{Nm}(\alpha) \pmod 8$ from $\{\alpha \in \mathcal{O} : (\alpha, 2) = \mathcal{O}\}$ to $(\mathbb{Z}/8\mathbb{Z})^\times$ is $(\mathbb{Z}/8\mathbb{Z})^\times$ ($D_K \equiv 1 \pmod 4$), $\{\bar{1}, \bar{5}\}$ ($D_K \equiv 4 \pmod 8$), $\{\bar{1}, \overline{1 - D_K/4}\}$ ($D_K \equiv 0 \pmod 8$).*

(3) *The image of the same map from $\{\alpha \in \mathcal{O} : \alpha \equiv 1 \pmod 2\}$ to $(\mathbb{Z}/8\mathbb{Z})^\times$ is $(\mathbb{Z}/8\mathbb{Z})^\times$ ($D_K \equiv 1 \pmod 4$), $\{\bar{1}, \bar{5}\}$ ($D_K \equiv 4 \pmod 8$), $\{\bar{1}\}$ ($D_K \equiv 0 \pmod 8$).*

(4) *The image of the same map from $\{\alpha \in \mathcal{O} : \alpha \equiv 1 \pmod 4\}$ to $(\mathbb{Z}/8\mathbb{Z})^\times$ is $\{\bar{1}, \bar{5}\}$ if $D_K \equiv 1 \pmod 4$.*

Even if the domains of the maps in Lemma 4 are replaced by the subsets consisting of totally positive elements, the images do not change.

Let $\mathbb{D}$ denote the set of integers of the form u^2D' with $u \in \mathbb{N}$ and D' the discriminants of a quadratic field or 1. We note that once an integer is of this form, such an expression is unique. The set $\mathbb{D}$ is closed under multiplication. If $D' = 1$, then $\chi_{D'}$ denotes the trivial character, and otherwise it denotes the Kronecker–Jacobi–Legendre symbol. For $D = u^2D' \in \mathbb{D}$, we define χ_D to be the character

$$\chi_D(m) = \begin{cases} \chi_{D'}(m) & ((D, m) = 1), \\ 0 & ((D, m) \neq 1). \end{cases}$$

LEMMA 5. *Let $D \in \mathbb{D}$ with $D = u^2D'$, where D' is 1 or a discriminant and $(u, D') = 1$, and let D_K be a positive discriminant.*

(1) *Let $N = |D'| \prod_{p|u} p$ ($v_2(D'D_K) \le 3$), $N = \frac{1}{2}|D'| \prod_{p|u} p$ ($v_2(D'D_K) = 4, 5$) and $N = \frac{1}{4}|D'| \prod_{p|u} p$ ($v_2(D'D_K) = 6$). Then $\chi_D \circ \mathrm{Nm}$ is in $\mathbf{C}^*_{(N)}$.*

(2) *Let $u = 1$. Then a necessary and sufficient condition for N to be the minimal natural number in the conductor of $\chi_D \circ \mathrm{Nm}$ is*

(i) *D and D_K have no common odd prime factor*, *and*
(ii) *neither $v_2(DD_K) = 4$ nor $DD_K/64 \equiv 1 \pmod 4$.*

P r o o f. (1) It is enough to show the assertion in case $u = 1$. Let $Z_N := \{(\mu) : \mu \in \mathcal{O},\ \mu \succ 0,\ \mu \equiv 1 \pmod N\}$. This is the identity element of $\mathbf{C}_{(N)}$. We must show that $\chi_D \circ \mathrm{Nm}$ is trivial on Z_N. If D is odd, then there is nothing to prove. Let $D \equiv 4 \pmod 8$. Lemma 4(2), (3) implies that $\chi_D \circ \mathrm{Nm}$ is trivial on $Z_{D/4}$ ($D_K \equiv 4 \pmod 8$), or on $Z_{D/2}$ ($D_K \equiv 0 \pmod 8$), and hence $\chi_D \circ \mathrm{Nm}$ is trivial on Z_N. Let $D \equiv 0 \pmod 8$. For i odd, let $\bar{i}$ denote the class in $\mathbb{Z}/(D)$ which is congruent to $i \pmod 8$ and to $1 \pmod{D/8}$. Then $\chi_D(\bar{5}) = -1$, $\chi_D(\bar{3}) = -(-1)^{(D/8-1)/2}$, $\chi_D(\bar{7}) = (-1)^{(D/8-1)/2}$. By

Lemma 4(2)–(4), $\chi_D\circ\mathrm{Nm}$ is trivial on $Z_{D/2}$ ($D_K\equiv 4 \pmod 8$), or on $Z_{D/4}$ ($D_K\equiv 0 \pmod 8$). Thus $\chi_D\circ\mathrm{Nm}$ is trivial on Z_N also in this case, which shows our assertion.

(2) Let p be a prime with $p\mid N$. We must show that $\chi\circ\mathrm{Nm}$ is nontrivial on $Z_{N/p}$ for any p if and only if D and D_K satisfy the condition. Since D is a discriminant, χ_D is a primitive character $\bmod D$. Let p be odd. If $p\nmid D_K$, then the image of the map $\mathfrak{A}\to\mathrm{Nm}(\mathfrak{A}) \pmod p$ from $Z_{N/p}$ to $(\mathbb{Z}/p\mathbb{Z})^\times$ is surjective by Lemma 4, and hence χ_D is nontrivial on $Z_{N/p}$ by primitiveness. If $p\mid D_K$, then χ_D is trivial on $Z_{N/p}$ again by Lemma 4. Hence (i) follows. Let $p=2$. By a similar argument to (1), we can show that $\chi_D\circ\mathrm{Nm}$ is nontrivial on $Z_{N/2}$ except for the case (ii). ∎

Let $D=2^w da$, $D_K=2^w da'$ ($w=0,2,3,\ 2\nmid d,\ 2\nmid a,\ 2\nmid a',\ (a,a')=1$) be distinct discriminants, where $aa'\equiv 1 \pmod 4$ if $w=3$. We note that $a\equiv a' \pmod 4$ and that aa' is a discriminant. Let $\widetilde{\chi}$ be the multiplicative function defined by $\widetilde{\chi}(p)=\chi_D(p)$ ($p\nmid 2^w d$) and $\widetilde{\chi}(p)=\chi_{aa'}(p)$ ($p\mid 2^w d$). Then $\widetilde{\chi}\circ\mathrm{Nm}$ is in $\mathbf{C}^*_{(a)}$ and its restriction to $\mathbf{C}_{(D)}$ is equal to the character $\chi_D\circ\mathrm{Nm}$. Let $\psi:=\chi_D\circ\mathrm{Nm}$ and $\widetilde{\psi}:=\widetilde{\chi}\circ\mathrm{Nm}$. Then

$$L_K(1-k,\widetilde{\psi})=\prod_{\mathfrak{P}\supset 2^w d}(1-\chi_{aa'}(\mathrm{Nm}(\mathfrak{P}))\,\mathrm{Nm}(\mathfrak{P})^{k-1})^{-1}L_K(1-k,\psi).$$

Hence

$$\begin{aligned}L(1-k,\widetilde{\chi})&L(1-k,\widetilde{\chi}\chi_K)\\ &=\prod_{p\mid 2^w d}(1-\chi_{aa'}(p)p^{k-1})^{-1}L(1-k,\chi_D)L(1-k,\chi_D\chi_K)\\ &=L(1-k,\chi_D)L(1-k,\chi_{aa'}).\end{aligned}$$

More generally, for $M\in\mathfrak{f}_\psi$, we have

$$L(1-k,\chi_{(M)})L(1-k,\chi_{(M)}\chi_K)=L(1-k,\chi_D)L(1-k,\chi_{M^2aa'}).$$

Let $D\in\mathbb{D}$, and $k\in\mathbb{N}$ with $(-1)^kD>0$. Put $\lambda_{2k,D_K,D}:=\lambda_{2k,\chi_D\circ\mathrm{Nm}}$ and $\lambda^{(D)}_{2k,D_K}:=\lambda_{2k}^{\chi_D\circ\mathrm{Nm}}$ for a positive discriminant D_K. Further, put $\lambda_{2k,1,D}:=(G_{k,\chi_D})^2$ ($k\neq 2$ or $D\neq 1$), and $\lambda^{(D)}_{2k,1}:=(G_k^{\chi_D})^2$ ($k\neq 2$ or D is not a square). In the following corollary we treat the case $k>1$. The case $k=1$ is considered in Section 7.

Corollary to Theorem 2. *Let D_K be 1 or the discriminant of a real quadratic field, and let $D\in\mathbb{D}$, u and D' be as in Lemma 5. Let $k>1$ with $(-1)^kD>0$. Let N be $|D'|\prod_{p\mid u}p$ if $D_K=1$ and as in Lemma 5(1) otherwise. Put $D'':=4D'D_K/(D',D_K)^2$ and $E:=2|D'/(D',D_K)|$ in case $v_2(D'D_K)=5$ or $D'D_K/64\equiv 3 \pmod 4$, and put $D'':=D'D_K/(D',D_K)^2$ and $E:=|D'/(D',D_K)|$ in any other case.*

(1) *Suppose that* $D \neq 1$ *if* $k = 2$ *and* $D_K = 1$. *Then*

$$\begin{aligned}\lambda_{2k,D_K,D}(z) = L(1-k,\chi_D)&L(1-k,\chi_{DD_K})\\&+4\sum_{n=1}^{\infty}\sum_{0<d\mid n}\chi_{DD_K}(d)d^{k-1}\\&\times\sum_{m\in\mathbb{Z}}\sigma_{k-1,\chi_D}\left(\frac{(n/d)^2D_K-m^2}{4}\right)\mathsf{e}(nz)\end{aligned}$$

is in $\mathbf{M}_{2k}(N)$. *For* M *with* $M \mid N$ *and* $(M, D'') = (N, D'')$, *the* 0*th Fourier coefficient at a cusp* i/M, $(i, M) = 1$, *is equal to*

$$\prod_{p\mid(N/M)}(1-p^{-1})(1-\chi_{D_K}(p)p^{-1})L(1-k,\chi_{M^2D'})L(1-k,\chi_{M^2D''}).$$

The modular form is in $\mathbf{M}^{\infty}_{2k}(N)$ *if* D, D_K *satisfy the conditions in Lemma* 5(2).

(2) *Let* $D \neq 1$. *Suppose that* D *is not a square if* $k = 2$ *and* $D_K = 1$. *Then*

$$\lambda^{(D)}_{2k,D_K}(z) = 4\sum_{n=1}^{\infty}\sum_{0<d\mid n}\chi_{DD_K}(d)d^{k-1}\sum_{m\in\mathbb{Z}}\sigma^{\chi_D}_{k-1}\left(\frac{(n/d)^2D_K-m^2}{4}\right)\mathsf{e}(nz)$$

is in $\mathbf{M}^{0}_{2k}(N)$. *The* 0*th Fourier coefficient at the cusp* 0 *is*

$$\begin{aligned}(-1)^kE^{-2k+1}\prod_{p\mid u}(1-\chi_{D'}(p)p^{-k})\prod_{p\mid(DD_K/D'')}&(1-\chi_{D''}(p)p^{-k})\\&\times L(1-k,\chi_{D'})L(1-k,\chi_{D''}).\end{aligned}$$

Proof. First let D_K be a discriminant. Then the assertions (1), (2) follow immediately from Theorem 2 and Lemma 5, except for the 0th Fourier coefficient at the cusp 0. We have the equality

$$L(k,\chi_D) = \prod_{p\mid u}(1-\chi_{D'}(p)p^{-k})L(k,\chi_{D'}).$$

D'' is a discriminant with $D'D_K = t^2D''$, and

$$L(k,\chi_{DD_K}) = \prod_{p\mid(DD_K/D'')}(1-\chi_{D''}(p)p^{-k})L(k,\chi_{D''}).$$

Then the functional equations of L-functions of primitive Dirichlet characters give our 0th Fourier coefficient. Now let $D_K = 1$. Our Fourier expansions are obtained by Lemma 3, and the assertions follow from Propositions 1 and 2 in case $K = \mathbb{Q}$. ∎

6. We give some applications of the Corollary to Theorem 2. Let

$$G_k(z) := 1 + \frac{(-1)^{k/2}2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)\mathsf{e}(nz) \quad \text{for even } k \ge 4,$$

where B_k denotes the kth Bernoulli number. This is a normalized Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ of weight k. The following lemma is elementary.

LEMMA 6. (i) *There is no nontrivial cusp form in* $\mathbf{M}_k(N)$ *if* $k < 12$ *and* $N = 1$ *or if* $(k, N) = (4,2), (4,3), (4,4), (6,2)$.

(ii) *Let* $k \ge 4$ *be even. If* N *is prime, then* $(1/(N^k-1))(N^kG_k(Nz) - G_k(z)) \in \mathbf{M}_k^{\infty}(N)$ *and* $(N^k/(N^k-1))(G_k(z) - G_k(Nz)) \in \mathbf{M}_k^0(N)$. *The former* (*resp. the latter*) *has* 1 *as its* 0*th coefficient at the cusp* $\sqrt{-1}\,\infty$ (*resp.* 0).

LEMMA 7. *Let* $a \in \mathbb{N}$ *be square-free. Let* a^* *be* a *or* $4a$ *according as* $a \equiv 1 \pmod 4$ *or not. Denote by* μ *the Möbius function. Let* $k \ge 2$ *be even and let* N *be* 1 *or a prime. Then, up to* $O(a^{k/2-1/28+\varepsilon}n^{k-1+\varepsilon})$,

$$\sum_{m\in\mathbb{Z}}\sigma_{k-1,\chi_{N^2}}\Big(\frac{n^2a^*-m^2}{4}\Big)$$

$$= \begin{cases} \dfrac{(-1)^{k/2}B_kL(1-k,\chi_{a^*})}{B_{2k}}\displaystyle\sum_{d|n}\mu(d)\chi_{a^*}(d)d^{k-1}\sigma_{2k-1}\Big(\frac{n}{d}\Big) & (N=1),\\[2ex] \dfrac{(-1)^{k/2}B_kL(1-k,\chi_{a^*})}{B_{2k}(N^k+1)}\displaystyle\sum_{d|n}\mu(d)\chi_{N^2a^*}(d)d^{k-1} & \\ \times\Big[\{N^k - N^{k-1} + 1 - \chi_{a^*}(N)N^{k-1}\}\sigma_{2k-1}\Big(\dfrac{n}{d}\Big) & \\ + N^{2k-2}\{-N + \chi_{a^*}(N)(N^k - N + 1)\}\sigma_{2k-1}\Big(\dfrac{n}{Nd}\Big)\Big] & (N \text{ prime}) \end{cases}$$

where there is an additional term $-\frac{1}{2}n^2$ *if* $N = 1$, $a = 1$ *and* $k = 2$. *The term* $O(a^{k/2-1/28+\varepsilon}n^{k-1+\varepsilon})$ *is* 0 *if* k *and* N *are as in Lemma* 5(1).

Proof. Let $a \equiv 1 \pmod 4$. Suppose $N \ne 1$ or $k \ne 2$. Put

$$c_0 := (1-N^{k-1})(1-\chi_a(N)N^{k-1})\zeta(1-k)L(1-k,\chi_a),$$
$$c_0' := (1-N^{-1})(1-\chi_a(N)N^{-1})\zeta(1-k)L(1-k,\chi_a).$$

Then by the Corollary to Theorem 2, λ_{2k,a,N^2} is in $\mathbf{M}_{2k}^{\infty,0}(N)$ with c_0 (resp. c_0') as its 0th Fourier coefficient at $\sqrt{-1}\,\infty$ (resp. 0). By Lemma 6(ii), $\lambda_{2k,a,N^2}(z) = (c_0/(N^k-1))(N^kG_k(Nz) - G_k(z)) + (c_0'N^k/(N^k-1))(G_k(z) - G_k(Nz))$ plus some cusp form. Comparing the Fourier coefficients and using the Möbius inversion formula we obtain the formula. The error term vanishes if $\mathbf{M}_{2k}(N)$ contains no nontrivial cusp form. A similar argument works also for other cases except for the case $N=1$, $a=1$, $k=2$ in (1) of the Corollary to Theorem 2, where nonexistence of $\lambda_{4,1,1}$ causes difficulty. For this, we refer to Cohen [5], Theorem 3.6. The Ramanujan–Petersson conjecture proved by Deligne and Iwaniec's result [14] gives the estimate of the error term. ∎

We give arithmetic expressions for values of $L(1-k,\chi_D)$ $(k=2,3,4)$ with D being discriminants of quadratic fields.

Example 1. Let D be a positive discriminant. Then

$$L(-1,\chi_D) = -\frac{1}{5}\sum_{m\in\mathbb{Z}}\sigma_1\biggl(\frac{D-m^2}{4}\biggr) = \frac{-1}{4-\chi_D(2)}\sum_{m\in\mathbb{Z}}\sigma_1^{\chi_4}\biggl(\frac{D-m^2}{4}\biggr) = \frac{-2}{9-\chi_D(3)}\sum_{m\in\mathbb{Z}}\sigma_1^{\chi_9}\biggl(\frac{D-m^2}{4}\biggr),$$

$$L(-3,\chi_D) = \sum_{m\in\mathbb{Z}}\sigma_3\biggl(\frac{D-m^2}{4}\biggr).$$

These equalities are obtained by substituting $n=1$ in Lemma 7. Let D be a negative discriminant. Then

$$L(-2,\chi_D) = \begin{cases} \dfrac{1}{31+4(-1)^{(D+1)/2}}\displaystyle\sum_{m\in\mathbb{Z}}\sigma_{2,\chi_{-4}}(|D|-m^2) & (2\nmid D),\\ -\displaystyle\sum_{m\in\mathbb{Z}}\sigma_{2,\chi_{-4}}\biggl(\frac{|D|-m^2}{4}\biggr) & (v_2(D)\ge 2).\end{cases}$$

Indeed, let $D_K = -4D$ $(2\nmid D)$, $-D/4$ $(v_2(D)=2)$, $-D$ $(v_2(D)=3)$. Then $\lambda_{6,D_K,-4}$ is in $\mathbf{M}_6(2)$, $\mathbf{M}_6^\infty(4)$, $\mathbf{M}_6^\infty(2)$ in the respective cases. From $\{8,-512,-1\}\in \mathrm{LR}_6(2)$, $\{8,0,-1\}\in\mathrm{LR}_6(4)$, $\{8,-1\}\in\mathrm{LR}_6'(2)$, the formula follows.

For a positive definite integral quadratic form f, we denote by $r_f(a)$ the number of integral representations of a by f. If f is a sum of k squares, then we denote it by $r_k(a)$. For a square-free a, we can have a formula for $r_{2k+1}(n^2a)$ up to $O(a^{k/2-1/28+\varepsilon}n^{k-1+\varepsilon})$ (cf. van Asch [1]). However, we treat several other quadratic forms here.

Let S be a positive even symmetric matrix of size $2k$ $(k\ge 2)$ with square determinant M^2 $(M\in\mathbb{N})$ with level N, that is, N is the least number in $\mathbb{N}$ such that NS^{-1} is even. Suppose that k is even and $N=1$ or a prime. The theta series

$$\Theta_S(z) = \sum_{r\in\mathbb{Z}^{2k}}\mathsf{e}\biggl(\frac{1}{2}{}^t rSrz\biggr)$$

associated with S is in $\mathbf{M}_k(N)$. The theta series takes the value $(-1)^{k/2}/M$ at the cusp 0 by the inversion formulas for theta series. It is written as a sum of Eisenstein series in Lemma 6 up to cusp forms. Let $g = \frac{1}{2}{}^t\mathbf{x}S\mathbf{x}$ with ${}^t\mathbf{x} = (x_1, \ldots, x_{2k})$. By the expression of Θ_S, $r_g(n)$ $(n \in \mathbb{N})$ is shown to be equal, up to $O(n^{(k-1)/2+\varepsilon})$, to

$$\frac{2k}{B_k}\sigma_{k-1}(n) \quad (4\,|\,k,\ N=1),$$

$$\frac{2k}{(N^k-1)B_k}[\{M^{-1}N(N^{k-1}-1)+(-1)^{k/2}(N-1)\} \times \sigma_{k-1}(n) + N(M^{-1}-(-1)^{k/2})\sigma_{k-1,\chi_{N^2}}(n)] \quad (2\,|\,k,\ \text{prime } N).$$

Here we note that for N prime,

$$\sigma_{k-1}(n/N) = N^{-k+1}(\sigma_{k-1}(n) - \sigma_{k-1,\chi_{N^2}}(n)),$$
$$\sigma_{k-1}^{\chi_{N^2}}(n) = (1-N^{-k+1})\sigma_{k-1}(n) + N^{-k+1}\sigma_{k-1,\chi_{N^2}}(n).$$

Let $f = g + x_{2k+1}^2$. Then $r_f(n) = \sum_{m\in\mathbb{Z}} r_g(n-m^2)$. By the above formulas for r_g, $r_f(n)$ is written in terms of $\sigma_{k-1}(n-m^2)$, $\sigma_{k-1,\chi_{N^2}}(n-m^2)$ up to $O(n^{k/2+\varepsilon})$. Then Lemma 7 gives a formula for r_f. For a square-free $a \in \mathbb{N}$, $r_g(n^2a)$ is equal, up to $O(a^{k/2-1/28+\varepsilon}n^{k-1+\varepsilon})$, to

$$\frac{2kL(1-k,\chi_{a^*})}{B_{2k}}\sum_{d|n^*}\mu(d)\chi_{a^*}(d)d^{k-1}\sigma_{2k-1}(n^*/d) \quad (4\,|\,k,\ N=1),$$

$$\begin{aligned}\frac{(-1)^{k/2}2kL(1-k,\chi_{a^*})}{M(N^{2k}-1)B_{2k}}\sum_{N\nmid d|n^*}&\mu(d)\chi_{a^*}(d)d^{k-1}[\{N^{2k}-(-1)^{k/2}M\\ &-\chi_{a^*}(N)N^k(1-(-1)^{k/2}M)\}\sigma_{2k-1}(n^*/d)\\ &-N^{k-1}\{N^{k+1}(1-(-1)^{k/2}M)+\chi_{a^*}(N)(-N+(-1)^{k/2}\\ &\times M(N^{2k}+N-1))\}\sigma_{2k-1}(n^*/Nd)] \quad (N \text{ being prime}),\end{aligned}$$

where n^* denotes $2n$ or n according as $a \equiv 1 \pmod 4$ or not and where in the latter formula there is an additional term $-240(N-M)M^{-1}(N+1)^{-1}n^2$ if $k=2$ and $a=1$.

Example 2. Let $g = x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_1x_2+x_3x_4$. Then $k=2$, $N=M=3$, and

$$\begin{aligned}r_g(n^2a) = 6L(-1,\chi_{a^*})\sum_{3\nmid d|n^*}\mu(d)\chi_{a^*}(d)d\{&(-7+3\chi_{a^*}(3))\sigma_3(n^*/d)\\ &+9(3-7\chi_{a^*}(3))\sigma_3(n^*/(3d))\}.\end{aligned}$$

Since $\mathbf{M}_4(3)$ contains no nontrivial cusp form, there appears no error term.

Example 3. Let A_{8k} ($k \in \mathbb{N}$) be a positive even unimodular matrix of size $8k$, and let $g = \frac{1}{2}{}^t\mathbf{x}A_{8k}\mathbf{x} + x_{8k+1}^2$ with ${}^t\mathbf{x} = (x_1, \ldots, x_{8k})$. For a square-free integer $a \in \mathbb{N}$,

$$r_g(n^2a) = \frac{8kL(1-4k, \chi_{a^*})}{B_{8k}} \sum_{d|n^*} \mu(d)\chi_{a^*}(d)d^{4k-1}\sigma_{8k-1}(n^*/d) + O(a^{2k-1/28+\varepsilon}n^{4k-1+\varepsilon}).$$

If $k = 1$, then

$$r_g(n^2a) = -240L(-3, \chi_{a^*}) \sum_{d|n^*} \mu(d)\chi_{a^*}(d)d^3\sigma_7(n^*/d),$$

since there is no nontrivial cusp form in $\mathbf{M}_8(1)$.

Finally, we give a formula in the case of a quadratic form with nonsquare discriminant.

Example 4. Let $g = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_5^2$. Let $a \in \mathbb{N}$ be square-free. Then

$$r_g(n^2a) = \begin{cases} -8L(-1, \chi_{8a}) \sum_{d|n} \mu(d)\chi_{8a}(d)d\left\{\sigma_3\left(\frac{n}{d}\right) - 16\sigma_3\left(\frac{n}{4d}\right)\right\} & (2\nmid a), \\ -8L(-1, \chi_{2a}) \sum_{d|n} \mu(d)\chi_{2a}(d)d\left\{3\sigma_3\left(\frac{n}{d}\right) - 8\sigma_3\left(\frac{n}{2d}\right)\right\} & (a \equiv 6 \pmod 8), \\ -8L(-1, \chi_{a/2}) \sum_{d|n} \mu(d)\chi_{2a}(d)d\left\{(19 - 6\chi_{a/2}(2))\sigma_3\left(\frac{n}{d}\right) + 8(-3 + 2\chi_{a/2}(2))\sigma_3\left(\frac{n}{2d}\right)\right\} & (a \equiv 2 \pmod 8). \end{cases}$$

Let f denote a quaternary form $x_1^2 + x_2^2 + x_3^2 + 2x_4^2$. By a standard argument, we have $r_f(n) = 2(4\sigma_1^{\chi_8}(n) - \sigma_{1,\chi_8}(n))$. Since $r_g(n) = \sum_{m\in\mathbb{Z}} r_f(n - m^2)$, we have

$$r_g(n) = 2\sum_{m\in\mathbb{Z}} (4\sigma_1^{\chi_8}(n - m^2) - \sigma_{1,\chi_8}(n - m^2)).$$

Let $a \equiv 1 \pmod 4$. By Corollary to Theorem 2, $\lambda_{4,a,8} \in \mathbf{M}_4^\infty(8)$ and $\lambda_{4,a}^{(8)} \in \mathbf{M}_4^0(8)$, and hence their U_2-images are in $\mathbf{M}_4^\infty(4)$ and $\mathbf{M}_4^0(4)$ respectively. Now $U_2(\lambda_{4,a}^{(8)})$ has $-2^{-6}L(-1, \chi_{8a})$ at its 0th Fourier coefficient at 0. We have

$$2\lambda_{4,a}^{(8)}(z) - \frac{1}{2}\lambda_{4,a,8}(z)$$
$$= -\frac{1}{2}L(-1,\chi_8)L(-1,\chi_{8a}) + 2\sum_{n=1}^{\infty}\sum_{d|n}\chi_{8a}(d)d$$
$$\times \sum_{m\in\mathbb{Z}}\left(4\sigma_1^{\chi_8}\left(\frac{(n/d)^2a-m^2}{4}\right) - \sigma_{1,\chi_8}\left(\frac{(n/d)^2a-m^2}{4}\right)\right)\mathsf{e}(nz)$$

and so,

$$U_2\left(2\lambda_{4,a}^{(8)} - \frac{1}{2}\lambda_{4,a,8}\right)(z)$$
$$= \frac{1}{2}L(-1,\chi_{8a}) + \sum_{n=1}^{\infty}\sum_{d|n}\chi_{8a}(d)dr_f((n/d)^2a)\mathsf{e}(nz),$$

which is equal to

$$2^{-5}L(-1,\chi_{8a})\left\{\frac{16}{15}(16G_4(4z)-G_4(2z)) - \frac{16}{15}(G_4(z)-G_4(2z))\right\}.$$

By comparing Fourier coefficients, we obtain the formula in this case. By a similar argument we can obtain formulas for $a \not\equiv 1 \pmod 4$.

7. In this section we consider a modular form $\lambda_{2k,\psi}$ in case $k = 1$. Its 0th coefficient is essentially a product of two class numbers of imaginary quadratic number fields. Costa's result [6] has already shown that modular forms are effective in the study of class numbers. Our purpose is different and we investigate a relation between ternary forms and class numbers. For m nonsquare, let $h(m)$ and $w(m)$ denote the class number of $\mathbb{Q}(\sqrt{m})$ and the number of roots of unity, respectively. Let D be a negative discriminant. Then $L(0,\chi_D)$ equals $2h(D)/w(D)$. The number $w(D)$ is 4 ($D=-4$), 6 ($D=-3$), or 2 (otherwise).

Let $N > 1$. Let $l \in \mathbb{N}$ be a divisor of N^m for some $m \in \mathbb{N}$. Let $\mathbf{M}_2(N,l)$ denote the subspace consisting of modular forms f in $\mathbf{M}_2(N)$ for which

$$\Big(U_l\prod_{p|N}(U_p-1)\Big)(f) = 0.$$

When N is prime, $\mathbf{M}_2(N,1)$ denotes the subspace in $\mathbf{M}_2(N)$ consisting of modular forms invariant under U_N. Obviously if $l\,|\,l'$, then $\mathbf{M}_2(N,l) \subset \mathbf{M}_2(N,l')$, and if $p^2\,|\,N$, then $U_p(\mathbf{M}_2(N,l)) \subset \mathbf{M}_2(N/p,l/(l,p))$. For the first several prime N, a basis of the space of cusp forms in $\mathbf{M}_2(N,1)$ and their Fourier coefficients are computed in [21].

PROPOSITION 5. (1) *Let the notation be as in Theorem* 2. *Suppose that* $k = 1$ *and that* χ *is a real-valued odd Dirichlet character with conductor* N'. *Let* l *be a natural number such that* $N' \mid ((l\prod_{p\mid N} p)^2 D_K)$ $(2\nmid N')$, *or* $N' \mid ((l\prod_{p\mid N} p)^2 D_K/4)$ $(2\mid N')$. *Then* $\lambda_{2,\psi}$ *is in* $\mathbf{M}_2(N,l)$.

(2) *Let the notation be as in Proposition* 4(2). *Let* $\psi = \chi \circ \mathrm{Nm}$. *If* χ *is real-valued*, *that is*, ψ *is a genus character*, *then* $\lambda_{2,\psi}$ *is in* $\mathbf{M}_2(p,1)$.

Proof. (1) Put

$$s(n) = \sum_{m\in\mathbb{Z}} \sigma_{0,\chi}\Big(\frac{n^2 D_K - m^2}{4}\Big).$$

Let $c \in \mathbb{N}$ be so that $\chi(c) = -1$. In particular, c is not a square. Then $\sigma_{0,\chi}(c)$ vanishes because for $d \mid c$, the equality $\chi(d) + \chi(d') = 0$ holds, d' being the complementary divisor. This shows that $\sigma_{0,\chi}(\frac{(ln\prod_{p\mid N} p)^2 D_K - m^2}{4})$ vanishes if $(N', m) = 1$, or if $2 \mid N'$ and $(N', m/2) = 1$. Thus

$$s\Big(ln\prod_{p\mid N} p\Big) = \sum_{p_1\mid N} s\Big(ln\prod_{p\neq p_1} p\Big) - \sum_{p_1,p_2\mid N} s\Big(ln\prod_{p\neq p_1,p_2} p\Big) + \ldots,$$

where p, p_i are primes. Putting $a(n) = \sum_{0<d\mid n} \chi_K(d)\chi(d)s(n/d)$, we have

$$a\Big(ln\prod_{p\mid N} p\Big) = \sum_{p_1\mid N} a\Big(ln\prod_{p\neq p_1} p\Big) - \sum_{p_1,p_2\mid N} a\Big(ln\prod_{p\neq p_1,p_2} p\Big) + \ldots$$

Since $a(n)$ $(n > 0)$ is the higher Fourier coefficient of $\lambda_{2,\psi}$, we have shown that the modular form is in $\mathbf{M}_2(N,l)$. Thus our assertion follows.

(2) The higher Fourier coefficient of $\lambda_{2,\psi}$ is obtained in Proposition 4(2). If $\chi_K(p) = 0$, then its nth and pnth coefficients are obviously equal for any $n \in \mathbb{N}$, that is, $\lambda_{2,\psi}$ is invariant under U_p. Suppose $\chi_K(p) \neq 0$. Let $c = p^r c'$ with $(c', p) = 1$. Since $\chi(p) = -1$, $\sigma_{0,\chi}(c)$ is equal to 0 if r is odd, and to $\sigma_{0,\chi}(c')$ otherwise. So

$$\sum_{m\in\mathbb{Z}} \sigma_{0,\chi}\Big(\frac{(pn)^2 D_K - m^2}{4p}\Big) = \sum_{m\in\mathbb{Z}} \sigma_{0,\chi}\Big(\frac{n^2 D_K - m^2}{4p}\Big),$$

which shows that $\lambda_{2,\psi}$ is invariant under U_p. ∎

THEOREM 3. (1) *Let* D *and* D_1 *be negative discriminants. Let* $a \in \mathbb{N}$ *be square-free. Let* a^* *denote* a *or* $4a$ *according as* $a \equiv 1 \pmod 4$ *or not. Assume that*

(i) *there is* $u \in \mathbb{N}$ *such that* $a^* D_1 = u^2 D$, *and*
(ii) $\chi_D(p) \neq 1$ *for any prime factor* p *of* u.

Let t *denote the cardinality of* $\{p : p \mid u,\ \chi_D(p) = -1\}$. *Let* $N = |D_1|$

$(2\nmid a^*$ or $2\nmid D_1)$, $\frac{1}{2}|D_1|$ $(v_2(a^*D_1)=4,5)$, $\frac{1}{4}|D_1|$ $(v_2(a^*D_1)=6)$. Then

$$\lambda_{2,a^*,D_1}(z)=2^{t+2}h(D_1)h(D)/w(D_1)w(D) + 4\sum_{n=1}^{\infty}\sum_{0<d|n}\chi_{u^2D}(d)\sum_{m\in\mathbb{Z}}\sigma_{0,\chi_{D_1}}\left(\frac{(n/d)^2a^*-m^2}{4}\right)\mathsf{e}(nz)$$

is a modular form in $\mathbf{M}_2(N,l)$, where $l=2^u$ with the least integer $u\geq\max\{0,(v_2(D_1)-v_2(a^*))/2\}$. If D_1 and a^* have no common odd prime factor and if neither $v_2(D_1a^*)=4$ nor $D_1a^*/64\equiv 1 \pmod 4$, then the modular form is also in $\mathbf{M}_2^{\infty,0}(N)$. Suppose otherwise. Let $M>1$ be a divisor of N. Then the 0th coefficient at a cusp i/M, $(i,M)=1$, is equal to 0 $((M,D)\neq 1)$,

$$2^{t_M+2}\prod_{p|(N/M)}(1-p^{-1})h(D_1)h(D)/w(D_1)w(D)\qquad((M,D)=1),$$

where t_M denotes the cardinality of $\{p:p\,|\,M,\ \chi_D(p)=-1\}$.

(2) *Let D and D_1 be negative discriminants such that $a^*=DD_1$ is the discriminant of a real quadratic field. Let p be a rational prime such that $\chi_{D_1}(p)=-1$ and $\chi_D(p)=0$ or -1, and let χ be a completely multiplicative function on $\mathbb{N}$ defined by $\chi(q)=\chi_{D_1}(q)$ for a prime q with $q\nmid D_1$, and $\chi(q)=\chi_D(q)$ for q dividing D_1. Then*

$$4h(D_1)h(D)/w(D_1)w(D) + \frac{2}{1-\chi_D(p)}\sum_{n=1}^{\infty}\sum_{\substack{0<d|n\\(d,pD_1)=1}}\chi_D(d)\sum_{m\in\mathbb{Z}}\sigma_{0,\chi}\left(\frac{(n/d)^2a^*-m^2}{4p}\right)\mathsf{e}(nz)$$

is a modular form in $\mathbf{M}_2(p,1)$.

Proof. (1) The 0th Fourier coefficient of λ_{2,a^*,D_1} is $L(0,\chi_{D_1})L(0,\chi_{u^2D})$, which is equal to $2^tL(0,\chi_{D_1})L(0,\chi_D)=2^{t+2}h(D_1)h(D)/(w(D_1)w(D))$. The 0th coefficients at other cusps are obtained as in the Corollary to Theorem 2. Thus by Lemma 5, Theorem 2(1) and Proposition 5, our assertion follows.

(2) Let K be a quadratic field with $D_K=a^*$, and let $\psi:=\chi\circ\mathrm{Nm}$ be a genus character corresponding to the decomposition $a^*=D\cdot D_1$. By Proposition 5(2), $\lambda_{2,\psi}$ is in $\mathbf{M}_2(p,1)$. Its 0th coefficient is equal to $(1-\psi(\mathfrak{P}))L_K(0,\psi)=2L(0,\chi_D)L(0,\chi_{D_1})$, and the higher coefficients are given in Proposition 4(2). Thus $\frac{1}{2}\lambda_{2,\psi}$ is the modular form in the theorem. ∎

In Theorem 3, the 0th coefficients at a cusp 0 are not presented. However, by Lemma 1, they can be obtained from the 0th coefficients at other cusps.

We give an application of Theorem 3(2).

Example. Let $r\equiv 3 \pmod 8 > 3$ be square-free, and let $-s$ be a negative discriminant with $s\not\equiv 7 \pmod 8$ and $(s,r)=1$. Let $p=2$, $D_1=$

$-r$ and $D=-s$ in Theorem 3(2). Then

$$h(-r)h(-s)+\frac{2}{1-\chi_{-s}(2)}\sum_{n=1}^{\infty}\sum_{\substack{d\mid n\\(d,2r)=1}}\chi_{-s}(d)\times\sum_{m\in\mathbb{Z}}\sigma_{0,\chi}\biggl(\frac{(n/d)^2rs-m^2}{8}\biggr)\mathsf{e}(nz)\in\mathbf{M}_2(2).$$

Since $\{24,-1\}\in \mathrm{LR}_2'(2)$, we have

$$h(-r)h(-s)=\frac{1}{12(1-\chi_{-s}(2))}\sum_{\substack{m\in\mathbb{Z}\\ m\equiv s\ (\mathrm{mod}\ 2)}}\sigma_{0,\chi}\biggl(\frac{rs-m^2}{8}\biggr).$$

If q is the minimal prime with $\chi(q)=1$, then $0\le\sigma_{0,\chi}(m)\le\log_q m$ (see the proof of Proposition 5). Thus we obtain the estimate

$$\begin{aligned}h(-r)h(-s)&\le\frac{1}{12(1-\chi_{-s}(2))}(\sqrt{rs}+2)\log_3(rs)\\&<\frac{1}{12}(\sqrt{rs}+1)(\log|r|+\log|s|).\end{aligned}$$

Note that this cannot be obtained from the usual estimate such as $h(-s)<C\sqrt{|s|}\log|s|$ with a constant C (see for example Newman [15]). A similar argument is possible for some other congruence conditions.

Let D be a discriminant. Then for $m\in\mathbb{N}$, $\sigma_{0,\chi_D}(m)$ is equal to the number of integral ideals in $\mathbb{Q}(\sqrt{D})$ with norm m. Hence for $D<0$, $w(D)\sigma_{0,\chi_D}(m)$ is equal to the number of representations of m by positive definite quadratic forms of discriminant D which form a complete system of representatives of the proper equivalence classes. It follows that higher Fourier coefficients of λ_{2,a^*,D_1} in Theorem 3(1) are closely related to representations of natural numbers by ternary forms.

We give an application of Theorem 3(1). We examine the case $D_1=-4$. Let a be square-free with $a\not\equiv 7\pmod 8$. Then $D=-4a$ ($a\equiv 1,2\pmod 4$), $D=-a$ ($a\equiv 3\pmod 8$) satisfy the conditions (i), (ii), where $t=0$ in the former, and $t=1$ in the latter. Since $h(-4)=1$ and $w(-4)=4$,

$$\lambda_{2,a^*,-4}=2^th(-a)/w(-a)+4\sum_{n=1}^{\infty}\sum_{d\mid n}\chi_{-4a}(d)\sum_{m\in\mathbb{Z}}\sigma_{0,\chi_{-4}}\biggl(\frac{(n/d)^2a^*-m^2}{4}\biggr)\mathsf{e}(nz).$$

Considering the norm form for $\mathbb{Q}(\sqrt{-1})$, we have

$$r_3(n)=4\sum_{m\in\mathbb{Z}}\sigma_{0,\chi_{-4}}(n-m^2)\qquad\text{for } n\in\mathbb{N}.$$

Here $U_2(\lambda_{2,a^*,-4})$ $(a \equiv 1 \pmod 4)$ and $\lambda_{2,a^*,-4}$ $(a \not\equiv 1 \pmod 4)$ are in $\mathbf{M}_2(2)$ and they have the expansion

$$2^t h(-a)/w(-a) + \sum_{n=1}^{\infty}\Big\{\sum_{d|n}\chi_{-4a}(d)r_f((n/d)^2a)\Big\}\mathsf{e}(nz).$$

Since $\{24,-1\}\in \mathrm{LR}'_2(2)$, we have shown that for a square-free $a>3$,

$$h(-a) = \begin{cases} \frac{1}{12}r_3(a) & (a\equiv 1,2 \pmod 4),\\ \frac{1}{24}r_3(a) & (a\equiv 3 \pmod 8),\end{cases}$$

which is known as "Gauss' three-square theorem". Since $\mathbf{M}_2(2)$ is spanned by $G_{2,\chi_4}(z)$, comparison of Fourier coefficients leads to

$$(2^{t+3}3h(-a)/w(-a))\sigma_{1,\chi_4}(n) = \sum_{d|n}\chi_{-4a}(d)r_3((n/d)^2a)$$

for any n. By the Möbius inversion formula, we obtain

$$r_3(n^2a) = (2^{t+3}3h(-a)/w(-a))\sum_{d|n}\mu(d)\chi_{-4a}(d)\sigma_{1,\chi_4}(n/d),$$

which is a classical result (Bachmann [3], Bateman [4]).

In this way we can obtain other such formulas by replacing D_1 by other negative discriminants. We state some of them as a corollary.

Corollary. *Let m be any natural number. Let $m=n^2a$ with a square-free. Let n^* be $2n$ or n according as $a\equiv 1 \pmod 4$ or not.*

(1) *Then* $r_3(m) = 0$ $(a\equiv 7 \pmod 8)$, *and* $r_3(m) = \delta_1(a)h(-a) \times \sum_{d|n}\mu(d)\chi_{-4a}(d)\sigma_{1,\chi_4}(n/d)$ (*otherwise*), *where* $\delta_1(a) = 6$ $(a=1)$, 8 $(a=3)$, 12 $(a\equiv 1,2 \pmod 4,\ a>1)$, 24 $(a\equiv 3 \pmod 8,\ a>3)$.

(2) *Let* $f = x^2+y^2+2z^2$. *Then* $r_f(m)=0$ *if* $a\equiv 14 \pmod{16}$. *Suppose otherwise. Then*

$$r_f(m) = \begin{cases} \delta_2(m)h(-2a)\sum_{d|n}\mu(d)\chi_{-8a}(d)\sigma_{1,\chi_4}(n/d) & (2\,|\,a \text{ or } 2\nmid n),\\ \delta_2(m)h(-2a)\sum_{d|n}\mu(d)\chi_{-8a}(d)\sigma_{1,\chi_4}(n/2d) & (2\nmid a \text{ and } 2\,|\,n),\end{cases}$$

where $\delta_2(m)$ *denotes* 6 $(a=2)$, 8 $(a=6)$, 12 $(a\equiv 2 \pmod 8,\ a>2)$, 24 $(a\equiv 6 \pmod{16},\ a>6)$, 4 $(2\nmid a,\ 2\nmid n)$, 12 $(2\nmid a,\ 2\,|\,n)$.

(3) *Let* $f = x^2+y^2+yz+z^2$. *Then*

$$r_f(m) = \begin{cases} 0 & (a\equiv 6 \pmod 9),\\ \delta_3(a)h(-3a)\sum_{d|n^*}\mu(d)\chi_{-3a^*}(d)\sigma_{1,\chi_9}(n^*/d) & (\text{otherwise}),\end{cases}$$

where $\delta_3(a)$ *denotes* 2 $(a=1)$, 3 $(a=3)$, $6(1+v_3(a))$ $(a\neq 1,3)$.

(4) *Let* $f = x^2 + y^2 + 3z^2$. *Then* $r_f(m) = 0$ *if* $a \equiv 6 \pmod 9$. *Suppose otherwise. Then*

$$r_f(m) = \begin{cases} \delta_3'(a)h(-3a)\sum_{d\mid n}\mu(d)\chi_{-12a}(d)\sigma_{1,\chi_9}(n/d) & (a \equiv 1 \pmod 8), \\ \delta_3'(a)h(-3a)\sum_{d\mid n}\mu(d)\chi_{-3a}(d)\sigma_{1,\chi_9}(n/d) & (a \equiv 5 \pmod 8), \\ \delta_3'(a)h(-3a)\sum_{d\mid n}\mu(d)\chi_{-12a}(d)\{\sigma_{1,\chi_9}(n/d) & \\ \qquad\qquad + 2\sigma_{1,\chi_9}(n/(2d))\} & (a \equiv 2, 3 \pmod 4), \end{cases}$$

where $\delta_3'(a) = 4$ $(a = 1)$, 2 $(a = 3)$, $12(1 + v_3(a))$ $(a \equiv 1 \pmod 8,\ a > 1)$, $8(1 + v_3(a))$ $(a \equiv 5 \pmod 8)$, $2(1 + v_3(a))$ $(a \equiv 2, 3 \pmod 4,\ a \neq 3)$.

(5) *Let* $f = x^2 + y^2 + yz + 2z^2$. *Then*

$$r_f(m) = \begin{cases} 0 & (a/7 \not\equiv 3, 5, 6 \pmod 7), \\ 2\delta_7(a)(1 + v_7(a))h(-7a) & \\ \quad \times \sum_{d\mid n^*}\mu(d)\chi_{-7a^*}(d)\sigma_{1,7}(n^*/d) & (\textit{otherwise}), \end{cases}$$

where $\delta_7(7) = 1/2$, $\delta_7(21) = 1/3$ *and* $\delta_7(a) = 1$ $(a \neq 7, 21)$.

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