On characterization of Dirichlet $L$-functions

by

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1. Introduction. Let $L(s, f)$ denote the Dirichlet series $\sum_{n=1}^{\infty} f(n)/n^s$. If $f$ is purely recurring, then $L(s, f)$ is absolutely convergent for $\text{Re}(s) > 1$ and

$$L(s, f) = \frac{1}{N^s} \sum_{n=1}^{N} f(n) \zeta(s, n/N),$$

where $N$ is a period of $f$ and

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^s}$$

is the Hurwitz zeta function. We know that $L(s, f)$ can be extended analytically to the whole plane as a meromorphic function of order one and has only a simple pole with residue $(f(1) + \ldots + f(N))/N$ at $s = 1$ unless $f(1) + \ldots + f(N) = 0$, in which case there exists no pole in the whole plane and $L(s, f)$ is convergent for $\text{Re}(s) > 0$. We call $f$ even (resp. odd) modulo $N$ if, extending it periodically to all integers, $f(-x) = (-1)^d f(x)$ with $d = 0$ (resp. $d = 1$). Schnee [6] showed the functional equation

$$\left(\frac{N}{\pi}\right)^{s/2} \Gamma\left(\frac{s+d}{2}\right) L(s, f) = i^{-d} \left(\frac{N}{\pi}\right)^{(1-s)/2} \Gamma\left(\frac{1-s+d}{2}\right) L(1-s, T_N f),$$

where

$$T_N f(x) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} f(n) \exp\left(\frac{2\pi inx}{N}\right).$$

We list some of the above properties of $L(s, f)$ as

(A) The Dirichlet series expansion of $L(s, f)$ is absolutely convergent for $\text{Re}(s) > 1$.

(B) $L(s, f)$ can be continued into the whole plane to a meromorphic function of finite order with a finite number of poles.
(C) For a non-negative integer \(d\) and a positive number \(N\) a functional equation holds in the form

\[
\left( \frac{N}{\pi} \right)^{s/2} \Gamma \left( \frac{s + d}{2} \right) L(s, f) = \left( \frac{N}{\pi} \right)^{(1-s)/2} \Gamma \left( \frac{1 - s + d}{2} \right) L(1 - s, g),
\]

where \(L(s, g)\) is convergent in a half-plane.  

In Section 2 we shall prove that (A), (B), and (C) characterize Dirichlet series with recurrent coefficients, following Chandrasekharan–Narashimhan [3] and modifying Siegel’s proof [7] of Hamburger’s theorem [4] on the Riemann zeta function. In Section 3 we characterize Dirichlet \(L\)-functions without using Euler products. We shall use Dirichlet \(L\)-functions in Section 4 to give a characterization of finite Dirichlet series in a way different from Toyozumi’s results in [8]. In Section 5 we shall extend the concept of equivalence and conductors of Dirichlet characters to general periodic functions.

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2. Characterization of recurring coefficients

**Lemma 2.1.** For functions \(f \neq 0\), properties (A), (B) and (C) imply that \(N\) is a positive integer, the number \(d\) is 0 or 1, \(f\) is purely recurring, even or odd modulo \(N\) according as \(d = 0\) or 1, and \(g = i^{-d}T_N f\).

**Proof** (for more details see Chandrasekharan–Narashimhan [3]). We put

\[
\phi(s) = (2N)^s L(2s - d, f), \quad \psi(s) = (2N)^s L(2s - d, g).
\]

The given functional equation becomes

\[
(2\pi)^{-s} \Gamma(s) \phi(s) = (2\pi)^{s-\delta} \Gamma(\delta - s) \psi(\delta - s),
\]

where \(\delta = d + 1/2\).

Let \(\alpha, \beta\) be positive numbers such that

\[
\sum_{n=1}^\infty \frac{f(n)}{n^{2\alpha-d}}, \quad \sum_{n=1}^\infty \frac{g(n)}{n^{2\beta-d}}
\]

converge absolutely. By (A) we may choose \(\alpha < 1 + d\) (in fact, any \(\alpha > (1 + d)/2\) would do).

We see from (B) and the functional equation that \(\phi(s)\) has at most a finite number of poles \(r\), all in the strip \(\delta - \beta < \text{Re}(r) < \alpha\).

We start off from the integral

\[
\frac{1}{2\pi i} \int_{(\alpha)} \Gamma(s) \phi(s) x^{-s} \, ds \quad (x > 0)
\]
over the vertical line \((\alpha)\) with real point \(\alpha\). By the formula
\[
\frac{1}{2\pi i} \int \frac{\Gamma(s)}{y^s} \, ds = e^{-y} \quad (y > 0),
\]
putting in the series representation of \(L(s, f)\), it is, on the one hand,
\[
\sum_{n=1}^{\infty} f(n)n^d e^{-n^2x/(2N)}.
\]

The series representations and the functional equation together with the Phragmén–Lindelöf principle, \(L(s, f)\) being of finite order, imply in a standard way that \(|\phi(s)|\) can be estimated by a power of \(|\text{Im}(s)|\) in any given vertical strip. This enables one, on the other hand, to push the line of integration to \((\delta - \beta)\).

Using the functional equation,
\[
\frac{1}{2\pi i} \int \frac{\Gamma(s)\phi(s)x^{-s}}{(\delta-\beta)} \, ds = \frac{1}{2\pi i} \int \frac{\Gamma(\delta - s)\psi(\delta - s)(2\pi)^{2s-\delta}x^{-s}}{\Gamma(s)} \, ds
\]
\[
= \frac{1}{2\pi i} \int \frac{\Gamma(s)\psi(s)}{(\beta)} (2\pi)^{\delta-s}x^{s-\delta} \, ds
\]
\[
= \left(\frac{2\pi}{x}\right)^\delta \sum_{n=1}^{\infty} g(n)n^d e^{-2\pi^2n^2/(Nx)}
\]
by a similar calculation in the last step as above.

It remains to collect the residues of \(\Gamma(s)\phi(s)x^{-s}\). At any given pole \(r\) of order \(q\) the residue is of the form
\[
x^{-r}P_r(\log x),
\]
where \(P_r\) is a polynomial of degree \(\leq q\) with constant coefficients. Denoting by \(P(x)\) their (finite) sum,
\[
P(x) = \sum_r x^{-r}P_r(\log x),
\]
we get
\[
su(\Sum_{n=1}^{\infty} f(n)n^d e^{-n^2x/(2N)} = \left(\frac{2\pi}{x}\right)^{d+1/2} \sum_{n=1}^{\infty} g(n)n^d e^{-2\pi^2n^2/(Nx)} + P(x).
\]
Following Siegel’s idea, we multiply \((\ast)\) throughout by \(x^d e^{-s^2x/(2N)}\) first with \(s > 0\), and integrate with respect to \(x\) over \((0, \infty)\). The left hand side
becomes
\[ F_1(s) = (2N)^{d+1} \Gamma(d+1) \sum_{n=1}^{\infty} \frac{f(n)n^d}{(s^2 + n^2)^{d+1}}, \]
and using the formula
\[ \int_0^\infty e^{-(ax+b)/x} \sqrt{x} \, dx = \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \quad (a, b > 0), \]
the first term on the right becomes
\[ F_2(s) = (2\pi)^{d+1} \sqrt{N} \sum_{n=1}^{\infty} s^{-1}g(n)n^de^{-2\pi ns/N}, \]
both the resulting series being absolutely convergent.

Finally, the second term on the right becomes
\[ F_3(s) = \int_0^\infty x^d P(x)e^{-s^2x/(2N)} \, dx. \]
The latter is a finite linear combination of integrals, absolutely convergent
by \( \text{Re}(d - r) > d - \alpha > -1 \), of the type
\[ \int_0^\infty x^{d-r}(\log x)^m e^{-s^2x/(2N)} \, dx = \int_0^\infty \left( \frac{y}{s^2} \right)^{d-r} (\log y - 2\log s)^m e^{-y/(2N)} \frac{dy}{s^2} \]
with integers \( m \geq 0 \). This is \( s^{2r-2d-2} \) multiplied by a polynomial in \( \log s \) and we see that \( F_3(s) \) can be extended to a single-valued regular function
in the whole plane with the non-positive real axis deleted.

Our formula for \( F_2(s) \) extends \( sF_2(s) \) to a function regular and periodic
with period \( iN \) for \( \text{Re}(s) > 0 \).

Finally, the series representation of \( F_1(s) \) does, in fact, converge for all
complex \( s \neq \pm in \) \( (n = 1, 2, \ldots) \) representing a meromorphic function in the
whole plane with poles of order \( d + 1 \) at \( \pm in \) only (unless \( f(n) = 0 \)).

From the periodicity of \( sF_1(s) - sF_3(s) = sF_2(s) \) we see that \( N \) is a
positive integer and
\[ \lim_{s \to in} F_1(s)s(s - in)^{d+1} = (-i)^d N^{d+1} \Gamma(d+1) f(n) \]
is periodic in \( n \) with period \( N \).

Denote by \( f_E \) and \( f_O \) the even and the odd part of \( f \) modulo \( N \), respectively. Using \( L(s, f) = L(s, f_E) + L(s, f_O) \) in (C), the functional equations
for \( L(s, f_E) \) and \( L(s, f_O) \) and the formula
\[ \frac{\Gamma(s/2)}{\Gamma((1-s)/2)} = \frac{2^{1-s}}{\sqrt{\pi}} \Gamma(s) \cos \frac{s\pi}{2}, \]
we get
\[ L(s, T_N f_E) \cos \frac{s\pi}{2} - i L(s, T_N f_O) \sin \frac{s\pi}{2} = G(s) L(s, g) \cos \frac{(s-d)\pi}{2}, \]
where
\[ G(s) = \begin{cases} \prod_{j=0}^{d-1} \frac{s + j}{s - d + 1 + 2j} & \text{if } d > 1, \\ 1 & \text{if } d = 0 \text{ or } 1. \end{cases} \]
Putting \( s = 4r + 1 + d \) for any positive integer \( r \) large enough, we get \( L(4r + 1 + d, h) = 0 \), where \( h = -\sin(d\pi/2)T_N f_E - i \cos(d\pi/2)T_N f_O \), implying \( h = 0 \). Therefore, \( T_N f_E = 0 \) or \( T_N f_O = 0 \) according as \( d \equiv 1 \) or \( 0 \mod 2 \) and
\[ L(s, g - i^{-d}T_N f) = (1 - G(s)) L(s, g). \]

The rational function \( 1 - G(s) \) is thus the quotient of two Dirichlet series. Such a quotient or its reciprocal tends to a finite limit with an exponential speed, \( O(e^{-as}) \) as \( s \to +\infty \), a speed a non-constant rational function cannot produce. Our \( G(s) \) is only constant, \( G(s) \equiv 1 \) if \( d = 0 \) or \( 1 \), implying also \( g - i^{-d}T_N f = 0 \). The proof of Lemma 2.1 is complete.

3. Characterization of Dirichlet \( L \)-functions. Apostol ([1], [2]) characterizes Dirichlet \( L \)-functions corresponding to primitive characters by functional equation and Euler product. We replace the latter by an algebraic condition.

**Proposition 3.1.** Let \( f \neq 0 \) satisfy (A), (B) and (C), the latter with \( g = W f \), where \( W \) is a constant. By Lemma 2.1, \( N \) is an integer and assume that \( f(n) = 0 \) if \( (n, N) > 1 \) and that the field \( Q_f \) generated by the values \( f(n) \) is algebraic over the rationals and is linearly disjoint from the \( N \)th cyclotomic field \( C_N \). Then \( f \) is a constant multiple of a primitive character \( \mod N \).

**Proof.** By Lemma 2.1 we also know that \( f \) is purely recurring with period \( N \) and \( T_N f = i^d W f \).

Our algebraic assumption means that for any \( m \) relatively prime to \( N \) there is an automorphism \( \tau_m \) of the composite field \( Q_f C_N \) such that \( \tau_m \) leaves \( Q_f \) invariant and \( \tau_m(e^{2\pi i/N}) = e^{2\pi im/N} \).

We get
\[ \tau_m(\sqrt{N}(T_N f)(k)) = \tau_m\left( \sum_{n=1}^{N} f(n)e^{2\pi ink/N} \right) = \sum_{n=1}^{N} f(n)e^{2\pi imnk/N} = \sqrt{N}(T_N f)(mk) \]
and by the identity $T_Nf = i^dW\overline{f}$,
$$\sqrt{N}i^dWf(mk) = \tau_m(\sqrt{N}i^dWf(k)) = \tau_m(\sqrt{N}i^dWf(k)).$$

Putting $k = 1$ here we get
$$\sqrt{N}i^dWf(m) = \tau_m(\sqrt{N}i^dWf(1)).$$

This shows that $f(1) \neq 0$, otherwise $f = 0$, a contradiction. We may assume $f(1) = 1$ and dividing the last two equations we have
$$f(mk) = f(m)f(k).$$

If $(m, N) > 1$ then this holds trivially, both sides vanishing. Hence $f$ is a character mod $N$ satisfying $(T_Nf)(1) = i^dWf(1) = i^dW$, i.e. $(T_Nf)(n) = (T_Nf)(1) \cdot f(n)$. Such a character is known to be primitive (see e.g. [1], Lemma 1 or [5]) and the proof is complete.

We remark that Dirichlet characters do not always satisfy the algebraic condition, but Proposition 3.1 enables us to characterize e.g. the Legendre symbol by assuming $f$ to be rational-valued.

4. Characterization of finite series. If in
$$F(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$$
corresponds to a Dirichlet character $\chi$, then
$$L(s, f) = F(s)\zeta(s)$$
has the purely recurring coefficients
$$f(n) = c_n * \chi(n) = \sum_{d|n} c_d \chi(n/d).$$

Conversely, we have

**Theorem 4.1.** If for each Dirichlet character $\chi$ there is an $N$ such that $f(n + N) = f(n)$ for $n$ large enough, then $F(s)$ is a finite series.

**Proof.** Denoting by $\mu$ the Möbius function we see that
$$F(s) = \frac{L(s, f)}{L(s, \chi)} = L(s, f)\zeta(s)$$
is a Dirichlet series absolutely convergent for $\Re(s) > 1$, representing a meromorphic function of order $\leq 1$ in the whole plane.

We first claim that for any given complex number $s$ ($\neq 1$) there is a Dirichlet character $\chi$ such that $L(s, \chi) \neq 0$. Since $L(s, f)$ can only have a first order pole at $s = 1$ as its only singularity, it will follow that $F(s)$ is regular for $s \neq 1$. Using the zeta function, $\zeta(s) = L(s, \chi)$ with $\chi = 1$, having
the same singularity at \( s = 1 \), we shall even find that \( F(s) \) is an entire function.

To prove the claim we first note that \( \zeta(s, x) \) \((0 < x < 1)\), also a regular function in \( s \) in the whole plane with the exception of \( s = 1 \), satisfies

\[
\frac{\partial^m \zeta(s, x)}{\partial x^m} = (-1)^m s(s + 1) \cdots (s + m - 1) \zeta(s + m, x)
\]

\[
= (-1)^m s(s + 1) \cdots (s + m - 1) \sum_{n=0}^{\infty} \frac{1}{(x + n)^{s+m}}
\]

for \( \text{Re}(s + m) > 1 \), implying

\[
\left| \frac{\partial^m \zeta(s, x)}{\partial x^m} \right| \sim \frac{|s(s + 1) \cdots (s + m - 1)|}{(x + 1)^{\text{Re}(s+m)}} \to \infty
\]

as \( m \to \infty \), provided that \( s \neq 1, 0, -1, -2, \ldots \). Hence \( \zeta(s, x) \) cannot vanish identically in \( x \) for such an \( s \) and there exists a rational number \( x = p/q \), \( 0 < p < q \), \((p, q) = 1\), such that \( \zeta(s, p/q) \neq 0 \). Now,

\[
\frac{1}{q^s} \zeta(s, \frac{p}{q}) = \sum_{n=0}^{\infty} \frac{1}{(nq + p)^s} = \sum_{k=1 \atop k \equiv p \pmod{q}}^{\infty} \frac{1}{k^s}
\]

can be represented as a linear combination of Dirichlet \( L \)-functions \( \pmod{q} \), showing that at least one of them does not vanish.

As to the remaining cases \( s = 0, -1, -2, \ldots \), we have \( \zeta(s) \neq 0 \) (\( s = 0, -1, -3, \ldots \)) and \( L(s, \chi) \neq 0 \) (\( s = -2, -4, \ldots \)) for any odd character \( \chi \).

For the rest of the proof we fix our Dirichlet \( L \)-function e.g. as \( \zeta(s) \) and use the single relation

\[
F(s) \zeta(s) = L(s, f).
\]

\( f \), being ultimately recurring, can be written as \( f_\infty + f_E + f_O \); here \( f_\infty(n) \) vanishes for \( n \) large enough, \( f_E \) and \( f_O \) are purely recurring with period \( N \), even and odd, respectively.

From the respective functional equations we have

\[
\zeta(-k) = L(-k, f_E) = 0
\]

for even, positive integers \( k \), implying

\[
0 = L(-k, f) = L(-k, f_E) + L(-k, f_\infty).
\]

From the functional equation of \( f_O \) we see that

\[
|L(-k, f_O)| > e^{\frac{1}{2} k \log k}
\]

for even \( k \) large enough, unless \( T_N f_O = 0, f_O = 0 \). The finite series \( L(s, f_\infty) \) also tends to infinity but at a smaller rate, only exponentially, as \( s \to -\infty \), unless it is a constant.
We conclude first that $f_{\Omega} = 0$ and then $f_\infty = 0$. Hence $f = f_E$ and
$$F(s) = \frac{L(s, f_E)}{\zeta(s)}.$$  

By the respective functional equations
$$F(s) = N^{1/2-s} \frac{L(1-s, T_N f_E)}{\zeta(1-s)},$$
implying for $\text{Re}(s) \leq -1$
$$|F(s)| \leq c N^{1/2 - \text{Re}(s)}.$$

An entire function of finite order, representable by a Dirichlet series for $\text{Re}(s) > 1$ and satisfying an estimate like this is a finite series. A proof of this standard fact runs e.g. as follows.

By the Phragmén–Lindelöf principle $F(s)$ is bounded in any fixed vertical strip. The coefficient formula,
$$c_n = \lim_{T \to \infty} \frac{1}{2T} \int_{\sigma-iT}^{\sigma+iT} F(s) n^s \, ds,$$
valid first for $\sigma > 1$, but by the above boundedness for any $\sigma$, implies
$$|c_n| \leq c N^{1/2 - \sigma} n^\sigma \quad (\sigma \leq -1),$$
and letting $\sigma \to -\infty$ gives $c_n = 0 \ (n > N)$. (This proof even allows for a finite number of singularities, compare with Toyoizumi [8].)

5. An equivalence relation. In the set of all convergent Dirichlet series, we define the equivalence $L(s, f) \sim L(s, g)$ if there exist two non-zero finite series $L(s, h_1)$ and $L(s, h_2)$ such that $L(s, h_0) = L(s, f)L(s, h_1) - L(s, g)L(s, h_2)$ is a finite series. If $D_i$ is the least common multiple of integers $d$ such that $h_i(d) \neq 0$, then this means
$$\sum_{d \mid (n, D_1)} f(n/d) h_1(d) - \sum_{d \mid (n, D_2)} g(n/d) h_2(d) = 0$$
for $n$ large enough. The conductor of $L(s, f)$ can be defined as the minimum of the primitive period of $g$ for which $L(s, g) \sim L(s, f)$ and $g$ is purely recurring.

**Theorem 5.1.** Our conductor of a Dirichlet $L$-function coincides with the ordinary conductor of the associated character.

**Proof.** Assume that $L(s, \chi) \sim L(s, f)$, that is, for $n$ large enough
$$\sum_{d \mid (n, D_1)} \chi(n/d) h_1(d) = \sum_{d \mid (n, D_2)} f(n/d) h_2(d).$$
Let $M$ denote the primitive period of $f$. By putting $rn$ as $n$ in the above identity, where $r \equiv 1 \pmod{M}$ and $(r, D_1D_2) = 1$, the right hand side is invariant and the left hand side is multiplied by $\chi(r)$. There exist infinitely many $n$ such that the left hand side is not zero, otherwise $L(s, \chi)L(s, h_1)$ would be a finite series. Therefore $\chi(r) = 1$.

If $\chi$ belongs to the modulus $q$ and $a_1 \equiv a_2 \pmod{(q, M)}$, $(a_1, q) = (a_2, q) = 1$, then we can find an $r$ with the above properties such that in addition $ra_1 \equiv a_2 \pmod{q}$. This implies $\chi(a_1) = \chi(a_2)$, i.e. $\chi$ can be defined mod$(q, M)$ and the conductor of $\chi$ is $\leq (q, M) \leq M$.

The rest is obvious.

Two characters are said to be equivalent if their corresponding primitive characters are the same.

**Corollary 5.2.** Dirichlet $L$-functions are equivalent if and only if their associated characters are equivalent.

**Proof.** Assume in the identity in Theorem 5.1 that $f$ is also a character. Putting $rn$ as $n$ with $(r, D_1D_2) = 1$, the left and right hand sides are multiplied by $\chi(r)$ and $f(r)$, respectively. Since the two sides are not identically zero, we have $\chi(r) = f(r)$ for $(r, D_1D_2) = 1$, so that $\chi$ and $f$ are equivalent.

**Proposition 5.3.** Any positive integer $N$ except for 2 is the conductor of a Dirichlet series.

**Proof.** According to Corollary 5.2 there exists a Dirichlet series with conductor $N$ if $N \equiv 0, 1$ or 3 mod 4. We show that the conductor of the Dirichlet series

$$L(s, f) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

is $N$ if $N \equiv 2 \pmod{4}$. Assume that $L(s, f) \sim L(s, g)$, where $g$ is purely recurring with primitive period $M < N$. We have

$$\sum_{d|(n, D_1)} f(n/d)h_1(d) = \sum_{d|(n, D_2)} g(n/d)h_2(d)$$

for $n$ large enough, but both sides being purely recurring, in fact for all $n$. This means

$$L(s, g)/L(s, f) = L(s, h_1)/L(s, h_2).$$

The left hand side is an ordinary Dirichlet series $\sum_{n=1}^{\infty} a(n)/n^s$ because $f(1) \neq 0$ and we see from the right hand side that $a(n) = 0$ if $(n, D_1D_2) = 1$. Let $d_1$ be the least integer such that $a(d_1) \neq 0$.

In any case except $2M = N$ we can find an integer (even a prime) $q$ satisfying $q \equiv 1 \pmod{M}$, $q \not\equiv 1 \pmod{N}$ and $(q, D_1D_2) = 1$. From the
identity
\[ g(n) = \sum_{d|n} a(d) f(n/d) \]
we get
\[ g(d_1) = a(d_1) f(1) = a(d_1) \neq 0, \]
\[ g(d_1q) = a(d_1) f(q) = a(d_1) \cdot 0 = 0, \]
contradicting the fact that by \( d_1 \equiv d_1q \pmod{M} \), \( g(d_1) = g(d_1q) \).

In the exceptional case \( 2M = N \) we have \( M \) odd since \( N \equiv 2 \pmod{4} \) and we can find an integer \( q \) satisfying \( 2q \equiv 1 \pmod{M} \), \((q, D_1D_2) = 1\). We get \( g(2d_1q) = g(d_1) \neq 0 \) as established above, contradicting the fact that
\[ g(2d_1q) = a(d_1) f(2q) + a(2d_1) f(q) = 0, \]
since \( 2q \not\equiv 1 \pmod{N} \), \( N \) being even and \( q \equiv (M + 1)/2 \neq 1 \pmod{N} \), provided that \( M > 1 \).

The identity
\[ a + \frac{b}{2^s} + \frac{a}{3^s} + \frac{b}{4^s} + \ldots = \left(a - \frac{a-b}{2^s}\right)\zeta(s) \]
shows that no series has conductor \( N = 2 \).

**Proposition 5.4.** Let \( f \) and \( g \) be purely recurring with period \( N \), such that \( f(n) = g(n) = 0 \) for \( (n, N) > 1 \). If \( L(s, f) \sim L(s, g) \) and \( g \neq 0 \), then \( f = \vartheta g \) with a constant \( \vartheta \).

**Proof.** Let \( \chi \) run over the characters \( \mod{N} \). Under our assumption we have the representations
\[ f = \sum_{\chi} c_\chi \chi, \quad g = \sum_{\chi} d_\chi \chi \]
with constants \( c_\chi, d_\chi \).

The relation
\[ L(s, f)L(s, h_1) - L(s, g)L(s, h_2) = L(s, h_0) \]
can be rewritten as
\[ \sum_{\chi} (c_\chi L(s, h_1) - d_\chi L(s, h_2))L(s, \chi) = \sum_{\chi} L(s, h_\chi) L(s, \chi) = L(s, h_0) \]
\((L(s, h_\chi) \text{ all denoting finite series}) \) or, in terms of the coefficients,
\[ \sum_{\chi} \sum_{d|n} h_\chi(d) \chi(n/d) = 0 \]
for \( n \) large enough.

Assuming that not all \( h_\chi = 0 \), let \( q \) be the least value such that there is a \( \chi \) with \( h_\chi(q) \neq 0 \). Applying the identity for \( n = pq \) with a prime \( p \) large
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enough, we get

$$\sum_{\chi} h_{\chi}(q)\chi(p) = \sum_{\chi} \sum_{d|pq} h_{\chi}(d)\chi(pq/d) = 0.$$ 

Since large primes $p$ represent all reduced residue classes mod $N$, it follows that $\sum_{\chi} h_{\chi}(q)\chi = 0$ and $h_{\chi}(q) = 0$ for all $\chi$, a contradiction. We infer that $L(s, h_{\chi}) = L(s, h_0) = 0$ for all $\chi$.

We get $c_\chi L(s, f) - d_\chi L(s, g) = 0$ for any $\chi$ and, since not all $d_\chi = 0$, the statement follows.

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