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When are global units norms of units?

by

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1. Introduction. Let L/K be a Galois extension of number fields, $\omega \in K$, H the Hilbert class field of L. Often it is possible to determine that $\omega = N_{L/K}(z)$, for some $z \in L$, while number theoretical questions often demand more: If ω is an element of $\mathcal{O}_K^*$, is it also the norm of an element $z \in \mathcal{O}_L^*$? In this paper, we prove that if ω is not the norm of a unit, then it is not even a global norm from H.

Throughout the paper, we will be using the following notation: For an arbitrary number field M, $\mathcal{O}_M$ will denote the ring of algebraic integers in M, $\mathcal{O}_M^*$ the group of units of $\mathcal{O}_M$, P_M the principal ideals of M. Further, A_M will denote the adeles of M, A_M^* the group of ideles of M, with I_M denoting the ideals of M. Finally, C_M and Cl_M will denote the idèle class group of M and the ideal class group of M, respectively.

Theorem 1. *Let L/K be a Galois extension of number fields with H the Hilbert class field of L. Then*

$$N_{H/K}(H^*) \cap \mathcal{O}_K^* \subset N_{L/K}(\mathcal{O}_L^*).$$

Proof. Let

$$G := \mathrm{Gal}(L/K) \quad \text{and} \quad \Gamma := \mathrm{Gal}(H/K),$$

with $\omega \in N_{H/K}(H^*) \cap \mathcal{O}_K^*$. We will also write ω for its image in the Tate cohomology group $\widehat{H}^0(\Gamma, \mathcal{O}_H^*)$. This is harmless, since we want to prove that $\omega \in N_{L/K}(\mathcal{O}_L^*)$ and the other representatives of the class of ω in $\widehat{H}^0(\Gamma, \mathcal{O}_H^*)$ differ from ω by an element of $N_{H/K}(\mathcal{O}_H^*) \subset N_{L/K}(\mathcal{O}_L^*)$. Consider the commutative diagram of Γ-modules, with exact rows and exact columns:

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$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_H^* & \longrightarrow & H^* & \longrightarrow & P_H & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_H & \longrightarrow & A_H^* & \longrightarrow & I_H & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_H/\mathcal{O}_H^* & \longrightarrow & C_H & \longrightarrow & Cl_H & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We thus derive the following diagram of $\varGamma$-cohomology groups:

$$\begin{array}{ccccc}
\widehat{H}^{-2}(\varGamma, Cl_H) & \longrightarrow & \widehat{H}^{-1}(\varGamma, U_H/\mathcal{O}_H^*) & \longrightarrow & \widehat{H}^{-1}(\varGamma, C_H) \\
\downarrow & & \downarrow{\scriptstyle \delta} & & \downarrow \\
\widehat{H}^{-1}(\varGamma, P_H) & \longrightarrow & \widehat{H}^{0}(\varGamma, \mathcal{O}_H^*) & \longrightarrow & \widehat{H}^{0}(\varGamma, H^*) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \widehat{H}^{0}(\varGamma, U_H) & \longrightarrow & \widehat{H}^{0}(\varGamma, A_H^*)
\end{array}$$

The zero in the lower left corner comes from the fact that $\widehat{H}^{-1}(\varGamma, I_H) = 0$, which follows from Shapiro's lemma (see [2], Lemma 2.11 for details).

We now do a diagram chase: $\omega \in \widehat{H}^0(\varGamma, \mathcal{O}_H^*)$ maps to 0 in $\widehat{H}^0(\varGamma, H^*)$. Therefore it "comes from" $z \in \widehat{H}^{-1}(\varGamma, P_H)$. Because of the zero in the left lower corner, z itself "comes from" $y \in \widehat{H}^{-2}(\varGamma, Cl_H)$. Let x denote the image of y in $\widehat{H}^{-1}(\varGamma, U_H/\mathcal{O}_H^*)$. Let $x \in U_H$ also denote any unit idèle whose class in $\widehat{H}^{-1}(\varGamma, U_H/\mathcal{O}_H^*)$ is our x. By the exactness of the diagram, the *class* x maps to 0 in $\widehat{H}^{-1}(\varGamma, C_H)$, hence the *element* $x \in H^* \cdot (A_H^*)^{I_\varGamma}$, where $I_\varGamma$ denotes the augmentation ideal of $Z[\varGamma]$, and $H^* \cdot (A_H^*)^{I_\varGamma}$ denotes the subgroup of A_H^* generated by the set $\{x^{1-\sigma} : x \in A_H^*,\ \sigma \in \varGamma\}$ and H^*. Moreover, by the commutativity of the diagram, the map labeled δ maps the *class* x to the class of ω in $\widehat{H}^0(\varGamma, \mathcal{O}_H^*)$. Recall the definition of the coboundary map $\delta : \widehat{H}^{-1}(\varGamma, U_H/\mathcal{O}_H^*) \to \widehat{H}^0(\varGamma, \mathcal{O}_H^*)$: Choose any element $x \in U_H$ representing the *class* $x \in \widehat{H}^{-1}(\varGamma, U_H/\mathcal{O}_H^*)$; then $\delta(x) =$ the class of $N_{H/K}(x)$ in $\widehat{H}^0(\varGamma, \mathcal{O}_H^*) \cong \mathcal{O}_K^*/N_{H/K}(\mathcal{O}_H^*)$. Thus we see that the class of the $N_{H/K}$-norm of the *element* x in $\widehat{H}^0(\varGamma, \mathcal{O}_H^*)$ is equal to that of ω. Combining this with the fact that $\widehat{H}^{-1}(\Gamma, U_H/\mathcal{O}_H^*)$ is a submodule of $U_H/(U_H)^{I_\Gamma}\mathcal{O}_H^*$, we see that we may modify the element $x \in U_H$ by an element of $\mathcal{O}_H^*$ so that the elements x and ω satisfy $N_{H/K}(x) = \omega$.

Let therefore $\alpha \in H^*$ such that $x\alpha^{-1} \in (A_H^*)^{I_\Gamma}$. Then

$$N_{H/L}(x\alpha^{-1}) \in N_{H/L}((A_H^*)^{I_\Gamma}) = (N_{H/L}(A_H^*))^{I_G} \subset (U_L \cdot L^*)^{I_G} = U_L^{I_G} \cdot (L^*)^{I_G}.$$

The first equality follows from the fact that $N_{H/L}$ is a central element in the group ring $Z[\Gamma]$, the inclusion from the fact that H is the Hilbert class field of L and the last equality is a generality. By I_G we denote the augmentation ideal of the group ring $Z[G]$.

So, we have $N_{H/L}(x\alpha^{-1}) = u\beta$, where $u \in U_L^{I_G}$ and $\beta \in (L^*)^{I_G}$. This implies that

$$\eta = N_{H/L}(x) \cdot u^{-1} = N_{H/L}(\alpha) \cdot \beta \in L^* \cap U_L = \mathcal{O}_L^*.$$

Since $N_{L/K}(u) = 1$, we have $N_{L/K}(\eta) = N_{H/K}(x) = \omega$, as required. ∎

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References

[1] J. W. S. Cassels and A. Fröhlich (eds.), *Algebraic Number Theory*, Academic Press, New York, 1967.

[2] A. Fröhlich, *Central Extensions, Galois Groups, and Ideal Class Groups of Number Fields*, American Mathematical Society, 1980.

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