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A certain power series associated with a Beatty sequence

by

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0. Introduction. We consider the function

$$f(\theta,\phi;x,y)=\sum_{k=1}^{\infty}\sum_{1\le m\le k\theta+\phi}x^k y^m. \tag{1}$$

Putting $y=1$ entails that

$$f(\theta,\phi;x,1)=\sum_{k=1}^{\infty}[k\theta+\phi]x^k. \tag{2}$$

The sequence $\{[k\theta+\phi]\}_{k=1}^{\infty}$, which appears in this power series, is called a *Beatty sequence*. In that context it is natural to consider the sequence of differences

$$\{[(k+1)\theta+\phi]-[k\theta+\phi]\}_{k=1}^{\infty}. \tag{3}$$

The function $f(\theta,0;x,y)$ and the sequence $\{[(k+1)\theta]-[k\theta]\}_{k=1}^{\infty}$ in the homogeneous case have been treated independently by many authors (see e.g. [1], [7], [8] and [2], [10] respectively). The inhomogeneous case of (3) has also been treated by several authors (see e.g. [3]–[5]).

In 1992 Nishioka, Shiokawa and Tamura [9] described the sequence (3) in the inhomogeneous case by using the characteristic properties of (1), but their result (Theorem 3 of [9]) is incorrect. The arguments only hold when ϕ is an integer or when $b_n=1$ for all positive integers n (for the definition of b_n see the next section).

In this paper we base on the arguments corrected by the author [6] and describe the sequence (3) completely in the new form. Of course, Theorem 2 of [9] holds because $\phi=0$. Lemmas 2 and 3 of [9], which were used to prove Theorem 3 of [9], work and have meaning only in the original context. After

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correcting the arguments properly, both lemmas are no longer useful and we need different new arguments to obtain a correction to Theorem 3 of [9].

1. Preliminary remarks and notation. Throughout this paper $\theta > 0$ is irrational and $k\theta + \phi$ is never integral for any positive integer k. As usual, $\theta = [a_0, a_1, a_2, \ldots]$ denotes the continued fraction expansion of θ, where

$$\theta = a_0 + \theta_0, \quad a_0 = [\theta],$$
$$1/\theta_{n-1} = a_n + \theta_n, \quad a_n = [1/\theta_{n-1}] \quad (n = 1, 2, \ldots).$$

The nth convergent $p_n/q_n = [a_0, a_1, \ldots, a_n]$ of θ is then given by the recurrence relations

$$p_n = a_n p_{n-1} + p_{n-2} \quad (n = 0, 1, \ldots), \quad p_{-2} = 0, \quad p_{-1} = 1,$$
$$q_n = a_n q_{n-1} + q_{n-2} \quad (n = 0, 1, \ldots), \quad q_{-2} = 1, \quad q_{-1} = 0.$$

One now expands ϕ in terms of the sequence $\{\theta_0, \theta_1, \ldots\}$ by setting

$$\phi = b_0 - \phi_0, \quad b_0 = \lceil \phi \rceil,$$
$$\phi_{n-1}/\theta_{n-1} = b_n - \phi_n, \quad b_n = \lceil \phi_{n-1}/\theta_{n-1} \rceil \quad (n = 1, 2, \ldots).$$

Furthermore, the quantities s_n and t_n are defined by

$$s_n = \sum_{\nu=0}^{n} b_\nu p_{\nu-1} \quad (n = 0, 1, \ldots), \quad s_n = 0 \quad (n < 0),$$
$$t_n = \sum_{\nu=0}^{n} b_\nu q_{\nu-1} \quad (n = 0, 1, \ldots), \quad t_n = 0 \quad (n < 0).$$

We can assume $0 < \theta, \phi < 1$ without loss of generality. As shown in Sections 1 and 2 of [6],

$$f(\theta, \phi; x, y) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{t_n} y^{s_n}}{(1 - x^{q_n} y^{p_n})(1 - x^{q_{n-1}} y^{p_{n-1}})},$$

which yields

$$\sum_{k=0}^{\infty} ([(k+1)\theta + \phi] - [k\theta + \phi]) x^k = \frac{1}{x} \lim_{n \to \infty} P_n^*(x), \qquad |x| < 1.$$

Here, $P_n^*(x)$ is defined recursively by

$$P_n^*(x) = A_n^*(x) P_{n-1}^*(x) + x^{b_n q_{n-1}} P_{n-2}^*(x) \quad (n \ge 1)$$

with $P_{-1}^*(x) = 1$, $P_0^*(x) = 0$, where

$$A_n^*(x) = \frac{1 - x^{q_n} - x^{b_n q_{n-1}}(1 - x^{q_{n-2}})}{1 - x^{q_{n-1}}} \quad (n \ge 1).$$

Let $P_n^*(x) = d_1x + d_2x^2 + d_3x^3 + \ldots$ be the power series expansion. Put $P_n^* = d_1d_2d_3\ldots$, which is the string of coefficients of the power series beginning from that of x^1.

Define

$$\Gamma_n = \{a_3 - b_3, a_4 - b_4, \ldots, a_n - b_n\} \quad (n \geq 3)$$

and write $\pi_n = a_n - b_n$ if $a_n > b_n$, $\varpi_n = a_n - b_n$ if $a_n \geq b_n$—to account for the case when the entry 0 is permitted.

We consider the following situations:

$\Gamma_n \in \mathcal{O}$		if $\Gamma_n = \varpi_3\varpi_4\ldots\varpi_n$,
$\Gamma_n \in \mathcal{A}_{k,l}$	(or simply $\mathcal{A}$)	if Γ_n ends in $(-1)0^{2k-1}\pi_{n-l}\underbrace{\varpi_{n-l+1}\ldots\varpi_n}_{l}$,
$\Gamma_n \in \mathcal{B}_k$	(or simply $\mathcal{B}$)	if Γ_n ends in $(-1)0^{2k-1}$,
$\Gamma_n \in \mathcal{C}_k$	(or simply $\mathcal{C}$)	if Γ_n ends in $(-1)0^{2k-2}$ $(k \geq 2)$,
$\Gamma_n \in \mathcal{C}_1$		if Γ_n ends in $\pi_{n-l-1}\underbrace{\varpi_{n-l}\ldots\varpi_{n-1}}_{l}(-1)$,
$\Gamma_n \in \mathcal{D}_k$	(or simply $\mathcal{D}$)	if Γ_n ends in $(-1)0^{2k-2}(-1)$,

where k is a positive integer and l is a non-negative integer. (Note that $\Gamma_3 \in \mathcal{O}$ if $a_3 \geq b_3$ and $\Gamma_3 \in \mathcal{C}$ if $a_3 = b_3 - 1$.)

Let $\beta_n = t_n - q_{n-1} - b_1 + 1 = (b_n - 1)q_{n-1} + b_{n-1}q_{n-2} + \ldots + b_2q_1 + 1$. We define the words u, v and Δ_n as

$$u = \underbrace{0\ldots0}_{a_1-1}1, \quad v = \underbrace{0\ldots0}_{b_1-1}1 \quad \text{and} \quad \Delta_n = \underbrace{0\ldots0}_{\beta_n-2}(-1)^{n+1}(-1)^n.$$

2. Main results. Our main result, which replaces the alleged Theorem 3 of [9], is

THEOREM. *Let θ be irrational with $0 < \theta, \phi < 1$. Then either*

$$\{[(k+1)\theta + \phi] - [k\theta + \phi]\}_{k=0}^{\infty} = \lim_{n\to\infty} P_n^*$$

or

$$\{[(k+1)\theta + \phi] - [k\theta + \phi]\}_{k=1}^{\infty} = \lim_{n\to\infty} \underbrace{0\ldots01}_{b_1-1} w_n.$$

Here (w_n) is the sequence of words of respective lengths q_n, with letters 0 *or* 1, *given inductively by*

$$w_1 = u, \quad w_2 = w_1^{b_2-1}0w_1^{a_2-b_2+1}, \quad w_n = w_{n-1}^{c_n}w_{n-2}w_{n-1}^{a_n-c_n},$$

where

$$c_n = \begin{cases} b_n + 1 & \text{if } \Gamma_{n-1} \in \mathcal{B} \text{ and } a_n > b_n, \\ 0 & \text{if } \Gamma_{n-1} \in \mathcal{C}, \\ 1 & \text{if } \Gamma_{n-1} \in \mathcal{D}, \\ \min(a_n, b_n) & \text{otherwise.} \end{cases}$$

Remark. By Lemma 1 below, $a_n \le b_n$ if $\Gamma_{n-1} \in \mathcal{C}, \mathcal{D}$. Other possible cases are limited to $\Gamma_{n-1} \in \mathcal{B}$ and $a_n = b_n$, and $\Gamma_{n-1} \in \mathcal{O}, \mathcal{A}$.

The Theorem is a direct consequence of the following Proposition, which describes P_n^*. From now on the underline means to add (-1) to the last one part in that word. For example, if $W = 00101$, then $\underline{W} = 00100$. If $W = 00100$, then $\underline{W} = 0010(-1)$, $\underline{W^2} = 00100\,0010(-1)$ and $(\underline{W})^2 = 0010(-1)0010(-1)$.

Proposition. *For every* $n = 1, 2, \ldots$, *we have* $P_n^* = v\underline{w_n}w_n''$. *Here*, $|w_n| = q_n$ *for every* n, *and* $w_1 = u$, $w_2 = u^{b_2-1}0u^{a_2-b_2+1}$; w_1'' *and* w_2'' *are empty*; *and* w_n *and* w_n'' $(n \ge 3)$ *are determined as follows*:

(1) *If* $n = 3$ *and* $\Gamma_{n-1} \in \mathcal{O}$ *or* $\mathcal{A}$ $(n \ge 4)$, *then*

$$\begin{cases} w_n = w_{n-1}^{b_n} w_{n-2} w_{n-1}^{a_n-b_n} \quad \text{and} \quad w_n'' = \text{empty} & \text{if } a_n \ge b_n, \\ w_n = w_{n-1}^{a_n} w_{n-2} \quad \text{and} \quad w_n'' = \Delta_{n-1} & \text{if } a_n = b_n - 1. \end{cases}$$

(2) *If* $\Gamma_{n-1} \in \mathcal{B}$ $(n \ge 5)$, *then*

$$\begin{cases} w_n = w_{n-1}^{b_n+1} w_{n-2} w_{n-1}^{a_n-b_n-1} \quad \text{and} \quad w_n'' = \text{empty} & \text{if } a_n > b_n, \\ w_n = w_{n-1}^{a_n} w_{n-2} \quad \text{and} \quad w_n'' = \Delta_{n-2k-1} & \text{if } a_n = b_n, \end{cases}$$

$(k = 1$ *if* $\Gamma_{n-2} \in \mathcal{D})$.

(3) *If* $\Gamma_{n-1} \in \mathcal{C}$ $(n \ge 4)$, *then*

$$w_n = w_{n-2} w_{n-1}^{a_n} \quad \text{and} \quad w_n'' = \begin{cases} \text{empty} & \text{if } a_n = b_n, \\ \Delta_{n-1} & \text{if } a_n = b_n - 1. \end{cases}$$

(4) *If* $\Gamma_{n-1} \in \mathcal{D}$ $(n \ge 5)$, *then*

$$w_n = w_{n-1} w_{n-2} w_{n-1}^{a_n-1} \quad \text{and} \quad w_n'' = \begin{cases} \text{empty} & \text{if } a_n = b_n, \\ \Delta_{n-1} & \text{if } a_n = b_n - 1. \end{cases}$$

We detail the initial cases $n = 1, 2, 3$ here. We notice that

$$A_n^*(x) = 1 + x^{q_{n-1}} + \ldots + x^{(b_n-1)q_{n-1}} + \begin{cases} x^{b_n q_{n-1} + q_{n-2}}(1 + x^{q_{n-1}} + \ldots + x^{(a_n-b_n-1)q_{n-1}}) & \text{if } a_n > b_n, \\ 0 & \text{if } a_n = b_n, \\ -x^{q_n} & \text{if } a_n = b_n - 1. \end{cases}$$

Since $P_1^*(x) = x^{b_1}$, we have $P_1^* = v = v\underline{u}$. Since $P_2^*(x) = x^{b_1} A_2^*(x)$, we have

$$P_2^* = \begin{cases} vu^{b_2-1}0u^{a_2-b_2} = vu^{b_2-1}0u^{a_2-b_2}\underline{u} & \text{if } a_2 > b_2, \\ vu^{b_2-1} = vu^{b_2-1}0\underline{u} & \text{if } a_2 = b_2, \\ vu^{b_2-1}(-1) & \text{if } a_2 = b_2 - 1. \end{cases}$$

Thus, $w_2 = u^{b_2-1}0u^{a_2-b_2+1}$. Since $P_3^*(x) = x^{b_1}(A_3^*(x)A_2^*(x)+1)$, we have

$$P_3^* = \begin{cases} vw_2^{b_3}uw_2^{a_3-b_3-1}u^{b_2-1}0u^{a_2-b_2} & \text{if } a_3 > b_3, \\ vw_2^{b_3} = vw_2^{b_3}\underline{u} & \text{if } a_3 = b_3, \\ vw_2^{a_3}\underbrace{00\ldots\ldots00}_{a_1+\beta_2-2}(-1)1 & \text{if } a_3 = b_3 - 1. \end{cases}$$

3. Lemmas. We need the following lemmas to complete the proof of the Proposition.

LEMMA 1. (1) *If* $\Gamma_{n-1} \in \mathcal{C}$ *or* $\mathcal{D}$, *then* $a_n \le b_n$.
(2) *If* $\Gamma_{n-1} \in \mathcal{B}$, *then* $a_n \ge b_n$.

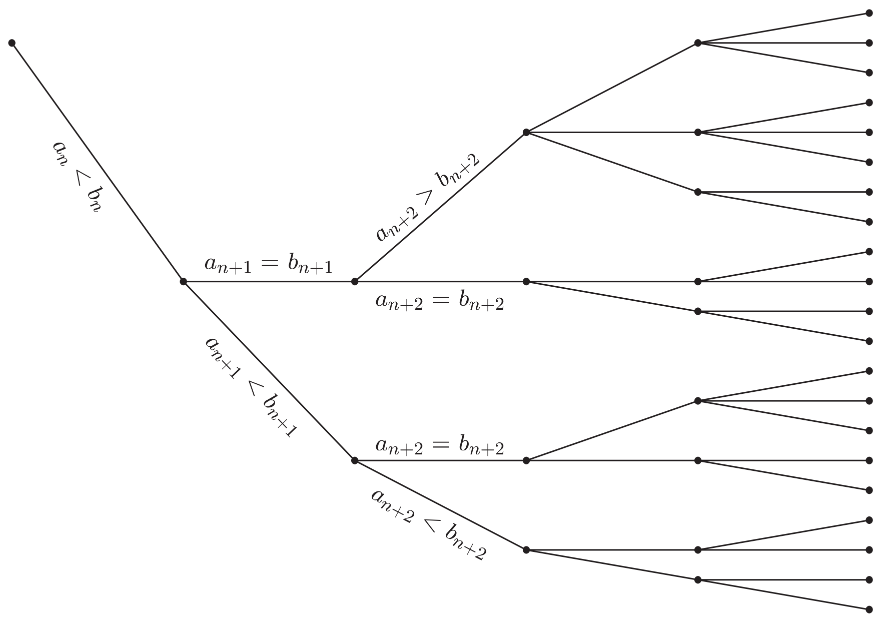

Fig. 1

P r o o f. We prove (1) and (2) together. Notice that as long as $a_i \ge b_i$ for $i = 3, 4, \ldots$, always $\Gamma_i \in \mathcal{O}$. Suppose that $a_3 \ge b_3, \ldots, a_{n-2} \ge b_{n-2}$ and $a_{n-1} = b_{n-1} - 1$ for some fixed $n \ge 4$, which means $\Gamma_{n-1} \in \mathcal{C}_1$. From the definition we have

$$\theta_{n-1} + \phi_{n-1} = \left(\frac{1}{\theta_{n-2}} - a_{n-1}\right) + \left(b_{n-1} - \frac{\phi_{n-2}}{\theta_{n-2}}\right) = \frac{1-\phi_{n-2}}{\theta_{n-2}} + 1 > 1$$

or

$$0 < \frac{1}{\theta_{n-1}} - \frac{\phi_{n-1}}{\theta_{n-1}} < 1.$$

Therefore,

$$a_n = \left[\frac{1}{\theta_{n-1}}\right] \le b_n = \left\lceil \frac{\phi_{n-1}}{\theta_{n-1}} \right\rceil.$$

The case $\Gamma_{n-1} \in \mathcal{C}_1$ is proved.

If $a_n = b_n$, that is, $\Gamma_n \in \mathcal{B}_1$, we get

$$\theta_n + \phi_n = \left(\frac{1}{\theta_{n-1}} - a_n\right) + \left(b_n - \frac{\phi_{n-1}}{\theta_{n-1}}\right) = \frac{1}{\theta_{n-1}} - \frac{\phi_{n-1}}{\theta_{n-1}} < 1.$$

Therefore,

$$a_{n+1} = \left[\frac{1}{\theta_n}\right] \ge b_{n+1} = \left\lceil \frac{\phi_n}{\theta_n} \right\rceil.$$

The case $\Gamma_n \in \mathcal{B}_1$ is proved. If $a_{n+1} > b_{n+1}$, $\Gamma_{n+1} \in \mathcal{A}_{1,0}$. If $a_{n+1} = b_{n+1}$, $\Gamma_{n+1} \in \mathcal{C}_2$.

If $a_n < b_n$, that is, $\Gamma_n \in \mathcal{D}_1$, similarly to the case $\Gamma_{n-1} \in \mathcal{C}_1$, we get $\theta_n + \phi_n > 1$ and $a_{n+1} \le b_{n+1}$. The case $\Gamma_n \in \mathcal{D}_1$ is proved. If $a_{n+1} = b_{n+1}$, $\Gamma_{n+1} \in \mathcal{B}_1$. If $a_{n+1} < b_{n+1}$, $\Gamma_{n+1} \in \mathcal{D}_1$ again.

Now, we consider each case for an arbitrary positive integer k (≥ 2). Let $\Gamma_{i-1} \in \mathcal{C}_k$ for some integer i (≥ 6). Since $\Gamma_{i-2} \in \mathcal{B}_{k-1}$,

$$\frac{1}{\theta_{i-2}} - \frac{\phi_{i-2}}{\theta_{i-2}} > 1.$$

Hence,

$$\theta_{i-1} + \phi_{i-1} = \left(\frac{1}{\theta_{i-2}} - a_{i-1}\right) + \left(b_{i-1} - \frac{\phi_{i-2}}{\theta_{i-2}}\right) = \frac{1 - \phi_{i-2}}{\theta_{i-2}} > 1$$

or

$$0 < \frac{1}{\theta_{i-1}} - \frac{\phi_{i-1}}{\theta_{i-1}} < 1.$$

Therefore, $a_i \le b_i$. If $a_i = b_i$, $\Gamma_i \in \mathcal{B}_k$. If $a_i < b_i$, $\Gamma_i \in \mathcal{D}_k$.

The general case $\Gamma_i \in \mathcal{B}_k$ or $\Gamma_i \in \mathcal{D}_k$ is treated similarly.

The situation in Lemma 1 is illustrated in Figure 1.

LEMMA 2. (1) *If* $\Gamma_{n-1} \in \mathcal{O}$ *or* $\mathcal{A}$, *then* $\beta_{n-1} \le q_{n-1}$.

(2) *If* $\Gamma_{n-2} \in \mathcal{C}_k$ *and* $\Gamma_{n-1} \in \mathcal{B}_k$, *then* $\beta_{n-2k-1} \le q_{n-3}$.

(3) *If* $\Gamma_{n-2} \in \mathcal{D}_k$ *and* $\Gamma_{n-1} \in \mathcal{B}_1$, *then* $\beta_{n-3} \le q_{n-2} + q_{n-3}$.

(4) *If* $\Gamma_{n-1} \in \mathcal{C}_k$, *then* $\beta_{n-2k} \le q_{n-2}$.

(5) *If* $\Gamma_{n-1} \in \mathcal{D}_k$, *then* $\beta_{n-2} \le q_{n-1} + q_{n-2}$.

Proof. If $a_i \ge b_i$ for any $i = 3, 4, \ldots, n$, then

$$\begin{aligned}\beta_n &= (b_n - 1)q_{n-1} + b_{n-1}q_{n-2} + \ldots + b_3 q_2 + b_2 q_1 + 1 \\ &\le (a_n - 1)q_{n-1} + a_{n-1}q_{n-2} + \ldots + a_3 q_2 + (a_2 + 1)q_1 + 1 = q_n.\end{aligned}$$

The other cases will be proved inductively in the proof of the Proposition.

LEMMA 3. (1) *If* $\Gamma_{n-1} \in \mathcal{O}$ *or* $\mathcal{A}$ *and* $a_n \ge b_n$, *then*

$$w_n w_{n-1} - w_{n-1} w_n = \underbrace{0\ldots0}_{q_{n-1}} \Delta_n.$$

(2) *If* $\Gamma_{n-1} \in \mathcal{O}$ *or* $\mathcal{A}$ *and* $a_n < b_n$, *then*

$$-w_n w_{n-1} + w_{n-1} w_n = \underbrace{0\ldots0}_{q_n} \Delta_{n-1}.$$

(3) *If* $\Gamma_{n-1} \in \mathcal{B}_k$ *and* $a_n > b_n$, *then*

$$-w_n w_{n-1} + w_{n-1} w_n = \underbrace{00\ldots\ldots\ldots00}_{(b_n+1)q_{n-1}+q_{n-2}} \Delta_{n-2k-1}.$$

(4) *If* $\Gamma_{n-1} \in \mathcal{B}_k$ *and* $a_n = b_n$, *then*

$$-w_n w_{n-1} + w_{n-1} w_n = \underbrace{0\ldots0}_{q_n} \Delta_{n-2k-1}.$$

(5) *If* $\Gamma_{n-1} \in \mathcal{C}_k$ *and* $a_n \le b_n$, *then*

$$w_n w_{n-1} - w_{n-1} w_n = \underbrace{0\ldots0}_{q_{n-1}} \Delta_{n-2k}.$$

(6) *If* $\Gamma_{n-1} \in \mathcal{D}_k$ *and* $a_n \le b_n$, *then*

$$w_n w_{n-1} - w_{n-1} w_n = \underbrace{0\ldots0}_{q_{n-1}} \Delta_{n-2}.$$

Proof. Here, we shall prove only the case when $\Gamma_{n-1} \in \mathcal{O}$ and $a_n \ge b_n$. The others will be proved inductively in the proof of the Proposition. Both w''_{n-2} and w''_{n-1} are empty by induction. Set $X = x^{q_{n-1}}$ for brevity. If $a_n > b_n$, then

$$P_n^*(x) = (1 + X + \ldots + X^{b_n-1} + X^{b_n} x^{q_{n-2}} (1 + X + \ldots + X^{a_n-b_n-1})) \\ \times P_{n-1}^*(x) + X^{b_n} P_{n-2}^*(x),$$

which yields

$$P_n^* = v w_{n-1}^{b_n} w_{n-2} \underline{w_{n-1}^{a_n-b_n}}.$$

If $a_n = b_n$, we have $P_n^*(x) = (1+X+\ldots+X^{b_n-1})P_{n-1}^*(x)+X^{b_n} P_{n-2}^*(x)$, yielding $P_n^* = v w_{n-1}^{b_n} \underline{w_{n-2}}$. Hence, we have $w_n = w_{n-1}^{b_n} w_{n-2} w_{n-1}^{a_n-b_n}$. Therefore, if n is odd, then

$$\begin{aligned} w_n w_{n-1} - w_{n-1} w_n &= w_{n-1}^{b_n} w_{n-2} w_{n-1}^{a_n-b_n} w_{n-1} - w_{n-1} w_{n-1}^{b_n} w_{n-2} w_{n-1}^{a_n-b_n} \\ &= \underbrace{0\ldots0}_{b_n q_{n-1}} (w_{n-2} w_{n-1} - w_{n-1} w_{n-2}) \end{aligned}$$

$$
\begin{aligned}
&= \underbrace{0\dots0}_{b_nq_{n-1}}(w_{n-2}w_{n-2}^{b_{n-1}}w_{n-3}w_{n-2}^{a_{n-1}-b_{n-1}} - w_{n-2}^{b_{n-1}}w_{n-3}w_{n-2}^{a_{n-1}-b_{n-1}}w_{n-2})\\
&= \underbrace{00\dots\dots\dots00}_{b_nq_{n-1}+b_{n-1}q_{n-2}}(w_{n-2}w_{n-3} - w_{n-3}w_{n-2}) = \dots\\
&= \underbrace{000\dots\dots\dots\dots\dots000}_{b_nq_{n-1}+b_{n-1}q_{n-2}+\dots+b_3q_2}(w_1w_2 - w_2w_1)\\
&= \underbrace{000\dots\dots\dots\dots\dots000}_{b_nq_{n-1}+b_{n-1}q_{n-2}+\dots+b_3q_2}(uu^{b_2-1}0u^{a_2-b_2+1} - u^{b_2-1}0u^{a_2-b_2+1}u)\\
&= \underbrace{0000\dots\dots\dots\dots\dots\dots\dots0000}_{b_nq_{n-1}+b_{n-1}q_{n-2}+\dots+b_3q_2+(b_2-1)q_1}(\underbrace{0\dots0}_{q_1-1}10 - 0\underbrace{0\dots0}_{q_1-1}1)\\
&= \underbrace{00\dots\dots00}_{q_{n-1}+\beta_n-2}1(-1).
\end{aligned}
$$

If n is even, then w_1 and w_2 above are interchanged, so we obtain

$$\underbrace{00\dots\dots00}_{q_{n-1}+\beta_n-2}(-1)1.$$

4. Proof of Proposition. We prove the Proposition together with Lemmas 2 and 3. We write $[B_{k-1}C_kD_k]$ for brevity when $\Gamma_{n-3} \in \mathcal{B}_{k-1}$, $\Gamma_{n-2} \in \mathcal{C}_k$ and $\Gamma_{n-1} \in \mathcal{D}_k$. From Lemma 1 all cases are classified into one of $\mathcal{O}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and the number of patterns like $[B_{k-1}C_kD_k]$ is limited.

We denote by S the sequence of the patterns of $[\Gamma_{n-3}, \Gamma_{n-2}, \Gamma_{n-1}]$.

4.1. *Case* $\Gamma_{n-1} \in \mathcal{O}$. The only possible pattern is $[OOO]$. Then, both w''_{n-2} and w''_{n-1} are empty. As we have already seen in the proof of Lemma 3,

$$w_n = w_{n-1}^{b_n}w_{n-2}w_{n-1}^{a_n-b_n} \quad \text{and} \quad w''_n = \text{empty} \quad \text{if } a_n \ge b_n.$$

If $a_n = b_n - 1$, by using Lemma 3(1) with $\Gamma_{n-3} \in \mathcal{O}$ and $\beta_{n-1} = (b_{n-1}-1)q_{n-2} + q_{n-3} + \beta_{n-2}$ we have $P_n^*(x) = (1 + X + \dots + X^{b_n-1} - x^{q_n})P_{n-1}^*(x) + X^{b_n}P_{n-2}^*(x)$, which yields

$$
\begin{aligned}
P_n^* &= vw_{n-1}^{b_n}\underline{w_{n-2}} - \underbrace{0\dots0}_{q_n}v\underline{w_{n-1}} = v\,\underbrace{\overset{b_n}{\underline{w_{n-1}}}}_{\text{first } q_n}w_{n-2} - \underbrace{0\dots0}_{b_1+q_n}w_{n-1}\\
&= vw_{n-1}^{a_n}\underline{w_{n-2}}\,\underbrace{00\dots\dots00}_{(b_{n-1}-1)q_{n-2}}(w_{n-3}w_{n-2} - w_{n-2}w_{n-3})\\
&= vw_{n-1}^{a_n}\underline{w_{n-2}}\Delta_{n-1}.
\end{aligned}
$$

Therefore, $w_n = w_{n-1}^{a_n}w_{n-2}$ and $w''_n = \Delta_{n-1}$.

Using the results here and Lemma 3(1) with $\Gamma_{n-2} \in \mathcal{O}$, we obtain Lemma 3(2), that is,

$$\begin{aligned} -w_n w_{n-1} + w_{n-1} w_n &= -w_{n-1}^{a_n} w_{n-2} w_{n-1} + w_{n-1}^{a_n} w_{n-1} w_{n-2} \\ &= \underbrace{0\ldots0}_{a_n q_{n-1}} \underbrace{0\ldots0}_{q_{n-2}} \Delta_{n-1} = \underbrace{0\ldots0}_{q_n} \Delta_{n-1}. \end{aligned}$$

As long as $a_i \geq b_i$ for $i = 3, 4, \ldots$, there is no other pattern. But once $a_n < b_n$ for some n, the pattern $[OOC_1]$ follows $[OOO]$ in the sequence S and the loop starts. The situation after this can be seen in Figure 2. "//" stands for $a_i \geq b_i$, "/" for $a_i > b_i$, "−" for $a_i = b_i$, "\" for $a_i < b_i$. Once we encounter C_1 (or B_1, D_1) again, the situation after that is the same as the situation after the first C_1 (or B_1, D_1).

We shall indicate the loop in all patterns according to the class of Γ_{n-1}. Some initial cases are omitted, but it is easy to see that they are special cases of the general ones and they are included in them.

4.2. *Case* $\Gamma_{n-1} \in \mathcal{C}$. From Lemma 1 the possible patterns are

$$[OOC_1],\ [B_k A_{k,0} C_1],\ [A_{k,l-1} A_{k,l} C_1],\ [C_k B_k C_{k+1}],\ [D_k B_1 C_2].$$

• $[OOC_1]$. This follows $[OOO]$ in the sequence S.

Since $\Gamma_{n-1} = \varpi_3 \ldots \varpi_{n-2}(-1)$ ($\Gamma_3 = (-1)$ when $n = 4$), w''_{n-2} is empty and $w_{n-1} = w_{n-2}^{a_{n-1}} w_{n-3}$ and $w''_{n-1} = \Delta_{n-2}$.

If $a_n = b_n$, we have $P_n^*(x) = (1+X+\ldots+X^{a_n-1})P_{n-1}^*(x) + X^{a_n} P_{n-2}^*(x)$. Since the string of coefficients of

$$(1 + X + \ldots + X^{b_n-1}) \times x^{b_1} X((-1)^{n-1} x^{\beta_{n-2}-1} + (-1)^{n-2} x^{\beta_{n-2}})$$

is

$$\underbrace{0\ldots0}_{b_1+\beta_{n-2}} (\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{a_n},$$

we obtain

$$P_n^* = v w_{n-1}^{a_n} \underline{w_{n-2}} + \underbrace{0\ldots0}_{b_1+\beta_{n-2}} (\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{a_n}.$$

From Lemma 2(1) with $\Gamma_{n-2} \in \mathcal{O}$ we get $0 < \beta_{n-2} \leq q_{n-2}$. Therefore, w''_n is empty and the conclusion of Lemma 2(4) is satisfied.

If $a_n = b_n - 1$, we have $P_n^*(x) = (1 + X + \ldots + X^{b_n-1} - x^{q_n}) P_{n-1}^*(x) + X^{b_n} P_{n-2}^*(x)$. Since the string of coefficients of

$$(1 + X + \ldots + X^{b_n-1} - x^{q_n}) \times x^{b_1} X((-1)^{n-1} x^{\beta_{n-2}-1} + (-1)^{n-2} x^{\beta_{n-2}})$$

is

$$\underbrace{0\ldots0}_{b_1+\beta_{n-2}} (\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{b_n} \underbrace{0\ldots0}_{q_{n-2}-2}(-1)^n(-1)^{n-1},$$

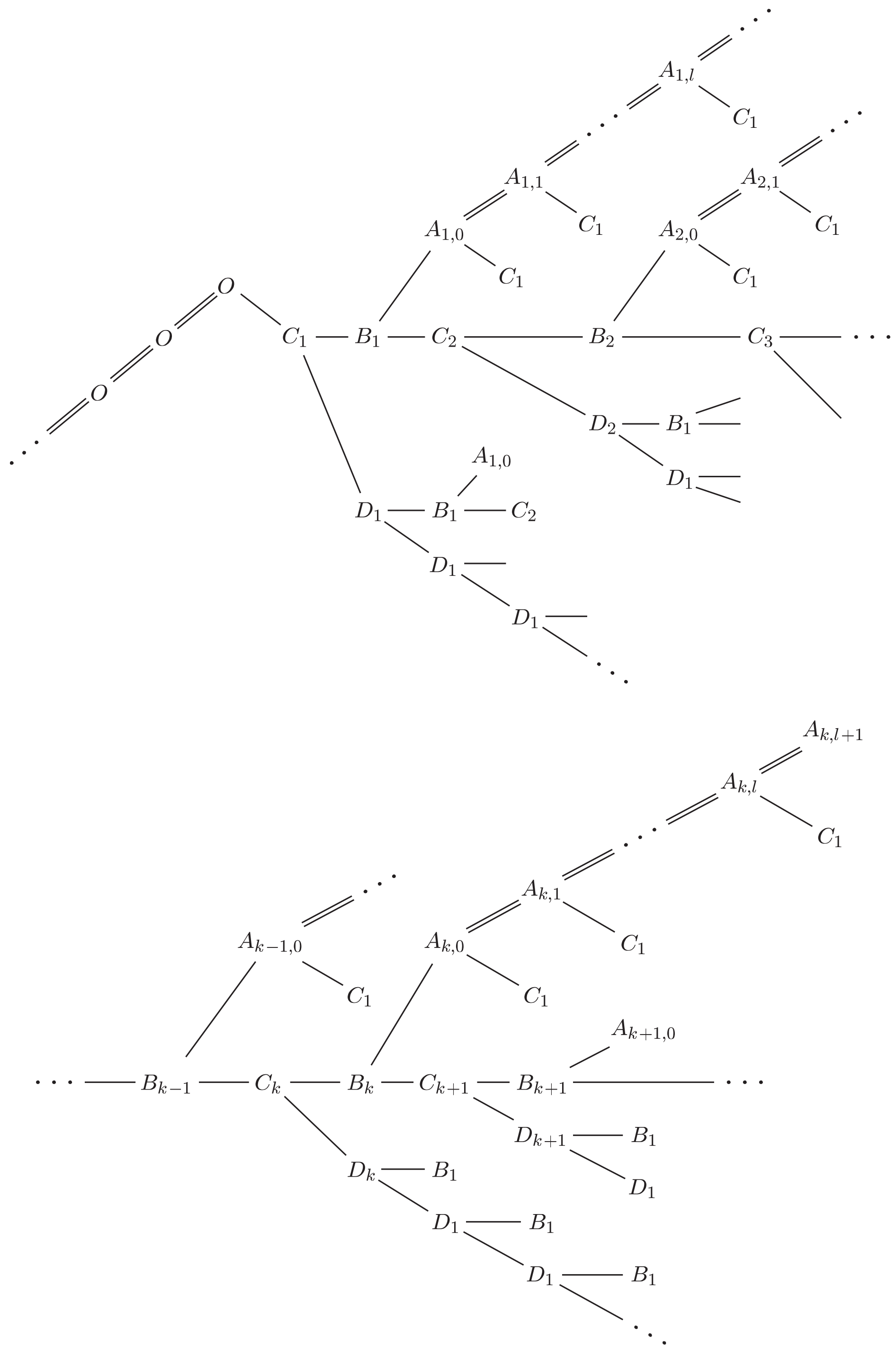
$A_{1,l}$
C_1
$A_{1,1}$
$A_{2,1}$
$A_{1,0}$
C_1
$A_{2,0}$
C_1
C_1
C_1
O
O
O
C_1
B_1
C_2
B_2
C_3
D_2
B_1
$A_{1,0}$
D_1
D_1
B_1
C_2
D_1
D_1
$A_{k,l+1}$
$A_{k,l}$
C_1
$A_{k,1}$
$A_{k-1,0}$
$A_{k,0}$
C_1
C_1
C_1
$A_{k+1,0}$
B_{k-1}
C_k
B_k
C_{k+1}
B_{k+1}
D_{k+1}
B_1
D_k
B_1
D_1
D_1
B_1
D_1
B_1

Fig. 2

we obtain

$$P_n^* = vw_{n-1}^{b_n}\underline{w_{n-2}} - \underbrace{0\ldots0}_{q_n} v\underline{w_{n-1}}$$
$$+ \underbrace{0\ldots0}_{b_1+\beta_{n-2}}(\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{b_n}\underbrace{0\ldots0}_{q_{n-2}-2}(-1)^n(-1)^{n-1}.$$

Using Lemma 3(1) with $\Gamma_{n-3}\in\mathcal{O}$ gives

$$w_{n-1}^{b_n}\underline{w_{n-2}} - \underbrace{0\ldots01}_{q_n}\underline{w_{n-1}} = \underbrace{\underline{w_{n-1}^{b_n}}}_{\text{first } q_n} w_{n-2} - \underbrace{0\ldots0}_{q_n} w_{n-1}$$
$$= w_{n-1}^{a_n}\underline{w_{n-2}}(w_{n-2}^{a_{n-1}-1}w_{n-3}w_{n-2} - w_{n-2}^{a_{n-1}-1}w_{n-2}w_{n-3})$$
$$= -w_{n-1}^{a_n}\underline{w_{n-2}}\ \underbrace{00\ldots\ldots00}_{(a_{n-1}-1)q_{n-2}}\ \underbrace{0\ldots0}_{q_{n-3}}\Delta_{n-2}.$$

Since $a_nq_{n-1}+q_{n-2}+(a_{n-1}-1)q_{n-2}+q_{n-3}+\beta_{n-2} = \beta_{n-2}+b_nq_{n-1}$, $\beta_{n-2}+a_nq_{n-1}\le q_{n-2}+a_nq_{n-1}=q_n$ and $\beta_{n-2}+b_nq_{n-1}+q_{n-2}=q_n+\beta_{n-1}$, we get

$$w_n = w_{n-1}^{a_n}w_{n-2} + \underbrace{0\ldots0}_{\beta_{n-2}}(\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{a_n} \quad\text{and}\quad w_n'' = \Delta_{n-1}.$$

Using Lemma 3(1) with $\Gamma_{n-3}\in\mathcal{O}$ again, we finally obtain

$$w_n = w_{n-2}(w_{n-2}^{a_{n-1}-1}w_{n-3}w_{n-2})^{a_n} + \underbrace{0\ldots0}_{\beta_{n-2}}(\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{a_n}$$
$$= w_{n-2}(w_{n-2}^{a_{n-1}-1}(w_{n-3}w_{n-2} + \underbrace{0\ldots0}_{q_{n-3}}\Delta_{n-2}))^{a_n}$$
$$= w_{n-2}(w_{n-2}^{a_{n-1}-1}w_{n-2}w_{n-3})^{a_n} = w_{n-2}w_{n-1}^{a_n}.$$

The conclusion of Lemma 3(5) is proved in this case, because

$$w_nw_{n-1} - w_{n-1}w_n = w_{n-2}w_{n-1}^{a_n}w_{n-1} - w_{n-1}w_{n-2}w_{n-1}^{a_n}$$
$$= w_{n-2}w_{n-1} - w_{n-1}w_{n-2} = \underbrace{0\ldots0}_{q_{n-1}}\Delta_{n-2}.$$

• $[B_kA_{k,0}C_1]$. This follows $[C_kB_kA_{k,0}]$ or $[D_kB_1A_{1,0}]$ in the sequence S.

Since Γ_{n-1} ends in $(-1)0^{2k-1}\pi_{n-2}(-1)$, w_{n-2}'' is empty and $w_{n-1}'' = \Delta_{n-2}$ and $w_{n-1} = w_{n-2}^{a_{n-1}}w_{n-3}$. From Lemma 2(1) with $\Gamma_{n-2}\in\mathcal{A}$, $\beta_{n-2} = b_{n-2}q_{n-3}+q_{n-4}+\beta_{n-2k-3}\le q_{n-2}$.

If $a_n=b_n$, then similarly to $[OOC_1]$,

$$w_n = w_{n-1}^{b_n}w_{n-2} + \underbrace{0\ldots0}_{\beta_{n-2}}(\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{b_n} = w_{n-2}w_{n-1}^{a_n}.$$

Here we used instead

$$w_{n-3}w_{n-2}+\underbrace{0\ldots0}_{q_{n-3}}\Delta_{n-2}=w_{n-3}w_{n-2}-\underbrace{00\ldots\ldots\ldots\ldots00}_{(b_{n-2}+1)q_{n-3}+q_{n-4}}\Delta_{n-2k-3}$$
$$=w_{n-2}w_{n-3}$$

from Lemma 3(3) with $\Gamma_{n-3}\in\mathcal{B}_k$.

If $a_n<b_n$, then by using Lemma 3(3) with $\Gamma_{n-3}\in\mathcal{B}_k$,

$$\begin{aligned}P_n^*&=vw_{n-1}^{a_n}\underline{w_{n-2}}(w_{n-2}^{a_{n-1}-1}w_{n-3}w_{n-2}-w_{n-2}^{a_{n-1}-1}w_{n-2}w_{n-3})\\&\quad+\underbrace{0\ldots0}_{b_1+\beta_{n-2}}(\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{b_n}\underbrace{0\ldots0}_{q_{n-2}-2}(-1)^n(-1)^{n-1}\\&=vw_{n-1}^{a_n}\underline{w_{n-2}}\underbrace{00\ldots\ldots\ldots00}_{(a_{n-1}-1)q_{n-2}}\underbrace{00\ldots\ldots\ldots\ldots00}_{(b_{n-2}+1)q_{n-3}+q_{n-4}}\Delta_{n-2k-3}\\&\quad+\underbrace{0\ldots0}_{b_1+\beta_{n-2}}(\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{b_n}\underbrace{0\ldots0}_{q_{n-2}-2}(-1)^n(-1)^{n-1}\\&=vw_{n-1}^{a_n}\underline{w_{n-2}}\Delta_{n-1}+\underbrace{0\ldots0}_{b_1+\beta_{n-2}}(\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{a_n}\\&=vw_{n-2}\underline{w_{n-1}^{a_n}}\Delta_{n-1},\end{aligned}$$

because $b_nq_{n-1}+\beta_{n-2}=q_n+(a_{n-1}-1)q_{n-2}+(b_{n-2}+1)q_{n-3}+q_{n-4}+\beta_{n-2k-3}$, $\beta_{n-2}<q_{n-2}$ and $b_nq_{n-1}+q_{n-2}+\beta_{n-2}=q_n+\beta_{n-1}$.

Thus the assertion of Lemma 3(5) is proved because by Lemma 3(2) with $\Gamma_{n-2}\in\mathcal{A}$,

$$w_nw_{n-1}-w_{n-1}w_n=w_{n-2}w_{n-1}^{a_n}w_{n-1}-w_{n-1}w_{n-2}w_{n-1}^{a_n}=\underbrace{0\ldots0}_{q_{n-1}}\Delta_{n-2}.$$

• $[A_{k,l-1}A_{k,l}C_1]$. This follows $[B_kA_{k,0}A_{k,1}]$ or $[A_{k,l-2}A_{k,l-1}A_{k,l}]$ in the sequence S.

We use Lemma 3(1) with $\Gamma_{n-3}\in\mathcal{A}$ instead of Lemma 3(3) with $\Gamma_{n-3}\in\mathcal{B}_k$. The rest of the proof is much the same as in the case $[B_kA_{k,0}C_1]$.

• $[C_kB_kC_{k+1}]$. This follows $[OC_1B_1]$, $[A_{k,l}C_1B_1]$ or $[B_{k-1}C_kB_k]$ in the sequence S.

Since Γ_{n-1} ends in $(-1)0^{2k}$, w''_{n-2} is empty and $w_{n-1}=w_{n-2}^{b_{n-1}}w_{n-3}$ and $w''_{n-1}=\Delta_{n-2k-2}$. Moreover, $\beta_{n-2k-2}=\beta_{n-1}-q_{n-1}\le q_{n-4}$ from Lemma 2(2) with $\Gamma_{n-3}\in\mathcal{C}_k$. So, Lemma 2(4) with $\Gamma_{n-1}\in\mathcal{C}_{k+1}$ holds.

If $a_n=b_n$, then from Lemma 3(5) with $\Gamma_{n-3}\in\mathcal{C}_k$,

$$w_n=w_{n-1}^{b_n}w_{n-2}+\underbrace{0\ldots0}_{\beta_{n-2k-2}}(\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{b_n}=w_{n-2}w_{n-1}^{a_n}$$

and w''_n is empty.

If $a_n = b_n - 1$, then

$$P_n^* = vw_{n-1}^{a_n}\underline{w_{n-2}}(w_{n-2}^{b_{n-1}-1}w_{n-3}w_{n-2} - w_{n-2}^{b_{n-1}-1}w_{n-2}w_{n-3}) + \underbrace{00\ldots\ldots00}_{b_1+\beta_{n-2k-2}}(\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{b_n}\underbrace{0\ldots0}_{q_{n-2}-2}(-1)^n(-1)^{n-1}.$$

Since $q_n + (b_{n-1}-1)q_{n-2} + \beta_{n-2k-2} + q_{n-3} = \beta_{n-2k-2} + b_nq_{n-1}$, $\beta_{n-2k-2} + a_nq_{n-1} < q_{n-2} + a_nq_{n-1} = q_n$ and $\beta_{n-2k-2} + b_nq_{n-1} + q_{n-2} = q_n + \beta_{n-1}$, similarly using Lemma 3(5) with $\Gamma_{n-3} \in \mathcal{C}_k$ we can obtain the result.

The assertion of Lemma 3(5) holds in this case, because by Lemma 3(4) with $\Gamma_{n-2} \in \mathcal{B}_k$,

$$w_nw_{n-1} - w_{n-1}w_n = w_{n-2}w_{n-1}^{a_n}w_{n-1} - w_{n-1}w_{n-2}w_{n-1}^{a_n} = \underbrace{0\ldots0}_{q_{n-1}}\Delta_{n-2k-2}.$$

• $[D_kB_1C_2]$. This follows $[C_kD_kB_1]$ or $[D_kD_1B_1]$ in the sequence S.

Since Γ_{n-1} ends in $(-1)0^{2k-2}(-1)00$, w''_{n-2} is empty and $w''_{n-1} = \Delta_{n-4}$ and $w_{n-1} = w_{n-2}^{a_{n-1}}w_{n-3}$. From Lemma 2(3) with $\Gamma_{n-3} \in \mathcal{D}_k$, $\beta_{n-4} \le q_{n-3} + q_{n-4} \le q_{n-2}$. We use Lemma 3(6) with $\Gamma_{n-3} \in \mathcal{C}_k$ instead of Lemma 3(5) with $\Gamma_{n-3} \in \mathcal{D}_l$. The rest of the proof is much the same as in the case $[C_kB_kC_{k+1}]$ when $k = 1$.

4.3. *Case* $\Gamma_{n-1} \in \mathcal{D}$. From Lemma 1 the possible patterns are

$$[OC_1D_1],\ [A_{k,l}C_1D_1],\ [B_{k-1}C_kD_k],\ [C_kD_kD_1],\ [D_kD_1D_1].$$

• $[OC_1D_1]$. This follows $[OOC_1]$ in the sequence S.

Since $\Gamma_{n-1} = \varpi_3\ldots\varpi_{n-3}(-1)(-1)$, we get $w''_{n-2} = \Delta_{n-3}$, $w_{n-1} = w_{n-3}w_{n-2}^{a_{n-1}}$ and $w''_{n-1} = \Delta_{n-2}$.

If $a_n = b_n$, then

$$P_n^* = vw_{n-1}^{b_n}\underline{w_{n-2}} + \underbrace{00\ldots\ldots\ldots00}_{b_1+b_nq_{n-1}+q_{n-2}}\Delta_{n-3} + \underbrace{0\ldots0}_{b_1+\beta_{n-2}}(\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{b_n}.$$

Since $\beta_{n-2} = q_{n-2} + \beta_{n-3} \le q_{n-2} + q_{n-3} < q_{n-1} + q_{n-2}$ (so, the assertion of Lemma 2(5) holds) and $\beta_{n-2} + (b_n - 1)q_{n-1} < q_n$, we obtain

$$w_n = w_{n-1}^{a_n}w_{n-2} + \underbrace{0\ldots0}_{\beta_{n-2}}(\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{a_n-1} \quad \text{and} \quad w''_n = \text{empty}.$$

If $a_n = b_n - 1$, then

$$P_n^* = v\underbrace{w_{n-1}^{b_n}}_{\text{first } q_n}w_{n-2} - \underbrace{0\ldots0}_{b_1+q_n}w_{n-1} + \underbrace{00\ldots\ldots\ldots00}_{b_1+b_nq_{n-1}+q_{n-2}}\Delta_{n-3} + \underbrace{0\ldots0}_{b_1+\beta_{n-2}}(\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{b_n}\underbrace{0\ldots0}_{q_{n-2}-2}(-1)^n(-1)^{n-1}.$$

Since from Lemma 3(2) with $\Gamma_{n-3} \in \mathcal{O}$,

$$\begin{aligned} & w_{n-1}^{b_n} w_{n-2} - \underbrace{0\ldots0}_{q_n} w_{n-1} \\ &\quad = w_{n-1}^{a_n} w_{n-3} w_{n-2}^{a_{n-1}} w_{n-2} - \underbrace{0\ldots0}_{q_n} w_{n-3} w_{n-2}^{a_{n-1}} \\ &\quad = w_{n-1}^{a_n} (w_{n-2} w_{n-3} + \underbrace{0\ldots0}_{q_{n-2}} \Delta_{n-3}) w_{n-2}^{a_{n-1}} - \underbrace{0\ldots0}_{q_n} w_{n-3} w_{n-2}^{a_{n-1}}, \end{aligned}$$

we have

$$\underbrace{\underline{w_{n-1}^{b_n}}}_{\text{first } q_n} w_{n-2} - \underbrace{0\ldots0}_{q_n} w_{n-1} = w_{n-1}^{a_n} \underline{w_{n-2}} \Delta_{n-3}.$$

Since $\beta_{n-2} + b_n q_{n-1} = b_n q_{n-1} + q_{n-2} + \beta_{n-3}$, $\beta_{n-2} + a_n q_{n-1} = q_n + \beta_{n-3}$, $\beta_{n-2} + (a_n - 1) q_{n-1} < q_n$ and $\beta_{n-2} + b_n q_{n-1} + q_{n-2} = q_n + \beta_{n-1}$, we get

$$w_n = w_{n-1}^{a_n} w_{n-2} + \underbrace{0\ldots0}_{\beta_{n-2}} (\underbrace{0\ldots0}_{q_{n-1}-2} (-1)^{n-1} (-1)^n)^{a_n - 1} \quad \text{and} \quad w_n'' = \text{empty}.$$

Using Lemma 3(2) with $\Gamma_{n-3} \in \mathcal{O}$ again and $-\Delta_{n-2} = \underbrace{0\ldots0}_{q_{n-2}} \Delta_{n-3}$, we finally obtain

$$\begin{aligned} w_n &= w_{n-1}((w_{n-3} w_{n-2} + \Delta_{n-2}) w_{n-2}^{a_{n-1}-1})^{a_n - 1} w_{n-2} \\ &= w_{n-1}(w_{n-2} w_{n-3} w_{n-2}^{a_{n-1}-1})^{a_n-1} w_{n-2} \\ &= w_{n-1} w_{n-2} (w_{n-3} w_{n-2}^{a_{n-1}})^{a_n - 1} = w_{n-1} w_{n-2} w_{n-1}^{a_n - 1}. \end{aligned}$$

The assertion of Lemma 3(6) holds in this case, because by Lemma 3(5) with $\Gamma_{n-2} \in \mathcal{C}_1$,

$$\begin{aligned} w_n w_{n-1} - w_{n-1} w_n &= w_{n-1} w_{n-2} w_{n-1}^{a_n - 1} w_{n-1} - w_{n-1} w_{n-1} w_{n-2} w_{n-1}^{a_n - 1} \\ &= \underbrace{0\ldots0}_{q_{n-1}} (w_{n-2} w_{n-1} - w_{n-1} w_{n-2}) \\ &= -\underbrace{0\ldots0}_{q_{n-1}} \underbrace{0\ldots0}_{q_{n-2}} \Delta_{n-3} = \underbrace{0\ldots0}_{q_{n-1}} \Delta_{n-2}. \end{aligned}$$

• $[A_{k,l} C_1 D_1]$. This follows $[A_{k,l-1} A_{k,l} C_1]$ or $[B_k A_{k,0} C_1]$ in the sequence S.

This case is similar to $[O C_1 D_1]$.

• $[B_{k-1} C_k D_k]$. This follows $[C_k B_k C_{k+1}]$ or $[D_k B_1 C_2]$ in the sequence S.

Since Γ_{n-1} ends in $(-1)0^{2k-2}(-1)$, we have $w_{n-2}'' = \Delta_{n-2k-1}$, $w_{n-1}'' = \Delta_{n-2}$ and $w_{n-1} = w_{n-3} w_{n-2}^{a_{n-1}}$. From Lemma 2(4) with $\Gamma_{n-2} \in \mathcal{C}_k$ we have

$$\beta_{n-2} = q_{n-2} + \beta_{n-2k-1} \le q_{n-2} + q_{n-3} \le q_{n-1} + q_{n-2}.$$

We use

$$w_{n-3}w_{n-2} + \Delta_{n-2} = w_{n-3}w_{n-2} - \underbrace{0\ldots0}_{q_{n-2}}\Delta_{n-2k-1} = w_{n-2}w_{n-3}$$

from Lemma 3(4) with $\Gamma_{n-3} \in \mathcal{B}_{k-1}$. The rest of the proof is much the same as in the case $[OC_1D_1]$.

• $[C_kD_kD_1]$. This follows $[OC_1D_1]$, $[A_{k,l}C_1D_1]$ or $[B_{k-1}C_kD_k]$ in the sequence S.

Since Γ_{n-1} ends in $(-1)0^{2k-2}(-1)(-1)$, we have $w''_{n-2} = \Delta_{n-3}$, $w''_{n-1} = \Delta_{n-2}$ and $w_{n-1} = w_{n-2}w_{n-3}w_{n-2}^{a_{n-1}-1}$. We also have

$$\beta_{n-2} = q_{n-2} + \beta_{n-3} \le q_{n-2} + q_{n-2} + q_{n-3} \le q_{n-1} + q_{n-2}.$$

So the assertion of Lemma 2(5) is satisfied.

If $a_n = b_n$, then

$$\begin{aligned} P_n^* &= vw_{n-1}^{a_n}\underline{w_{n-2}} + \underbrace{00\ldots\ldots\ldots00}_{b_1+b_nq_{n-1}+q_{n-2}}\Delta_{n-3} \\ &\quad + \underbrace{0\ldots0}_{b_1+\beta_{n-2}}(\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{a_n} \\ &= vw_{n-1}^{a_n}\underline{w_{n-2}} + \underbrace{0\ldots0}_{b_1+\beta_{n-2}}(\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{a_n-1}. \end{aligned}$$

Since $\beta_{n-2} = q_{n-2} + q_{n-3} + \beta_{n-2k-2}$, we have

$$\begin{aligned} w_n &= w_{n-1}w_{n-2}((w_{n-3}w_{n-2} + \underbrace{0\ldots0}_{q_{n-3}}\Delta_{n-2k-2})w_{n-2}^{a_{n-1}-1})^{a_n-1} \\ &= w_{n-1}w_{n-2}(w_{n-2}w_{n-3}w_{n-2}^{a_{n-1}-1})^{a_n-1} = w_{n-1}w_{n-2}w_{n-1}^{a_n-1} \end{aligned}$$

and w''_n is empty.

If $a_n < b_n$, by Lemma 3(5) with $\Gamma_{n-3} \in \mathcal{C}_k$ and $\beta_{n-3} = q_{n-3} + \beta_{n-2k-2}$ we get

$$\begin{aligned} &\underbrace{w_{n-1}^{b_n}}_{\text{first } q_n} w_{n-2} - \underbrace{0\ldots0}_{q_n} w_{n-1} \\ &\qquad = w_{n-1}^{a_n}\underline{w_{n-2}}(w_{n-2}^{a_{n-1}-1}w_{n-3}w_{n-2} - w_{n-2}^{a_{n-1}-1}w_{n-2}w_{n-3} \\ &\qquad\quad + \underbrace{0\ldots0}_{\beta_{n-3}-2}(-1)^n(-1)^{n-1}\,\underbrace{00\ldots\ldots\ldots00}_{(a_{n-1}-1)q_{n-2}-2}(-1)^{n-1}(-1)^n) \end{aligned}$$

$$= w_{n-1}^{a_n}\underline{w_{n-2}}(-\underbrace{00\ldots\ldots\ldots\ldots00}_{(a_{n-1}-1)q_{n-2}+q_{n-3}}\Delta_{n-2k-2}$$

$$+\Delta_{n-3}\underbrace{00\ldots\ldots\ldots00}_{(a_{n-1}-1)q_{n-2}-2}(-1)^{n-1}(-1)^n)$$

$$= w_{n-1}^{a_n}\underline{w_{n-2}}\Delta_{n-3}.$$

Hence,

$$P_n^* = vw_{n-1}^{a_n}\underline{w_{n-2}}\Delta_{n-3} + \underbrace{00\ldots\ldots\ldots00}_{b_1+b_nq_{n-1}+q_{n-2}}\Delta_{n-3}$$

$$+\underbrace{0\ldots0}_{b_1+\beta_{n-2}}(\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{b_n}\underbrace{0\ldots0}_{q_{n-2}-2}(-1)^n(-1)^{n-1}$$

$$= vw_{n-1}^{a_n}\underline{w_{n-2}}\Delta_{n-1} + \underbrace{0\ldots0}_{b_1+\beta_{n-2}}(\underbrace{0\ldots0}_{q_{n-1}-2}(-1)^{n-1}(-1)^n)^{a_n-1},$$

since $\beta_{n-2} = q_{n-2} + \beta_{n-3}$ and $\beta_{n-1} = q_{n-1} + \beta_{n-2}$. The remaining part is similarly shown. Lemma 3(6) holds in this case, because by Lemma 3(6) with $\Gamma_{n-2} \in \mathcal{D}_k$,

$$w_nw_{n-1} - w_{n-1}w_n$$

$$= w_{n-1}w_{n-2}w_{n-1}^{a_n-1}w_{n-1} - w_{n-1}w_{n-1}w_{n-2}w_{n-1}^{a_n-1}$$

$$= \underbrace{0\ldots0}_{q_{n-1}}(w_{n-2}w_{n-1} - w_{n-1}w_{n-2})$$

$$= \underbrace{0\ldots0}_{q_{n-1}}\underbrace{0\ldots0}_{q_{n-2}}\underbrace{0\ldots0}_{\beta_{n-3}-2}(-1)^{n-3}(-1)^{n-2} = \underbrace{0\ldots0}_{q_{n-1}}\Delta_{n-2}.$$

• $[D_kD_1D_1]$. This follows $[C_kD_kD_1]$ or $[D_kD_1D_1]$ in the sequence S.

Since Γ_{n-1} ends in $(-1)0^{2k-2}(-1)(-1)(-1)$, we have $w''_{n-2} = \Delta_{n-3}$, $w''_{n-1} = \Delta_{n-2}$ and $w_{n-1} = w_{n-2}w_{n-3}w_{n-2}^{a_{n-1}-1}$. We use Lemma 3(6) with $\Gamma_{n-3} \in \mathcal{D}_k$ instead of Lemma 3(5) with $\Gamma_{n-3} \in \mathcal{C}_k$. The rest of the proof is much the same as in the case $[C_kD_kD_1]$ when $k = 1$.

4.4. *Case* $\Gamma_{n-1} \in \mathcal{B}$. From Lemma 1 the possible patterns are

$$[OC_1B_1],\ [A_{k,l}C_1B_1],\ [B_{k-1}C_kB_k],\ [C_kD_kB_1],\ [D_kD_1B_1].$$

• $[OC_1B_1]$. This follows $[OOC_1]$ in the sequence S.

Since $\Gamma_{n-1} = \varpi_3\ldots\varpi_{n-3}(-1)0$, we have $w''_{n-2} = \Delta_{n-3}$, $w_{n-1} = w_{n-3}w_{n-2}^{a_{n-1}}$ and w''_{n-1} is empty.

If $a_n > b_n$, we have

$$P_n^*(x) = (1 + X + \ldots + X^{b_n-1} + X^{b_n}x^{q_{n-2}}(1 + X + \ldots + X^{a_n-b_n-1}))$$

$$\times P_{n-1}^*(x) + X^{b_n}P_{n-2}^*(x),$$

which yields

$$P_n^* = vw_{n-1}^{b_n} w_{n-2} \underline{w_{n-1}^{a_n-b_n}} + \underbrace{00\ldots\ldots\ldots00}_{b_1+b_n q_{n-1}+q_{n-2}} \Delta_{n-3}.$$

Since $\beta_{n-3} \le q_{n-3} < (a_n - b_n)q_{n-1}$ from Lemma 2(4) with $\Gamma_{n-2} \in \mathcal{C}_1$, using Lemma 3(2) with $\Gamma_{n-3} \in \mathcal{O}$ we obtain

$$\begin{aligned} w_n &= w_{n-1}^{b_n}(w_{n-2}w_{n-3} + \underbrace{0\ldots0}_{q_{n-2}} \Delta_{n-3}) w_{n-2}^{a_{n-1}} w_{n-1}^{a_n-b_n-1} \\ &= w_{n-1}^{b_n} w_{n-3} w_{n-2} w_{n-2}^{a_{n-1}} w_{n-1}^{a_n-b_n-1} \\ &= w_{n-1}^{b_n+1} w_{n-2} w_{n-1}^{a_n-b_n-1} \end{aligned}$$

and the assertion of Lemma 2(2) is satisfied.

The conclusion of Lemma 3(3) holds in this case, because by Lemma 3(5) with $\Gamma_{n-2} \in \mathcal{C}_1$ we get

$$\begin{aligned} &-w_n w_{n-1} + w_{n-1} w_n \\ &\quad = -w_{n-1}^{b_n+1} w_{n-2} w_{n-1}^{a_n-b_n-1} w_{n-1} + w_{n-1} w_{n-1}^{b_n+1} w_{n-2} w_{n-1}^{a_n-b_n-1} \\ &\quad = \underbrace{0\ldots\ldots\ldots0}_{(b_n+1)q_{n-1}}(w_{n-1}w_{n-2} - w_{n-2}w_{n-1}) = \underbrace{00\ldots\ldots\ldots00}_{(b_n+1)q_{n-1}+q_{n-2}} \Delta_{n-3}. \end{aligned}$$

If $a_n = b_n$, then

$$P_n^*(x) = (1 + X + \ldots + X^{b_n-1}) P_{n-1}^*(x) + X^{b_n} P_{n-2}^*(x),$$

yielding

$$P_n^* = vw_{n-1}^{b_n} \underline{w_{n-2}} + \underbrace{00\ldots\ldots\ldots00}_{b_1+b_n q_{n-1}+q_{n-2}} \Delta_{n-3}.$$

From $b_n q_{n-1} + q_{n-2} = q_n$ we obtain

$$w_n = w_{n-1}^{a_n} w_{n-2} \quad \text{and} \quad w_n'' = \text{empty}.$$

The conclusion of Lemma 3(4) holds in this case, because by Lemma 3(5) with $\Gamma_{n-2} \in \mathcal{C}_1$

$$\begin{aligned} -w_n w_{n-1} + w_{n-1} w_n &= -w_{n-1}^{b_n} w_{n-2} w_{n-1} + w_{n-1}^{b_n} w_{n-1} w_{n-2} \\ &= \underbrace{0\ldots0}_{b_n q_{n-1}} \underbrace{0\ldots0}_{q_{n-2}} \Delta_{n-3} = \underbrace{0\ldots0}_{q_n} \Delta_{n-3}. \end{aligned}$$

• $[A_{k,l}C_1B_1]$. This follows $[A_{k,l-1}A_{k,l}C_1]$ or $[B_kA_{k,0}C_1]$ in the sequence S. This case is similar to $[OC_1B_1]$.

• $[B_{k-1}C_kB_k]$. This follows $[C_kB_kC_{k+1}]$ or $[D_kB_1C_2]$ in the sequence S. Since Γ_{n-1} ends in $(-1)0^{2k-1}$, we have $w_{n-2}'' = \Delta_{n-2k-1}$, $w_{n-1} = w_{n-3}w_{n-2}^{a_{n-1}}$ and w_{n-1}'' is empty. Lemma 2(2) is obvious from Lemma 2(4) with $\Gamma_{n-2} \in \mathcal{C}_k$.

If $a_n > b_n$, we use

$$w_{n-3}w_{n-2} + \varDelta_{n-2} = w_{n-3}w_{n-2} - \underbrace{0\ldots0}_{q_{n-2}}\varDelta_{n-2k-1} = w_{n-2}w_{n-3}$$

because of Lemma 3(4) with $\varGamma_{n-3} \in \mathcal{B}_{k-1}$ and $\beta_{n-2} = q_{n-2} + \beta_{n-2k-1}$. The rest of the proof is much the same as in the case $[OC_1B_1]$ but with $\varDelta_{n-2k-1}$ instead of $\varDelta_{n-3}$.

• $[C_kD_kB_1]$. This follows $[OC_1D_1]$, $[A_{k,l}C_1D_1]$ or $[B_{k-1}C_kD_k]$ in the sequence S.

Since $\varGamma_{n-1}$ ends in $(-1)0^{2k-2}(-1)0$, we have $w''_{n-2} = \varDelta_{n-3}$, w''_{n-1} is empty and $w_{n-1} = w_{n-2}w_{n-3}w_{n-2}^{a_{n-1}-1}$. We also have $\beta_{n-3} = q_{n-3}+\beta_{n-2k-2} \le q_{n-2} + q_{n-3}$. So, the conclusion of Lemma 2(3) is satisfied.

It is clear that

$$P_n^* = vw_{n-1}^{b_n}\underline{w_{n-2}w_{n-1}^{a_n-b_n}} + \underbrace{00\ldots\ldots\ldots00}_{b_1+b_nq_{n-1}+q_{n-2}}\varDelta_{n-3}.$$

Thus, if $a_n > b_n$, since

$$w_{n-2}w_{n-3} - w_{n-3}w_{n-2} = \underbrace{0\ldots0}_{q_{n-3}}\varDelta_{n-2k-2} = -\varDelta_{n-3}$$

from Lemma 3(5) with $\varGamma_{n-3} \in \mathcal{C}_k$, we obtain

$$\begin{aligned} w_n &= w_{n-1}^{b_n}w_{n-2}(w_{n-2}w_{n-3} + \varDelta_{n-3})w_{n-2}^{a_{n-1}-1}w_{n-1}^{a_n-b_n-1} \\ &= w_{n-1}^{b_n}w_{n-2}w_{n-3}w_{n-2}w_{n-2}^{a_{n-1}-1}w_{n-1}^{a_n-b_n-1} \\ &= w_{n-1}^{b_n+1}w_{n-2}w_{n-1}^{a_n-b_n-1} \end{aligned}$$

and w''_n is empty.

The assertion of Lemma 3(3) holds in this case, because by Lemma 3(6) with $\varGamma_{n-2} \in \mathcal{D}_k$,

$$\begin{aligned} &-w_nw_{n-1} + w_{n-1}w_n \\ &\quad= -w_{n-1}^{b_n+1}w_{n-2}w_{n-1}^{a_n-b_n-1}w_{n-1} + w_{n-1}w_{n-1}^{b_n+1}w_{n-2}w_{n-1}^{a_n-b_n-1} \\ &\quad= \underbrace{00\ldots\ldots00}_{(b_n+1)q_{n-1}}(w_{n-1}w_{n-2} - w_{n-2}w_{n-1}) = \underbrace{00\ldots\ldots\ldots00}_{(b_n+1)q_{n-1}+q_{n-2}}\varDelta_{n-3}. \end{aligned}$$

When $a_n = b_n$, $w_n = w_{n-1}^{b_n}w_{n-2}$ and $w''_n = \varDelta_{n-3}$.

The assertion of Lemma 3(4) holds in this case, because by Lemma 3(6) with $\varGamma_{n-2} \in \mathcal{D}_k$,

$$\begin{aligned} -w_nw_{n-1} + w_{n-1}w_n &= -w_{n-1}^{b_n}w_{n-2}w_{n-1} + w_{n-1}^{b_n}w_{n-1}w_{n-2} \\ &= \underbrace{0\ldots0}_{b_nq_{n-1}}\underbrace{0\ldots0}_{q_{n-2}}\varDelta_{n-3} = \underbrace{0\ldots0}_{q_n}\varDelta_{n-3}. \end{aligned}$$

• $[D_kD_1B_1]$. This follows $[C_kD_kD_1]$ or $[D_kD_1D_1]$ in the sequence S.

Since Γ_{n-1} ends in $(-1)0^{2k-2}(-1)(-1)0$, $w''_{n-2} = \Delta_{n-3}$ and w''_{n-1} is empty and $w_{n-1} = w_{n-2}w_{n-3}w_{n-2}^{a_{n-1}-1}$. It is clear that $\beta_{n-3} \le q_{n-2} + q_{n-3}$. We use Lemma 3(6) with $\Gamma_{n-3} \in \mathcal{D}_k$ instead of Lemma 3(5) with $\Gamma_{n-3} \in \mathcal{C}_k$. The rest of the proof is much the same as in the case $[C_kD_kB_1]$ when $k = 1$.

4.5. *Case* $\Gamma_{n-1} \in \mathcal{A}$. From Lemma 1 the possible patterns are

$$[C_kB_kA_{k,0}], [D_kB_1A_{1,0}], [B_kA_{k,0}A_{k,1}], [A_{k,l-2}A_{k,l-1}A_{k,l}].$$

• $[C_kB_kA_{k,0}]$. This follows $[OC_1B_1]$, $[A_{k,l}C_1B_1]$ or $[B_{k-1}C_kB_k]$ in the sequence S.

Since Γ_{n-1} ends in $(-1)0^{2k-1}\pi_{n-1}$, both w''_{n-2} and w''_{n-1} are empty and $w_{n-1} = w_{n-2}^{b_{n-1}+1}w_{n-3}w_{n-2}^{a_{n-1}-b_{n-1}-1}$. Now, from Lemma 2(2) with $\Gamma_{n-3} \in \mathcal{C}_k$,

$$\begin{aligned}\beta_{n-1} &= (b_{n-1}-1)q_{n-2} + b_{n-2}q_{n-3} + \ldots + b_{n-2k}q_{n-2k-1} \\ &\quad + b_{n-2k-1}q_{n-2k-2} + q_{n-2k-3} + \beta_{n-2k-2} \\ &= b_{n-1}q_{n-2} + q_{n-3} + \beta_{n-2k-2} \\ &\le (a_{n-1}-1)q_{n-2} + q_{n-3} + q_{n-4} < q_{n-1},\end{aligned}$$

so the conclusion of Lemma 2(1) is satisfied.

If $a_n \ge b_n$, then

$$P_n^* = vw_{n-1}^{b_n}\underline{w_{n-2}w_{n-1}^{a_n-b_n}}.$$

Hence, we have the result.

The conclusion of Lemma 3(1) holds in this case, because from Lemma 3(3) with $\Gamma_{n-2} \in \mathcal{B}_k$ we have

$$\begin{aligned}&w_nw_{n-1} - w_{n-1}w_n \\ &\qquad = w_{n-1}^{b_n}w_{n-2}w_{n-1}w_{n-1}^{a_n-b_n} - w_{n-1}^{b_n}w_{n-1}w_{n-2}w_{n-1}^{a_n-b_n} \\ &\qquad = \underbrace{0\ldots0}_{b_nq_{n-1}}\underbrace{00\ldots\ldots\ldots\ldots00}_{(b_{n-1}+1)q_{n-2}+q_{n-3}}\Delta_{n-2k-2} = \underbrace{0\ldots0}_{q_{n-1}}\Delta_n.\end{aligned}$$

If $a_n = b_n - 1$, then by Lemma 3(5) with $\Gamma_{n-3} \in \mathcal{C}_k$,

$$w_{n-3}w_{n-2} - w_{n-2}w_{n-3} = \underbrace{0\ldots0}_{q_{n-3}}\ \underbrace{0\ldots0}_{\beta_{n-2k}-2}(-1)^n(-1)^{n-1}.$$

Hence,

$$\begin{aligned}w''_n &= w_{n-2}^{b_{n-1}-1}w_{n-3}w_{n-2}w_{n-2}^{a_{n-1}-b_{n-1}} - w_{n-2}^{b_{n-1}-1}w_{n-2}w_{n-3}w_{n-2}^{a_{n-1}-b_{n-1}} \\ &\quad + \underbrace{00\ldots\ldots\ldots\ldots00}_{(b_{n-1}-1)q_{n-2}+q_{n-3}}\Delta_{n-2k-2} - \underbrace{00\ldots\ldots\ldots00}_{b_{n-1}q_{n-2}+q_{n-3}}\Delta_{n-2k-2} \\ &= \Delta_{n-1}.\end{aligned}$$

It is easy to get $w_n = w_{n-1}^{a_n}w_{n-2}$.

The conclusion of Lemma 3(2) holds in this case, because from Lemma 3(3) with $\Gamma_{n-2} \in \mathcal{B}_k$,

$$\begin{aligned} -w_n w_{n-1} + w_{n-1} w_n &= -\, w_{n-1}^{a_n} w_{n-2} w_{n-1} + w_{n-1}^{a_n} w_{n-1} w_{n-2} \\ &= -\, (\underbrace{000\ldots\ldots\ldots\ldots\ldots\ldots 000}_{a_n q_{n-1}+(b_{n-1}+1)q_{n-2}+q_{n-3}} \Delta_{n-2k-2}) \\ &= \underbrace{0\ldots 0}_{q_n} \Delta_{n-1}. \end{aligned}$$

• $[D_k B_1 A_{1,0}]$. This follows $[O C_1 B_1]$, $[A_{k,l} C_1 B_1]$ or $[B_{k-1} C_k B_k]$ in the sequence S.

Since Γ_{n-1} ends in $(-1)0^{2k-2}(-1)0\pi_{n-1}$, both w''_{n-2} and w''_{n-1} are empty and $w_{n-1} = w_{n-2}^{b_{n-1}+1} w_{n-3} w_{n-2}^{a_{n-1}-b_{n-1}-1}$. From Lemma 2(3) with $\Gamma_{n-3} \in \mathcal{D}_k$,

$$\begin{aligned} \beta_{n-1} &= b_{n-1} q_{n-2} + q_{n-3} + \beta_{n-4} \\ &\le q_{n-1} - q_{n-2} + q_{n-3} + q_{n-4} < q_{n-1}. \end{aligned}$$

We use Lemma 3(6) with $\Gamma_{n-3} \in \mathcal{D}_k$, which is the same as Lemma 3(5) with $\Gamma_{n-3} \in \mathcal{C}_k$ when $k = 1$. The rest of the proof is much the same as in the case $[C_k B_k A_{k,0}]$.

• $[B_k A_{k,0} A_{k,1}]$. This follows $[C_k B_k A_{k,0}]$ or $[D_k B_1 A_{1,0}]$ in the sequence S.

Since Γ_{n-1} ends in $(-1)0^{2k-1}\pi_{n-2}\varpi_{n-1}$, both w''_{n-2} and w''_{n-1} are empty and $w_{n-1} = w_{n-2}^{b_{n-1}} w_{n-3}$. Moreover, $\beta_{n-1} = (b_{n-1}-1)q_{n-2} + \beta_{n-2} + q_{n-3} \le q_{n-1} - q_{n-2} + \beta_{n-2} \le q_{n-1}$. So the conclusion of Lemma 2(1) is satisfied again.

If $a_n \ge b_n$, then it is clear that

$$w_n = w_{n-1}^{b_n} w_{n-2} w_{n-1}^{a_n-b_n} \quad \text{and} \quad w''_n \text{ is empty.}$$

The assertion of Lemma 3(1) holds in this case because by Lemma 3(1) with $\Gamma_{n-2} \in \mathcal{A}$ in the previous case,

$$\begin{aligned} w_n w_{n-1} - w_{n-1} w_n &= w_{n-1}^{b_n} w_{n-2} w_{n-1} w_{n-1}^{a_n-b_n} - w_{n-1}^{b_n} w_{n-1} w_{n-2} w_{n-1}^{a_n-b_n} \\ &= \underbrace{0\ldots 0}_{b_n q_{n-1}} \underbrace{00\ldots\ldots\ldots 00}_{q_{n-2}+\beta_{n-1}-2} (-1)^{n-1}(-1)^n = \underbrace{0\ldots 0}_{q_{n-1}} \Delta_n. \end{aligned}$$

If $a_n < b_n$, by using Lemma 3(3) with $\Gamma_{n-3} \in \mathcal{B}_k$ and $\beta_{n-1} = (b_{n-1}-1)q_{n-2} + (b_{n-2}+1)q_{n-3} + q_{n-4} + \beta_{n-2k-3}$ we obtain

$$\begin{aligned} P_n^* &= v w_{n-1}^{a_n} \underline{w_{n-2}} (w_{n-2}^{b_{n-1}-1} w_{n-3} w_{n-2} - w_{n-2}^{b_{n-1}-1} w_{n-2} w_{n-3}) \\ &= v w_{n-1}^{a_n} \underline{w_{n-2}} \underbrace{00\ldots\ldots\ldots 00}_{(b_{n-1}-1)q_{n-2}} \underbrace{00\ldots\ldots\ldots\ldots 00}_{(b_{n-2}+1)q_{n-3}+q_{n-4}} \Delta_{n-2k-3} \\ &= v w_{n-1}^{a_n} \underline{w_{n-2}} \Delta_{n-1}. \end{aligned}$$

The conclusion of Lemma 3(2) holds in this case because by Lemma 3(1) with $\Gamma_{n-2} \in \mathcal{A}$,

$$\begin{aligned} -w_n w_{n-1} + w_{n-1} w_n &= -w_{n-1}^{a_n} w_{n-2} w_{n-1} + w_{n-1}^{a_n} w_{n-1} w_{n-2} \\ &= \underbrace{0\ldots0}_{a_n q_{n-1}} \underbrace{0\ldots0}_{q_{n-2}} \Delta_{n-1} = \underbrace{0\ldots0}_{q_n} \Delta_{n-1}. \end{aligned}$$

• $[A_{k,l-2}A_{k,l-1}A_{k,l}]$. This follows $[A_{k,l-3}A_{k,l-2}A_{k,l-1}]$ in the sequence S. If $a_n < b_n$, we use Lemma 3(1) with $\Gamma_{n-3} \in \mathcal{A}$ and $\beta_{n-1} = (b_{n-1}-1)q_{n-2} + \beta_{n-2} + q_{n-3}$. The rest of the proof is much the same as in the case $[B_k A_{k,0} A_{k,1}]$.

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