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Almost all short intervals containing prime numbers

by

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1. Introduction. In 1937, Cramér [1] conjectured that every interval $(n, n+f(n)\log^2 n)$ contains a prime for some $f(n)\to 1$ as $n\to\infty$.

In 1943, assuming the Riemann Hypothesis, Selberg [19] showed that, for almost all n, the interval $(n, n+f(n)\log^2 n)$ contains a prime providing $f(n)\to\infty$ as $n\to\infty$. In the same paper, he also showed that, for almost all n, the interval $(n, n+n^{\frac{19}{77}+\varepsilon})$ contains a prime.

In 1971, Montgomery [16] improved the exponent $\frac{19}{77}$ to $\frac{1}{5}$. The zero density estimate of Huxley [7] gives the exponent $\frac{1}{6}$.

In 1982, Harman [3] used the sieve method to prove that, for almost all n, the interval $(n, n+n^{\frac{1}{10}+\varepsilon})$ contains a prime. Heath-Brown [5] and Harman [4] mentioned that the exponent $\frac{1}{12}$ can be achieved.

In [11], Jia Chaohua investigated the problem of the exceptional set of Goldbach numbers in the short interval. As a by-product, he proved that, for almost all n, the interval $(n, n+n^{\frac{1}{13}+\varepsilon})$ contains prime numbers. Li Hongze [14] improved the exponent $\frac{1}{13}$ to $\frac{2}{27}$.

Recently, Jia Chaohua [12] showed that, for almost all n, the interval $(n, n+n^{\frac{1}{14}+\varepsilon})$ contains prime numbers. Watt [20] also obtained the same result. Their methods are different. In [12] only classical methods are used and in [20] a new mean value estimate of Watt [21] is used in addition. Li Hongze [15] combined these methods to improve the exponent $\frac{1}{14}$ to $\frac{1}{15}$.

In this paper, we prove the following:

Theorem. *Suppose that B is a sufficiently large positive constant, ε is a sufficiently small positive constant and X is sufficiently large. Then for positive integers $n\in(X,2X)$, except for $O(X\log^{-B}X)$ values, the interval $(n, n+n^{\frac{1}{20}+\varepsilon})$ contains at least $0.005n^{\frac{1}{20}+\varepsilon}\log^{-1}n$ prime numbers.*

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We apply a mean value estimate of Deshouillers and Iwaniec [2] (see Lemma 2). Using the classical mean value estimate instead of that of Deshouillers and Iwaniec, we can get the exponent $\frac{1}{18}$. We refer to [13] and the explanation in [11].

Throughout this paper, we always suppose that B is a sufficiently large positive constant, ε is a sufficiently small positive constant and $\varepsilon_1 = \varepsilon^2$, $\delta = \varepsilon^{\frac{1}{3}}$. We also suppose that X is sufficiently large and that $x \in (X, 2X)$, $\eta = \frac{1}{2}X^{-\frac{19}{20}+\varepsilon}$. Let c, c_1 and c_2 denote positive constants which have different values at different places. $m \sim M$ means that there are positive constants c_1 and c_2 such that $c_1 M < m \le c_2 M$. We often use $M(s)$ (M may be another capital letter) to denote a Dirichlet polynomial of the form

$$M(s) = \sum_{m \sim M} \frac{a(m)}{m^s},$$

where $a(m)$ is a complex number with $a(m) = O(1)$.

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2. Mean value estimate (I)

LEMMA 1. *Suppose that* $X^\delta \ll H \ll X^{\frac{9}{85}}$, $MH = X$, $M(s)$ *is a Dirichlet polynomial and*

$$H(s) = \sum_{h \sim H} \frac{\Lambda(h)}{h^s}.$$

Let $b = 1 + 1/\log X$, $T_0 = \log^{\frac{B}{\varepsilon}} X$. *Then for* $T_0 \le T \le X$, *we have*

$$\min^2\left(\eta, \frac{1}{T}\right) \int_T^{2T} |M(b+it)H(b+it)|^2\, dt \ll \eta^2 \log^{-10B} x.$$

Proof. Let $s = b + it$. By the zero-free region of the ζ function, for $|t| \le 2X$ we have

$$\sum_{c_1 H < h \le c_2 H} \frac{\Lambda(h)}{h^s} = \frac{(c_2 H)^{1-s} - (c_1 H)^{1-s}}{1-s} + O(\log^{-\frac{2B}{\varepsilon}} x). \tag{1}$$

So, for $T_0 \le |t| \le 2X$,

$$H(s) \ll \log^{-\frac{B}{\varepsilon}} x. \tag{2}$$

According to the discussion in [6], there are $O(\log^2 X)$ sets $S(V, W)$, where $S(V, W)$ is the set of t_k ($k = 1, \ldots, K$) with the property $|t_r - t_s| \ge 1$ ($r \ne s$). Moreover,

$$V \le M^{\frac{1}{2}}|M(b+it_k)| < 2V, \quad W \le H^{\frac{1}{2}}|H(b+it_k)| < 2W,$$

where $X^{-1} \le M^{-\frac{1}{2}}V$, $X^{-1} \le H^{-\frac{1}{2}}W$ and $V \ll M^{\frac{1}{2}}$, $W \ll H^{\frac{1}{2}} \log^{-\frac{B}{\varepsilon}} x$. Then

$$\int_T^{2T} |M(b+it)H(b+it)|^2\,dt \ll V^2W^2x^{-1}\log^2 x|S(V,W)| + O(x^{-2\varepsilon_1}), \tag{3}$$

where $S(V,W)$ is one of sets with the above properties.

Assume $X^{\frac{1}{k+1}} \le H < X^{\frac{1}{k}}$, where k is a positive integer, $k \ge 9$ and $k\delta \ll 1$. Applying the mean value estimate (see Section 3 of [9] or Lemma 7 of [11]) to $M(s)$ and $H^k(s)$, we have

$$|S(V,W)| \ll V^{-2}(M+T)\log^d x,$$
$$|S(V,W)| \ll W^{-2k}(H^k+T)\log^d x,$$

where $d = c/\delta^2$. Applying the Halász method (see Section 3 of [9] or Lemma 7 of [11]) to $M(s)$ and $H^k(s)$, we have

$$|S(V,W)| \ll (V^{-2}M + V^{-6}MT)\log^d x,$$
$$|S(V,W)| \ll (W^{-2k}H^k + W^{-6k}H^kT)\log^d x.$$

Thus,

$$V^2W^2|S(V,W)| \ll V^2W^2F\log^d x,$$

where

$$F = \min\{V^{-2}(M+T), V^{-2}M + V^{-6}MT, W^{-2k}(H^k+T), W^{-2k}H^k + W^{-6k}H^kT\}.$$

It will be proved that

$$\min{}^2\left(\eta, \frac{1}{T}\right)V^2W^2F \ll \eta^2 x\log^{-\frac{B}{\varepsilon}} x. \tag{4}$$

We consider four cases.

(a) $F \le 2V^{-2}M$, $2W^{-2k}H^k$. Then

$$\begin{aligned} V^2W^2F &\ll V^2W^2\min\{V^{-2}M, W^{-2k}H^k\} \\ &\le V^2W^2(V^{-2}M)^{1-\frac{1}{2k}}(W^{-2k}H^k)^{\frac{1}{2k}} \\ &= V^{\frac{1}{k}}WM^{1-\frac{1}{2k}}H^{\frac{1}{2}} \ll x\log^{-\frac{B}{\varepsilon}} x \end{aligned}$$

and so

$$\min{}^2\left(\eta, \frac{1}{T}\right)V^2W^2F \ll \eta^2 x\log^{-\frac{B}{\varepsilon}} x.$$

(b) $F > 2V^{-2}M, 2W^{-2k}H^k$. Then

$$\begin{aligned} V^2W^2F &\ll V^2W^2\min\{V^{-2}T, V^{-6}MT, W^{-2k}T, W^{-6k}H^kT\} \\ &\le V^2W^2(V^{-2})^{1-\frac{3}{2k}}(V^{-6}M)^{\frac{1}{2k}}(W^{-2k})^{\frac{1}{k}}T = M^{\frac{1}{2k}}T. \end{aligned}$$

Since $k \geq 9$, we have $H \geq X^{\frac{1}{k+1}} \geq X^{1-\frac{k}{10}}$, $M^{\frac{1}{2k}} \ll X^{\frac{1}{20}}$, and so

$$\min^2\left(\eta, \frac{1}{T}\right) V^2W^2F \ll \frac{\eta}{T} x^{\frac{1}{20}} T \ll \eta^2 x^{1-\varepsilon_1}.$$

(c) $F \leq 2V^{-2}M, F > 2W^{-2k}H^k$. Then

$$\begin{aligned} V^2W^2F &\ll V^2W^2 \min\{V^{-2}M, W^{-2k}T, W^{-6k}H^kT\} \\ &\leq V^2W^2(V^{-2}M)^{1-\frac{1}{3k}}(W^{-6k}H^kT)^{\frac{1}{3k}} \\ &\ll MH^{\frac{1}{3}}T^{\frac{1}{3k}}, \end{aligned}$$

since $V \ll M^{\frac{1}{2}}$. As $H \geq X^{\frac{1}{k+1}} \geq X^{\frac{19}{20}\cdot\frac{1}{2k}+\varepsilon}$, we have

$$\min^2\left(\eta, \frac{1}{T}\right) V^2W^2F \ll \eta^{2-\frac{1}{3k}} T^{-\frac{1}{3k}} x^{1-\frac{19}{20}\cdot\frac{1}{3k}} T^{\frac{1}{3k}} x^{-\varepsilon_1} \ll \eta^2 x^{1-\varepsilon_1}.$$

(d) $F > 2V^{-2}M$, $F \leq 2W^{-2k}H^k$. Then

$$\begin{aligned} V^2W^2F &\ll V^2W^2 \min\{V^{-2}T, V^{-6}MT, W^{-2k}H^k\} \\ &\leq V^2W^2(V^{-2}T)^{1-\frac{3}{2k}}(V^{-6}MT)^{\frac{1}{2k}}(W^{-2k}H^k)^{\frac{1}{k}} \\ &= M^{\frac{1}{2k}}HT^{1-\frac{1}{k}}. \end{aligned}$$

If $k \geq 10$, then $H \leq X^{\frac{1}{k}} \leq X^{1-\frac{19(k-1)}{10(2k-1)}}$, $M \gg X^{\frac{19(k-1)}{10(2k-1)}}$, and so

$$\min^2\left(\eta, \frac{1}{T}\right) V^2W^2F \ll \eta^{1+\frac{1}{k}} x^{1-\frac{19}{20}(1-\frac{1}{k})} \ll \eta^2 x^{1-\varepsilon_1}.$$

If $k = 9$, then $X^{\frac{1}{10}} \leq H \ll X^{\frac{9}{85}}$, $M \gg X^{\frac{76}{85}}$, and so

$$\min^2\left(\eta, \frac{1}{T}\right) V^2W^2F \ll \eta^2 (x^{-\frac{19}{20}+\varepsilon})^{-\frac{8}{9}} x^{1-\frac{38}{45}} \ll \eta^2 x^{1-\varepsilon_1}.$$

Combining the above, we obtain (4). Hence, Lemma 1 follows.

LEMMA 2. *Suppose that $N(s)$ is a Dirichlet polynomial and*

$$L(s) = \sum_{c_1L<l\leq c_2L} \frac{1}{l^s}.$$

Let $T \geq 1$. Then

$$\begin{aligned} I &= \int_T^{2T} \left|L\left(\frac{1}{2}+it\right)\right|^4 \left|N\left(\frac{1}{2}+it\right)\right|^2 dt \\ &\ll \left(T + N^2T^{\frac{1}{2}} + N^{\frac{5}{4}}\left(T\min\left(L, \frac{T}{L}\right)\right)^{\frac{1}{2}} + NL^2T^{-2}\right)T^{\varepsilon_1}. \end{aligned}$$

Proof. First we assume $c_1L \le T^{\frac{1}{2}}$. If $N \le T$, then by the discussion in Section 2 of [2], we have

$$I \ll (T + N^2T^{\frac{1}{2}} + N^{\frac{5}{4}}(TL)^{\frac{1}{2}})T^{\varepsilon_1}.$$

If $N > T$, the mean value estimate yields

$$I \ll (L^2N + T)(LN)^{\varepsilon_1} \ll (T + N^2T^{\frac{1}{2}})T^{\varepsilon_1}.$$

Now we assume $T^{\frac{1}{2}} < c_1L \le 2T$. By the discussion in Section 2 of [2], we get

$$I \ll \left(T + N^2T^{\frac{1}{2}} + N^{\frac{5}{4}}\left(T \cdot \frac{T}{L}\right)^{\frac{1}{2}}\right)T^{\varepsilon_1}.$$

Lastly we assume $2T < c_1L$. It follows from Theorem 1 on page 442 of [18] that

$$\sum_{c_1L<l\le c_2L} \frac{1}{l^{\frac{1}{2}+it}} = \frac{(c_2L)^{\frac{1}{2}-it} - (c_1L)^{\frac{1}{2}-it}}{\frac{1}{2} - it} + O\left(\frac{1}{L^{\frac{1}{2}}}\right).$$

Hence,

$$L\left(\frac{1}{2} + it\right) \ll \frac{L^{\frac{1}{2}}}{|t|} + \frac{1}{L^{\frac{1}{2}}} \ll \frac{L^{\frac{1}{2}}}{|t|}$$

and

$$I \ll \frac{L^2}{T^4}\int_T^{2T}\left|N\left(\frac{1}{2} + it\right)\right|^2 dt \ll \frac{L^2}{T^4}(N + T) \ll \frac{NL^2}{T^2}.$$

Combining the above, we get Lemma 2.

LEMMA 3. *Suppose that $MNL = X$, $M(s)$, $N(s)$ are Dirichlet polynomials, and*

$$L(s) = \sum_{l\sim L} \frac{1}{l^s}.$$

Let $b = 1 + 1/\log X$, $T_1 = \sqrt{L}$. Assume further that M and N lie in one of the following regions:

$$(5)\qquad \begin{array}{lll} \text{(i)} & M \ll X^{\frac{9}{32}}, & N \ll M^{\frac{2}{3}}X^{\frac{1}{10}};\\ \text{(ii)} & X^{\frac{9}{32}} \ll M \ll X^{\frac{53}{160}}, & N \ll X^{\frac{23}{80}};\\ \text{(iii)} & X^{\frac{53}{160}} \ll M \ll X^{\frac{13}{32}}, & N \ll M^{-\frac{2}{3}}X^{\frac{61}{120}};\\ \text{(iv)} & X^{\frac{13}{32}} \ll M \ll X^{\frac{321}{680}}, & N \ll M^{-2}X^{\frac{21}{20}}. \end{array}$$

Then for $T_1 \le T \le X$, we have

$$\min{}^2\left(\eta, \frac{1}{T}\right)\int_T^{2T} |M(b + it)N(b + it)L(b + it)|^2\, dt \ll \eta^2x^{-2\varepsilon_1}.$$

P r o o f. It is sufficient to show that

$$I=\min^2\left(\eta,\frac{1}{T}\right)\int\limits_T^{2T}\left|M\left(\frac{1}{2}+it\right)N\left(\frac{1}{2}+it\right)L\left(\frac{1}{2}+it\right)\right|^2dt\ll\eta^2x^{1-2\varepsilon_1}.$$

We shall show that the above inequality holds, providing M and N satisfy the following conditions:

$$M^2N\ll X^{\frac{21}{20}},\quad M^2N^3\ll X^{\frac{61}{40}},\quad M^6N^7\ll X^{\frac{41}{10}},$$
$$N\ll X^{\frac{23}{80}},\quad N^3\ll M^2X^{\frac{3}{10}}.$$

Using the mean value estimate and Lemma 2, we have

$$\begin{aligned}\int\limits_T^{2T}&\left|M\left(\frac{1}{2}+it\right)N\left(\frac{1}{2}+it\right)L\left(\frac{1}{2}+it\right)\right|^2dt\\ &\ll\left(\int\limits_T^{2T}\left|M\left(\frac{1}{2}+it\right)\right|^4\left|N\left(\frac{1}{2}+it\right)\right|^2dt\right)^{\frac{1}{2}}\\ &\quad\times\left(\int\limits_T^{2T}\left|L\left(\frac{1}{2}+it\right)\right|^4\left|N\left(\frac{1}{2}+it\right)\right|^2dt\right)^{\frac{1}{2}}\\ &\ll(M^2N+T)^{\frac{1}{2}}(T+N^2T^{\frac{1}{2}}+N^{\frac{5}{4}}(TL)^{\frac{1}{2}}+NL^2T^{-2})^{\frac{1}{2}}T^{\varepsilon_1}.\end{aligned}$$

Hence,

$$\begin{aligned}I&\ll\min^2\left(\eta,\frac{1}{T}\right)(M^2N+T)^{\frac{1}{2}}(T+N^2T^{\frac{1}{2}}+N^{\frac{5}{4}}(TL)^{\frac{1}{2}})^{\frac{1}{2}}T^{\varepsilon_1}+\eta^2T_1^{-\frac{1}{2}}x^{1+\varepsilon_1}\\ &\ll\eta^2(M^2N+\eta^{-1})^{\frac{1}{2}}(\eta^{-1}+N^2\eta^{-\frac{1}{2}}+N^{\frac{5}{4}}(\eta^{-1}L)^{\frac{1}{2}})^{\frac{1}{2}}x^{\varepsilon_1}+\eta^2x^{1-2\varepsilon_1}\\ &\ll\eta^2x^{1-2\varepsilon_1}.\end{aligned}$$

In every region of (5), our conditions are satisfied. So Lemma 3 follows.

Lemma 4. *Under the assumptions of Lemma* 3, (5) *being replaced by the region*

$$M\ll X^{\frac{21}{40}},\quad N\ll X^{\frac{19}{160}},\tag{6}$$

for $T_1\le T\le X$, *we have*

$$\min^2\left(\eta,\frac{1}{T}\right)\int\limits_T^{2T}|M(b+it)N(b+it)L(b+it)|^2\,dt\ll\eta^2x^{-2\varepsilon_1}.$$

P r o o f. It is sufficient to show that

$$I=\min^2\left(\eta,\frac{1}{T}\right)\int\limits_T^{2T}\left|M\left(\frac{1}{2}+it\right)N\left(\frac{1}{2}+it\right)L\left(\frac{1}{2}+it\right)\right|^2dt\ll\eta^2x^{1-2\varepsilon_1}.$$

Using the mean value estimate and Lemma 2, we have

$$
\begin{aligned}
I &\ll \min{}^2\Big(\eta, \frac{1}{T}\Big)\Big(\int\limits_T^{2T}\Big|M\Big(\frac{1}{2}+it\Big)\Big|^4\,dt\Big)^{\frac{1}{2}} \\
&\quad\times\Big(\int\limits_T^{2T}\Big|L\Big(\frac{1}{2}+it\Big)\Big|^4\Big|N\Big(\frac{1}{2}+it\Big)\Big|^4\,dt\Big)^{\frac{1}{2}} \\
&\ll \min{}^2\Big(\eta, \frac{1}{T}\Big)(M^2+T)^{\frac{1}{2}}\Big(T+N^4T^{\frac{1}{2}} \\
&\quad + N^{\frac{5}{2}}\Big(T\min\Big(L, \frac{T}{L}\Big)\Big)^{\frac{1}{2}} + N^2L^2T^{-2}\Big)^{\frac{1}{2}}T^{\varepsilon_1} \\
&\ll \eta^2(M^2+\eta^{-1})^{\frac{1}{2}}(\eta^{-1}+N^4\eta^{-\frac{1}{2}})^{\frac{1}{2}}x^{\varepsilon_1} + \eta^2MN^{\frac{5}{4}}(\eta^{-1}L)^{\frac{1}{4}}x^{\varepsilon_1} \\
&\quad + N^{\frac{5}{4}}\eta^{2-\frac{7}{8}}x^{\varepsilon_1} + \eta^2T_1^{-\frac{1}{2}}x^{1+\varepsilon_1} \\
&\ll \eta^2x^{1-2\varepsilon_1},
\end{aligned}
$$

since $\min(L, T/L) \le T^{\frac{1}{2}}$. Thus Lemma 4 follows.

3. Mean value estimate (II)

LEMMA 5. *Suppose that* $MHK = X$ *and* $M(s)$, $H(s)$ *and* $K(s)$ *are Dirichlet polynomials*, *and* $G(s) = M(s)H(s)K(s)$. *Let* $b = 1 + 1/\log X$, $T_0 = \log^{\frac{B}{\varepsilon}} X$. *Assume further that for* $T_0 \le |t| \le 2X$, $M(b+it) \ll \log^{-\frac{B}{\varepsilon}} x$ *and* $H(b+it) \ll \log^{-\frac{B}{\varepsilon}} x$. *Moreover*, *suppose that* M *and* H *satisfy one of the following three conditions*:

1) $MH \ll X^{\frac{157}{290}}$, $X^{\frac{19}{110}} \ll H$, $M^{29}/H \ll X^{10}$, $X^{\frac{3}{10}} \ll M$, $H^{29}/M \ll X^{\frac{31}{5}}$, $X^{\frac{57}{10}} \ll M^{12}H^{11}$;

2) $MH \ll X^{\frac{26}{45}}$, $M^{29}H^{19} \ll X^{\frac{59}{4}}$, $X^{\frac{38}{45}} \ll M^2H$, $M^2H^{11} \ll X^{\frac{29}{10}}$, $X^{\frac{57}{2}} \ll M^{58}H^{49}$;

3) $MH \ll X^{\frac{25}{44}}$, $X^{\frac{19}{100}} \ll H$, $M^6H \ll X^{\frac{11}{5}}$, $X^{\frac{19}{70}} \ll M$, $MH^8 \ll X^{\frac{23}{10}}$, $X^{\frac{57}{20}} \ll M^6H^5$.

Then for $T_0 \le T \le X$, *we have*

$$
\min{}^2\Big(\eta, \frac{1}{T}\Big)\int\limits_T^{2T}|G(b+it)|^2\,dt \ll \eta^2\log^{-10B} x. \tag{7}
$$

P r o o f. Using the method of Lemma 1, we only show that for $T = 1/\eta = 2X^{\frac{19}{20}-\varepsilon}$,

$$
I = \int\limits_T^{2T}|G(b+it)|^2\,dt \ll \log^{-10B} x. \tag{8}
$$

I. First, we assume condition 1). On applying the mean value estimate and Halász method to $M^3(s)$, $H^5(s)$ and $K^2(s)$, we get

$$I \ll U^2V^2W^2x^{-1}F\log^c x,$$

where

$$F = \min\{V^{-6}(M^3+T), V^{-6}M^3+V^{-18}M^3T, W^{-10}(H^5+T), W^{-10}H^5+W^{-30}H^5T, U^{-4}(K^2+T), U^{-4}K^2+U^{-12}K^2T\}.$$

We discuss the following cases:

(a) $F \le 2V^{-6}M^3, 2W^{-10}H^5, 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-6}M^3, W^{-10}H^5, U^{-4}K^2\} \\ &\le U^2V^2W^2(V^{-6}M^3)^{\frac{3}{10}}(W^{-10}H^5)^{\frac{1}{5}}(U^{-4}K^2)^{\frac{1}{2}} \\ &= V^{\frac{1}{5}}M^{\frac{9}{10}}HK \ll x\log^{-11B}x. \end{aligned}$$

(b) $F \le 2V^{-6}M^3, 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-6}M^3, W^{-10}H^5, U^{-4}T, U^{-12}K^2T\} \\ &\le U^2V^2W^2(V^{-6}M^3)^{\frac{1}{3}}(W^{-10}H^5)^{\frac{1}{5}}(U^{-4}T)^{\frac{9}{20}}(U^{-12}K^2T)^{\frac{1}{60}} \\ &= T^{\frac{7}{15}}MHK^{\frac{1}{30}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(c) $F \le 2V^{-6}M^3, F > 2W^{-10}H^5, F \le 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-6}M^3, W^{-10}T, W^{-30}H^5T, U^{-4}K^2\} \\ &\le U^2V^2W^2(V^{-6}M^3)^{\frac{1}{3}}(W^{-10}T)^{\frac{3}{20}}(W^{-30}H^5T)^{\frac{1}{60}}(U^{-4}K^2)^{\frac{1}{2}} \\ &= T^{\frac{1}{6}}MKH^{\frac{1}{12}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(d) $F \le 2V^{-6}M^3, F > 2W^{-10}H^5, 2U^{-4}K^2$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\quad\ll U^2V^2W^2\min\{V^{-6}M^3, W^{-10}T, W^{-30}H^5T, U^{-4}T, U^{-12}K^2T\} \\ &\quad\le U^2V^2W^2(V^{-6}M^3)^{\frac{1}{3}}(W^{-10}T)^{\frac{1}{5}}(U^{-4}T)^{\frac{9}{20}}(U^{-12}K^2T)^{\frac{1}{60}} \\ &\quad= T^{\frac{2}{3}}MK^{\frac{1}{30}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(e) $F > 2V^{-6}M^3, F \le 2W^{-10}H^5, 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-6}T, V^{-18}M^3T, W^{-10}H^5, U^{-4}K^2\} \\ &\le U^2V^2W^2(V^{-6}T)^{\frac{17}{60}}(V^{-18}M^3T)^{\frac{1}{60}}(W^{-10}H^5)^{\frac{1}{5}}(U^{-4}K^2)^{\frac{1}{2}} \\ &= T^{\frac{3}{10}}M^{\frac{1}{20}}HK \ll x^{1-\varepsilon_1}. \end{aligned}$$

(f) $F > 2V^{-6}M^3, F \le 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\quad \ll U^2V^2W^2\min\{V^{-6}T, V^{-18}M^3T, W^{-10}H^5, U^{-4}T, U^{-12}K^2T\}\\
&\quad \le U^2V^2W^2(V^{-6}T)^{\frac13}(W^{-10}H^5)^{\frac15}(U^{-4}T)^{\frac{9}{20}}(U^{-12}K^2T)^{\frac{1}{60}}\\
&\quad = T^{\frac45}HK^{\frac{1}{30}} \ll x^{1-\varepsilon_1}.
\end{aligned}$$

(g) $F > 2V^{-6}M^3, 2W^{-10}H^5, F \le 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\quad \ll U^2V^2W^2\min\{V^{-6}T, V^{-18}M^3T, W^{-10}T, W^{-30}H^5T, U^{-4}K^2\}\\
&\quad \le U^2V^2W^2(V^{-6}T)^{\frac13}(W^{-10}T)^{\frac{3}{20}}(W^{-30}H^5T)^{\frac{1}{60}}(U^{-4}K^2)^{\frac12}\\
&\quad = T^{\frac12}H^{\frac{1}{12}}K \ll x^{1-\varepsilon_1}.
\end{aligned}$$

(h) $F > 2V^{-6}M^3, 2W^{-10}H^5, 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\quad \ll U^2V^2W^2\min\{V^{-6}, V^{-18}M^3, W^{-10}, W^{-30}H^5, U^{-4}, U^{-12}K^2\}T\\
&\quad \le U^2V^2W^2(V^{-6})^{\frac{17}{60}}(V^{-18}M^3)^{\frac{1}{60}}(W^{-10})^{\frac15}(U^{-4})^{\frac12}T\\
&\quad = TM^{\frac{1}{20}} \ll x^{1-\varepsilon_1},
\end{aligned}$$

since $M \ll X$.

II. Next, we assume condition 2). On applying the mean value estimate and Halász method to $M^2(s)H(s)$, $H^5(s)$ and $K^2(s)$, we get

$$I \ll U^2V^2W^2x^{-1}F\log^c x,$$

where

$$\begin{aligned}
F = \min\{&V^{-4}W^{-2}(M^2H+T), V^{-4}W^{-2}M^2H + V^{-12}W^{-6}M^2HT,\\
&W^{-10}(H^5+T), W^{-10}H^5 + W^{-30}H^5T, U^{-4}(K^2+T),\\
&U^{-4}K^2 + U^{-12}K^2T\}.
\end{aligned}$$

We consider several cases:

(a) $F \le 2V^{-4}W^{-2}M^2H, 2W^{-10}H^5, 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}W^{-2}M^2H, W^{-10}H^5, U^{-4}K^2\}\\
&\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac38}(W^{-10}H^5)^{\frac18}(U^{-4}K^2)^{\frac12}\\
&= V^{\frac12}M^{\frac34}HK \ll x\log^{-11B} x.
\end{aligned}$$

(b) $F \le 2V^{-4}W^{-2}M^2H, 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\qquad \ll U^2V^2W^2\min\{V^{-4}W^{-2}M^2H, W^{-10}H^5, U^{-4}T, U^{-12}K^2T\}\\
&\qquad \le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(W^{-10}H^5)^{\frac{1}{10}}(U^{-4}T)^{\frac{7}{20}}(U^{-12}K^2T)^{\frac{1}{20}}\\
&\qquad = T^{\frac{2}{5}}MHK^{\frac{1}{10}} \ll x^{1-\varepsilon_1}.
\end{aligned}$$

(c) $F \le 2V^{-4}W^{-2}M^2H, F > 2W^{-10}H^5, F \le 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}W^{-2}M^2H, W^{-10}T, W^{-30}H^5T, U^{-4}K^2\}\\
&\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(U^{-4}K^2)^{\frac{1}{2}}\\
&= WH^{\frac{1}{2}}MK \ll x\log^{-11B}x.
\end{aligned}$$

(d) $F \le 2V^{-4}W^{-2}M^2H, F > 2W^{-10}H^5, 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\quad \ll U^2V^2W^2\min\{V^{-4}W^{-2}M^2H, W^{-10}T, W^{-30}H^5T, U^{-4}T, U^{-12}K^2T\}\\
&\quad \le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(W^{-30}H^5T)^{\frac{1}{30}}(U^{-4}T)^{\frac{9}{20}}(U^{-12}K^2T)^{\frac{1}{60}}\\
&\quad = T^{\frac{1}{2}}MH^{\frac{2}{3}}K^{\frac{1}{30}} \ll x^{1-\varepsilon_1}.
\end{aligned}$$

(e) $F > 2V^{-4}W^{-2}M^2H, F \le 2W^{-10}H^5, 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\quad \ll U^2V^2W^2\min\{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-10}H^5, U^{-4}K^2\}\\
&\quad \le U^2V^2W^2(V^{-4}W^{-2}T)^{\frac{7}{20}}(V^{-12}W^{-6}M^2HT)^{\frac{1}{20}}(W^{-10}H^5)^{\frac{1}{10}}(U^{-4}K^2)^{\frac{1}{2}}\\
&\quad = T^{\frac{2}{5}}M^{\frac{1}{10}}H^{\frac{11}{20}}K \ll x^{1-\varepsilon_1}.
\end{aligned}$$

(f) $F > 2V^{-4}W^{-2}M^2H, F \le 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\quad \ll U^2V^2W^2\min\{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-10}H^5,\\
&\qquad\qquad U^{-4}T, U^{-12}K^2T\}\\
&\quad \le U^2V^2W^2(V^{-4}W^{-2}T)^{\frac{7}{20}}(V^{-12}W^{-6}M^2HT)^{\frac{1}{20}}(W^{-10}H^5)^{\frac{1}{10}}(U^{-4}T)^{\frac{1}{2}}\\
&\quad = T^{\frac{9}{10}}M^{\frac{1}{10}}H^{\frac{11}{20}} \ll x^{1-\varepsilon_1}.
\end{aligned}$$

(g) $F > 2V^{-4}W^{-2}M^2H, 2W^{-10}H^5, F \le 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F \\
&\quad \ll U^2V^2W^2 \min\{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-10}T, \\
&\qquad\qquad W^{-30}H^5T, U^{-4}K^2\} \\
&\quad \le U^2V^2W^2(V^{-4}W^{-2}T)^{\frac{9}{20}}(V^{-12}W^{-6}M^2HT)^{\frac{1}{60}}(W^{-30}H^5T)^{\frac{1}{30}}(U^{-4}K^2)^{\frac{1}{2}} \\
&\quad = T^{\frac{1}{2}}M^{\frac{1}{30}}H^{\frac{11}{60}}K \ll x^{1-\varepsilon_1}.
\end{aligned}$$

(h) $F > 2V^{-4}W^{-2}M^2H, 2W^{-10}H^5, 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F \\
&\quad \ll U^2V^2W^2 \min\{V^{-4}W^{-2}, V^{-12}W^{-6}M^2H, W^{-10}, W^{-30}H^5, \\
&\qquad\qquad U^{-4}, U^{-12}K^2\}T \\
&\quad \le U^2V^2W^2(V^{-4}W^{-2})^{\frac{9}{20}}(V^{-12}W^{-6}M^2H)^{\frac{1}{60}}(W^{-30}H^5)^{\frac{1}{30}}(U^{-4})^{\frac{1}{2}}T \\
&\quad = TM^{\frac{1}{30}}H^{\frac{11}{60}} \ll x^{1-\varepsilon_1}.
\end{aligned}$$

III. Lastly, we assume condition 3). On applying the mean value estimate and Halász method to $M^3(s)$, $H^4(s)$ and $K^2(s)$, we get

$$I \ll U^2V^2W^2x^{-1}F\log^c x,$$

where

$$\begin{aligned}
F = \min\{&V^{-6}(M^3+T), V^{-6}M^3 + V^{-18}M^3T, W^{-8}(H^4+T), W^{-8}H^4 \\
&+ W^{-24}H^4T, U^{-4}(K^2+T), U^{-4}K^2 + U^{-12}K^2T\}.
\end{aligned}$$

Consider the following cases:

(a) $F \le 2V^{-6}M^3, 2W^{-8}H^4, 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-6}M^3, W^{-8}H^4, U^{-4}K^2\} \\
&\le U^2V^2W^2(V^{-6}M^3)^{\frac{1}{4}}(W^{-8}H^4)^{\frac{1}{4}}(U^{-4}K^2)^{\frac{1}{2}} \\
&= V^{\frac{1}{2}}M^{\frac{3}{4}}HK \ll x\log^{-11B} x.
\end{aligned}$$

(b) $F \le 2V^{-6}M^3, 2W^{-8}H^4, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-6}M^3, W^{-8}H^4, U^{-4}T, U^{-12}K^2T\} \\
&\le U^2V^2W^2(V^{-6}M^3)^{\frac{1}{3}}(W^{-8}H^4)^{\frac{1}{4}}(U^{-4}T)^{\frac{3}{8}}(U^{-12}K^2T)^{\frac{1}{24}} \\
&= T^{\frac{5}{12}}MHK^{\frac{1}{12}} \ll x^{1-\varepsilon_1}.
\end{aligned}$$

(c) $F \le 2V^{-6}M^3, F > 2W^{-8}H^4, F \le 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-6}M^3, W^{-8}T, W^{-24}H^4T, U^{-4}K^2\} \\ &\le U^2V^2W^2(V^{-6}M^3)^{\frac{1}{3}}(W^{-8}T)^{\frac{1}{8}}(W^{-24}H^4T)^{\frac{1}{24}}(U^{-4}K^2)^{\frac{1}{2}} \\ &= T^{\frac{1}{6}}MKH^{\frac{1}{6}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(d) $F \le 2V^{-6}M^3, F > 2W^{-8}H^4, 2U^{-4}K^2$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\quad\ll U^2V^2W^2 \min\{V^{-6}M^3, W^{-8}T, W^{-24}H^4T, U^{-4}T, U^{-12}K^2T\} \\ &\quad\le U^2V^2W^2(V^{-6}M^3)^{\frac{1}{3}}(W^{-8}T)^{\frac{1}{8}}(W^{-24}H^4T)^{\frac{1}{24}}(U^{-4}T)^{\frac{1}{2}} \\ &\quad= T^{\frac{2}{3}}MH^{\frac{1}{6}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(e) $F > 2V^{-6}M^3, F \le 2W^{-8}H^4, 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-6}T, V^{-18}M^3T, W^{-8}H^4, U^{-4}K^2\} \\ &\le U^2V^2W^2(V^{-6}T)^{\frac{5}{24}}(V^{-18}M^3T)^{\frac{1}{24}}(W^{-8}H^4)^{\frac{1}{4}}(U^{-4}K^2)^{\frac{1}{2}} \\ &= T^{\frac{1}{4}}M^{\frac{1}{8}}HK \ll x^{1-\varepsilon_1}. \end{aligned}$$

(f) $F > 2V^{-6}M^3, F \le 2W^{-8}H^4, F > 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-6}T, V^{-18}M^3T, W^{-8}H^4, U^{-4}T, U^{-12}K^2T\} \\ &\le U^2V^2W^2(V^{-6}T)^{\frac{5}{24}}(V^{-18}M^3T)^{\frac{1}{24}}(W^{-8}H^4)^{\frac{1}{4}}(U^{-4}T)^{\frac{1}{2}} \\ &= T^{\frac{3}{4}}M^{\frac{1}{8}}H \ll x^{1-\varepsilon_1}. \end{aligned}$$

(g) $F > 2V^{-6}M^3, 2W^{-8}H^4, F \le 2U^{-4}K^2$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\quad\ll U^2V^2W^2 \min\{V^{-6}T, V^{-18}M^3T, W^{-8}T, W^{-24}H^4T, U^{-4}K^2\} \\ &\quad\le U^2V^2W^2(V^{-6}T)^{\frac{1}{3}}(W^{-8}T)^{\frac{1}{8}}(W^{-24}H^4T)^{\frac{1}{24}}(U^{-4}K^2)^{\frac{1}{2}} \\ &\quad= T^{\frac{1}{2}}H^{\frac{1}{6}}K \ll x^{1-\varepsilon_1}. \end{aligned}$$

(h) $F > 2V^{-6}M^3, 2W^{-8}H^4, 2U^{-4}K^2$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\quad\ll U^2V^2W^2 \min\{V^{-6}, V^{-18}M^3, W^{-8}, W^{-24}H^4, U^{-4}, U^{-12}K^2\}T \\ &\quad\le U^2V^2W^2(V^{-6})^{\frac{5}{24}}(V^{-18}M^3)^{\frac{1}{24}}(W^{-8})^{\frac{1}{4}}(U^{-4})^{\frac{1}{2}}T \\ &\quad= TM^{\frac{1}{8}} \ll x^{1-\varepsilon_1}, \end{aligned}$$

since $M \ll X^{\frac{2}{5}}$ (the latter follows from $MH \ll X^{\frac{25}{44}}$ and $X^{\frac{19}{100}} \ll H$).

Combining the above, we obtain (8). Hence, Lemma 5 follows.

LEMMA 6. *Under the assumption of Lemma* 5, M *and* H *lie in one of the following regions*:

(i) $X^{\frac{19}{70}} \ll M \ll X^{\frac{3}{10}}, \qquad M^{-\frac{6}{5}} X^{\frac{57}{100}} \ll H \ll M^{-\frac{1}{8}} X^{\frac{23}{80}}$;

(ii) $X^{\frac{3}{10}} \ll M \ll X^{\frac{19}{60}}, \qquad M^{-\frac{12}{11}} X^{\frac{57}{110}} \ll H \ll M^{-\frac{1}{8}} X^{\frac{23}{80}}$;

(iii) $X^{\frac{19}{60}} \ll M \ll X^{\frac{247}{770}}, \qquad X^{\frac{19}{110}} \ll H \ll M^{-\frac{1}{8}} X^{\frac{23}{80}}$;

(iv) $X^{\frac{247}{770}} \ll M \ll X^{\frac{359}{1100}}, \qquad X^{\frac{19}{110}} \ll H \ll M^{-1} X^{\frac{25}{44}}$;

(v) $X^{\frac{359}{1100}} \ll M \ll X^{\frac{481}{1450}}, \qquad X^{\frac{19}{110}} \ll H \ll M^{-6} X^{\frac{11}{5}}$;

(vi) $X^{\frac{481}{1450}} \ll M \ll X^{\frac{443}{1305}}, \qquad X^{\frac{19}{110}} \ll H \ll M^{-1} X^{\frac{157}{290}}$;

(vii) $X^{\frac{443}{1305}} \ll M \ll X^{\frac{19}{55}}, \qquad X^{\frac{19}{110}} \ll H \ll M^{-\frac{2}{11}} X^{\frac{29}{110}}$;

(viii) $X^{\frac{19}{55}} \ll M \ll X^{\frac{2143}{5620}}, \qquad M^{-\frac{58}{49}} X^{\frac{57}{98}} \ll H \ll M^{-\frac{2}{11}} X^{\frac{29}{110}}$;

(ix) $X^{\frac{2143}{5620}} \ll M \ll X^{\frac{3963}{9860}}, \qquad M^{-\frac{58}{49}} X^{\frac{57}{98}} \ll H \ll M^{-\frac{29}{19}} X^{\frac{59}{76}}$;

(x) $X^{\frac{3963}{9860}} \ll M \ll X^{\frac{4331}{9860}}, \qquad X^{\frac{9}{85}} \ll H \ll M^{-\frac{29}{19}} X^{\frac{59}{76}}$.

Then (7) *holds for* $T_0 \le T \le X$.

Proof. In the regions:

$$X^{\frac{3}{10}} \ll M \ll X^{\frac{19}{60}}, \quad M^{-\frac{12}{11}} X^{\frac{57}{110}} \ll H \ll M^{\frac{1}{29}} X^{\frac{31}{145}};$$
$$X^{\frac{19}{60}} \ll M \ll X^{\frac{19}{55}}, \quad X^{\frac{19}{110}} \ll H \ll M^{-1} X^{\frac{157}{290}},$$

we apply Lemma 5 with condition 1).

In the regions:

$$X^{\frac{443}{1305}} \ll M \ll X^{\frac{19}{55}}, \quad M^{-1} X^{\frac{157}{290}} \ll H \ll M^{-\frac{2}{11}} X^{\frac{29}{110}};$$
$$X^{\frac{19}{55}} \ll M \ll X^{\frac{2143}{5620}}, \quad M^{-\frac{58}{49}} X^{\frac{57}{98}} \ll H \ll M^{-\frac{2}{11}} X^{\frac{29}{110}};$$
$$X^{\frac{2143}{5620}} \ll M \ll X^{\frac{3963}{9860}}, \quad M^{-\frac{58}{49}} X^{\frac{57}{98}} \ll H \ll M^{-\frac{29}{19}} X^{\frac{59}{76}};$$
$$X^{\frac{3963}{9860}} \ll M \ll X^{\frac{4331}{9860}}, \quad X^{\frac{9}{85}} \ll H \ll M^{-\frac{29}{19}} X^{\frac{59}{76}},$$

we apply Lemma 5 with condition 2).

In the regions:

$$X^{\frac{19}{70}} \ll M \ll X^{\frac{3}{10}}, \quad M^{-\frac{6}{5}} X^{\frac{57}{100}} \ll H \ll M^{-\frac{1}{8}} X^{\frac{23}{80}};$$
$$X^{\frac{3}{10}} \ll M \ll X^{\frac{19}{60}}, \quad M^{\frac{1}{29}} X^{\frac{31}{145}} \ll H \ll M^{-\frac{1}{8}} X^{\frac{23}{80}};$$
$$X^{\frac{19}{60}} \ll M \ll X^{\frac{247}{770}}, \quad M^{-1} X^{\frac{157}{290}} \ll H \ll M^{-\frac{1}{8}} X^{\frac{23}{80}};$$
$$X^{\frac{247}{770}} \ll M \ll X^{\frac{359}{1100}}, \quad M^{-1} X^{\frac{157}{290}} \ll H \ll M^{-1} X^{\frac{25}{44}};$$
$$X^{\frac{359}{1100}} \ll M \ll X^{\frac{481}{1450}}, \quad M^{-1} X^{\frac{157}{290}} \ll H \ll M^{-6} X^{\frac{11}{5}},$$

we apply Lemma 5 with condition 3).

Putting together the above regions, we get Lemma 6.

LEMMA 7. *Under the assumption of Lemma* 5, *suppose that* M *and* H *also satisfy one of the following three conditions*:

1) $MH \ll X^{\frac{7}{10}}$, $X^{\frac{19}{110}} \ll H$, $M^{12}H \ll X^{\frac{63}{10}}$, $X^{\frac{133}{290}} \ll M$, $H^{19}/M \ll X^{\frac{19}{5}}$, $X^{\frac{38}{5}} \ll M^{12}H^{11}$;

2) $M^2H \ll X^{\frac{52}{45}}$, $M^{58}H^9 \ll X^{\frac{59}{2}}$, $X^{\frac{19}{45}} \ll M$, $MH^5 \ll X^{\frac{29}{20}}$, $X^{\frac{57}{4}} \ll M^{29}H^{10}$;

3) $MH \ll X^{\frac{51}{70}}$, $X^{\frac{19}{100}} \ll H$, $M^6H \ll X^{\frac{63}{20}}$, $X^{\frac{19}{44}} \ll M$, $H^7/M \ll X^{\frac{13}{10}}$, $X^{\frac{19}{5}} \ll M^6H^5$.

Then (7) *holds for* $T_0 \le T \le X$.

Proof. We only show that (8) holds for $T = 1/\eta = 2X^{\frac{19}{20}-\varepsilon}$.

I. First, we assume condition 1). We apply the mean value estimate and Halász method to $M^2(s)$, $H^5(s)$ and $K^3(s)$ to get

$$I \ll U^2V^2W^2x^{-1}F\log^c x,$$

where

$$F = \min\{V^{-4}(M^2+T), V^{-4}M^2 + V^{-12}M^2T, W^{-10}(H^5+T), W^{-10}H^5 + W^{-30}H^5T, U^{-6}(K^3+T), U^{-6}K^3 + U^{-18}K^3T\}.$$

We consider several cases:

(a) $F \le 2V^{-4}M^2, 2W^{-10}H^5, 2U^{-6}K^3$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-10}H^5, U^{-6}K^3\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(W^{-10}H^5)^{\frac{1}{6}}(U^{-6}K^3)^{\frac{1}{3}} \\ &= W^{\frac{1}{3}}H^{\frac{5}{6}}MK \ll x\log^{-11B} x. \end{aligned}$$

(b) $F \le 2V^{-4}M^2, 2W^{-10}H^5, F > 2U^{-6}K^3$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-10}H^5, U^{-6}T, U^{-18}K^3T\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(W^{-10}H^5)^{\frac{1}{5}}(U^{-6}T)^{\frac{17}{60}}(U^{-18}K^3T)^{\frac{1}{60}} \\ &= T^{\frac{3}{10}}MHK^{\frac{1}{20}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(c) $F \le 2V^{-4}M^2$, $F > 2W^{-10}H^5$, $F \le 2U^{-6}K^3$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-10}T, W^{-30}H^5T, U^{-6}K^3\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(W^{-10}T)^{\frac{3}{20}}(W^{-30}H^5T)^{\frac{1}{60}}(U^{-6}K^3)^{\frac{1}{3}} \\ &= T^{\frac{1}{6}}MKH^{\frac{1}{12}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(d) $F \le 2V^{-4}M^2$, $F > 2W^{-10}H^5$, $2U^{-6}K^3$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\quad \ll U^2V^2W^2 \min\{V^{-4}M^2, W^{-10}T, W^{-30}H^5T, U^{-6}T, U^{-18}K^3T\} \\ &\quad \le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(W^{-10}T)^{\frac{3}{20}}(W^{-30}H^5T)^{\frac{1}{60}}(U^{-6}T)^{\frac{1}{3}} \\ &\quad = T^{\frac{1}{2}}MH^{\frac{1}{12}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(e) $F > 2V^{-4}M^2, F \le 2W^{-10}H^5, 2U^{-6}K^3$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-4}T, V^{-12}M^2T, W^{-10}H^5, U^{-6}K^3\} \\ &\le U^2V^2W^2(V^{-4}T)^{\frac{9}{20}}(V^{-12}M^2T)^{\frac{1}{60}}(W^{-10}H^5)^{\frac{1}{5}}(U^{-6}K^3)^{\frac{1}{3}} \\ &= T^{\frac{7}{15}}M^{\frac{1}{30}}HK \ll x^{1-\varepsilon_1}. \end{aligned}$$

(f) $F > 2V^{-4}M^2, F \le 2W^{-10}H^5, F > 2U^{-6}K^3$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\quad \ll U^2V^2W^2 \min\{V^{-4}T, V^{-12}M^2T, W^{-10}H^5, U^{-6}T, U^{-18}K^3T\} \\ &\quad \le U^2V^2W^2(V^{-4}T)^{\frac{1}{2}}(W^{-10}H^5)^{\frac{1}{5}}(U^{-6}T)^{\frac{17}{60}}(U^{-18}K^3T)^{\frac{1}{60}} \\ &\quad = T^{\frac{4}{5}}HK^{\frac{1}{20}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(g) $F > 2V^{-4}M^2, 2W^{-10}H^5, F \le 2U^{-6}K^3$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\quad \ll U^2V^2W^2 \min\{V^{-4}T, V^{-12}M^2T, W^{-10}T, W^{-30}H^5T, U^{-6}K^3\} \\ &\quad \le U^2V^2W^2(V^{-4}T)^{\frac{1}{2}}(W^{-10}T)^{\frac{3}{20}}(W^{-30}H^5T)^{\frac{1}{60}}(U^{-6}K^3)^{\frac{1}{3}} \\ &\quad = T^{\frac{2}{3}}H^{\frac{1}{12}}K \ll x^{1-\varepsilon_1}. \end{aligned}$$

(h) $F > 2V^{-4}M^2, 2W^{-10}H^5, 2U^{-6}K^3$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\quad \ll U^2V^2W^2 \min\{V^{-4}, V^{-12}M^2, W^{-10}, W^{-30}H^5, U^{-6}, U^{-18}K^3\}T \\ &\quad \le U^2V^2W^2(V^{-4})^{\frac{1}{2}}(W^{-10})^{\frac{1}{5}}(U^{-6})^{\frac{17}{60}}(U^{-18}K^3)^{\frac{1}{60}}T \\ &\quad = TK^{\frac{1}{20}} \ll x^{1-\varepsilon_1}, \end{aligned}$$

since $K \ll X$.

II. Next, assume condition 2). We apply the mean value estimate and Halász method to $M^2(s)$, $H^5(s)$ and $K^2(s)H(s)$ to get

$$I \ll U^2V^2W^2x^{-1}F\log^c x,$$

where

$$F = \min\{V^{-4}(M^2+T), V^{-4}M^2 + V^{-12}M^2T, W^{-10}(H^5+T), W^{-10}H^5 + W^{-30}H^5T, U^{-4}W^{-2}(K^2H+T), U^{-4}W^{-2}K^2H + U^{-12}W^{-6}K^2HT\}.$$

Consider the following cases:

(a) $F \le 2V^{-4}M^2, 2W^{-10}H^5, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-4}M^2, W^{-10}H^5, U^{-4}W^{-2}K^2H\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(U^{-4}W^{-2}K^2H)^{\frac{1}{2}} \\ &= WH^{\frac{1}{2}}MK \ll x\log^{-11B} x. \end{aligned}$$

(b) $F \le 2V^{-4}M^2, 2W^{-10}H^5, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\ll U^2V^2W^2 \min\{V^{-4}M^2, W^{-10}H^5, U^{-4}W^{-2}T, U^{-12}W^{-6}K^2HT\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(W^{-10}H^5)^{\frac{1}{10}}(U^{-4}W^{-2}T)^{\frac{7}{20}}(U^{-12}W^{-6}K^2HT)^{\frac{1}{20}} \\ &= T^{\frac{2}{5}}MH^{\frac{11}{20}}K^{\frac{1}{10}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(c) $F \le 2V^{-4}M^2, F > 2W^{-10}H^5, F \le 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-4}M^2, W^{-10}T, W^{-30}H^5T, U^{-4}W^{-2}K^2H\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(U^{-4}W^{-2}K^2H)^{\frac{1}{2}} \\ &= WH^{\frac{1}{2}}MK \ll x\log^{-11B} x. \end{aligned}$$

(d) $F \le 2V^{-4}M^2, F > 2W^{-10}H^5, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\ll U^2V^2W^2 \min\{V^{-4}M^2, W^{-10}T, W^{-30}H^5T, U^{-4}W^{-2}T, \\ &\qquad U^{-12}W^{-6}K^2HT\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(W^{-30}H^5T)^{\frac{1}{30}}(U^{-4}W^{-2}T)^{\frac{9}{20}}(U^{-12}W^{-6}K^2HT)^{\frac{1}{60}} \\ &= T^{\frac{1}{2}}MH^{\frac{11}{60}}K^{\frac{1}{30}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(e) $F > 2V^{-4}M^2, F \le 2W^{-10}H^5, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\ll U^2V^2W^2 \min\{V^{-4}T, V^{-12}M^2T, W^{-10}H^5, U^{-4}W^{-2}K^2H\} \\ &\le U^2V^2W^2(V^{-4}T)^{\frac{7}{20}}(V^{-12}M^2T)^{\frac{1}{20}}(W^{-10}H^5)^{\frac{1}{10}}(U^{-4}W^{-2}K^2H)^{\frac{1}{2}} \\ &= T^{\frac{2}{5}}M^{\frac{1}{10}}HK \ll x^{1-\varepsilon_1}. \end{aligned}$$

(f) $F > 2V^{-4}M^2, F \le 2W^{-10}H^5, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}U^2V^2W^2F&\\ &\ll U^2V^2W^2\min\{V^{-4}T, V^{-12}M^2T, W^{-10}H^5, U^{-4}W^{-2}T,\\ &\qquad U^{-12}W^{-6}K^2HT\}\\ &\le U^2V^2W^2(V^{-4}T)^{\frac{7}{20}}(V^{-12}M^2T)^{\frac{1}{20}}(W^{-10}H^5)^{\frac{1}{10}}(U^{-4}W^{-2}T)^{\frac{1}{2}}\\ &= T^{\frac{9}{10}}M^{\frac{1}{10}}H^{\frac{1}{2}} \ll x^{1-\varepsilon_1}.\end{aligned}$$

(g) $F > 2V^{-4}M^2, 2W^{-10}H^5, F \le 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}&U^2V^2W^2F\\ &\ll U^2V^2W^2\min\{V^{-4}T, V^{-12}M^2T, W^{-10}T, W^{-30}H^5T, U^{-4}W^{-2}K^2H\}\\ &\le U^2V^2W^2(V^{-4}T)^{\frac{9}{20}}(V^{-12}M^2T)^{\frac{1}{60}}(W^{-30}H^5T)^{\frac{1}{30}}(U^{-4}W^{-2}K^2H)^{\frac{1}{2}}\\ &= T^{\frac{1}{2}}M^{\frac{1}{30}}H^{\frac{2}{3}}K \ll x^{1-\varepsilon_1}.\end{aligned}$$

(h) $F > 2V^{-4}M^2, 2W^{-10}H^5, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}, V^{-12}M^2, W^{-10}, W^{-30}H^5, U^{-4}W^{-2},\\ &\qquad U^{-12}W^{-6}K^2H\}T\\ &\le U^2V^2W^2(V^{-4})^{\frac{9}{20}}(V^{-12}M^2)^{\frac{1}{60}}(W^{-30}H^5)^{\frac{1}{30}}(U^{-4}W^{-2})^{\frac{1}{2}}T\\ &= TM^{\frac{1}{30}}H^{\frac{1}{6}} \ll x^{1-\varepsilon_1}.\end{aligned}$$

III. Lastly, we assume condition 3). We apply the mean value estimate and Halász method to $M^2(s)$, $H^4(s)$ and $K^3(s)$ to get

$$I \ll U^2V^2W^2x^{-1}F\log^c x,$$

where

$$\begin{aligned}F = \min\{&V^{-4}(M^2+T), V^{-4}M^2 + V^{-12}M^2T, W^{-8}(H^4+T), W^{-8}H^4\\ &+ W^{-24}H^4T, U^{-6}(K^3+T), U^{-6}K^3 + U^{-18}K^3T\}.\end{aligned}$$

Consider the following cases:

(a) $F \le 2V^{-4}M^2, 2W^{-8}H^4, 2U^{-6}K^3$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-8}H^4, U^{-6}K^3\}\\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(W^{-8}H^4)^{\frac{1}{6}}(U^{-6}K^3)^{\frac{1}{3}}\\ &= W^{\frac{2}{3}}H^{\frac{2}{3}}MK \ll x\log^{-11B} x.\end{aligned}$$

(b) $F \le 2V^{-4}M^2, 2W^{-8}H^4, F > 2U^{-6}K^3$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-4}M^2, W^{-8}H^4, U^{-6}T, U^{-18}K^3T\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(W^{-8}H^4)^{\frac{1}{4}}(U^{-6}T)^{\frac{5}{24}}(U^{-18}K^3T)^{\frac{1}{24}} \\ &= T^{\frac{1}{4}}MHK^{\frac{1}{8}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(c) $F \le 2V^{-4}M^2, F > 2W^{-8}H^4, F \le 2U^{-6}K^3$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-4}M^2, W^{-8}T, W^{-24}H^4T, U^{-6}K^3\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(W^{-8}T)^{\frac{1}{8}}(W^{-24}H^4T)^{\frac{1}{24}}(U^{-6}K^3)^{\frac{1}{3}} \\ &= T^{\frac{1}{6}}MKH^{\frac{1}{6}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(d) $F \le 2V^{-4}M^2, F > 2W^{-8}H^4, 2U^{-6}K^3$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\quad \ll U^2V^2W^2 \min\{V^{-4}M^2, W^{-8}T, W^{-24}H^4T, U^{-6}T, U^{-18}K^3T\} \\ &\quad \le U^2V^2W^2(V^{-4}M^2)^{\frac{1}{2}}(W^{-8}T)^{\frac{1}{8}}(W^{-24}H^4T)^{\frac{1}{24}}(U^{-6}T)^{\frac{1}{3}} \\ &\quad = T^{\frac{1}{2}}MH^{\frac{1}{6}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(e) $F > 2V^{-4}M^2, F \le 2W^{-8}H^4, 2U^{-6}K^3$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-4}T, V^{-12}M^2T, W^{-8}H^4, U^{-6}K^3\} \\ &\le U^2V^2W^2(V^{-4}T)^{\frac{3}{8}}(V^{-12}M^2T)^{\frac{1}{24}}(W^{-8}H^4)^{\frac{1}{4}}(U^{-6}K^3)^{\frac{1}{3}} \\ &= T^{\frac{5}{12}}M^{\frac{1}{12}}HK \ll x^{1-\varepsilon_1}. \end{aligned}$$

(f) $F > 2V^{-4}M^2, F \le 2W^{-8}H^4, F > 2U^{-6}K^3$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-4}T, V^{-12}M^2T, W^{-8}H^4, U^{-6}T, U^{-18}K^3T\} \\ &\le U^2V^2W^2(V^{-4}T)^{\frac{1}{2}}(W^{-8}H^4)^{\frac{1}{4}}(U^{-6}T)^{\frac{5}{24}}(U^{-18}K^3T)^{\frac{1}{24}} \\ &= T^{\frac{3}{4}}HK^{\frac{1}{8}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(g) $F > 2V^{-4}M^2, 2W^{-8}H^4, F \le 2U^{-6}K^3$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\quad \ll U^2V^2W^2 \min\{V^{-4}T, V^{-12}M^2T, W^{-8}T, W^{-24}H^4T, U^{-6}K^3\} \\ &\quad \le U^2V^2W^2(V^{-4}T)^{\frac{1}{2}}(W^{-8}T)^{\frac{1}{8}}(W^{-24}H^4T)^{\frac{1}{24}}(U^{-6}K^3)^{\frac{1}{3}} \\ &\quad = T^{\frac{2}{3}}H^{\frac{1}{6}}K \ll x^{1-\varepsilon_1}. \end{aligned}$$

(h) $F > 2V^{-4}M^2, 2W^{-8}H^4, 2U^{-6}K^3$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\quad \ll U^2V^2W^2\min\{V^{-4}, V^{-12}M^2, W^{-8}, W^{-24}H^4, U^{-6}, U^{-18}K^3\}T\\
&\quad \le U^2V^2W^2(V^{-4})^{\frac{1}{2}}(W^{-8})^{\frac{1}{4}}(U^{-6})^{\frac{5}{24}}(U^{-18}K^3)^{\frac{1}{24}}T\\
&\quad = TK^{\frac{1}{8}} \ll x^{1-\varepsilon_1},
\end{aligned}$$

since $X^{\frac{3}{5}} \ll MH$ (the latter follows from $X^{\frac{19}{44}} \ll M$ and $X^{\frac{19}{100}} \ll H$).

Combining the above, we obtain (8). Hence, Lemma 7 follows.

LEMMA 8. *Under the assumption of Lemma* 5, *suppose that* M *and* H *lie in one of the following regions*:

(i) $X^{\frac{19}{45}} \ll M \ll X^{\frac{19}{44}}, \quad M^{-\frac{29}{10}}X^{\frac{57}{40}} \ll H \ll M^{-\frac{1}{5}}X^{\frac{29}{100}}$;

(ii) $X^{\frac{19}{44}} \ll M \ll X^{\frac{897}{1972}}, \quad M^{-\frac{29}{10}}X^{\frac{57}{40}} \ll H \ll M^{-\frac{1}{5}}X^{\frac{29}{100}}$;

(iii) $X^{\frac{19}{44}} \ll M \ll X^{\frac{897}{1972}}, \quad M^{-\frac{6}{5}}X^{\frac{19}{25}} \ll H \ll M^{\frac{1}{7}}X^{\frac{13}{70}}$;

(iv) $X^{\frac{897}{1972}} \ll M \ll X^{\frac{133}{290}}, \quad X^{\frac{9}{85}} \ll H \ll M^{-\frac{1}{5}}X^{\frac{29}{100}}$;

(v) $X^{\frac{897}{1972}} \ll M \ll X^{\frac{133}{290}}, \quad M^{-\frac{6}{5}}X^{\frac{19}{25}} \ll H \ll M^{\frac{1}{7}}X^{\frac{13}{70}}$;

(vi) $X^{\frac{133}{290}} \ll M \ll X^{\frac{19}{40}}, \quad X^{\frac{9}{85}} \ll H \ll M^{\frac{1}{7}}X^{\frac{13}{70}}$;

(vii) $X^{\frac{19}{40}} \ll M \ll X^{\frac{53}{110}}, \quad X^{\frac{9}{85}} \ll H \ll M^{-1}X^{\frac{51}{70}}$;

(viii) $X^{\frac{53}{110}} \ll M \ll X^{\frac{339}{700}}, \quad X^{\frac{9}{85}} \ll H \ll M^{-\frac{58}{9}}X^{\frac{59}{18}}$;

(ix) $X^{\frac{53}{110}} \ll M \ll X^{\frac{339}{700}}, \quad X^{\frac{19}{110}} \ll H \ll M^{-1}X^{\frac{51}{70}}$;

(x) $X^{\frac{339}{700}} \ll M \ll X^{\frac{49}{100}}, \quad X^{\frac{9}{85}} \ll H \ll M^{-\frac{58}{9}}X^{\frac{59}{18}}$;

(xi) $X^{\frac{339}{700}} \ll M \ll X^{\frac{49}{100}}, \quad X^{\frac{19}{110}} \ll H \ll M^{-6}X^{\frac{63}{20}}$;

(xii) $X^{\frac{49}{100}} \ll M \ll X^{\frac{1}{2}}, \quad X^{\frac{19}{110}} \ll H \ll M^{-1}X^{\frac{7}{10}}$.

Then (7) *holds for* $T_0 \le T \le X$.

P r o o f. In the regions:

$$X^{\frac{133}{290}} \ll M \ll X^{\frac{19}{40}}, \quad M^{-\frac{12}{11}}X^{\frac{38}{55}} \ll H \ll M^{\frac{1}{19}}X^{\frac{1}{5}};$$
$$X^{\frac{19}{40}} \ll M \ll X^{\frac{1}{2}}, \quad X^{\frac{19}{110}} \ll H \ll M^{-1}X^{\frac{7}{10}},$$

we apply Lemma 7 with condition 1).

In the regions:

$$X^{\frac{19}{45}} \ll M \ll X^{\frac{897}{1972}}, \quad M^{-\frac{29}{10}}X^{\frac{57}{40}} \ll H \ll M^{-\frac{1}{5}}X^{\frac{29}{100}};$$
$$X^{\frac{897}{1972}} \ll M \ll X^{\frac{133}{290}}, \quad X^{\frac{9}{85}} \ll H \ll M^{-\frac{1}{5}}X^{\frac{29}{100}};$$
$$X^{\frac{133}{290}} \ll M \ll X^{\frac{19}{40}}, \quad X^{\frac{9}{85}} \ll H \ll M^{-\frac{12}{11}}X^{\frac{38}{55}};$$

$$X^{\frac{19}{40}} \ll M \ll X^{\frac{53}{110}}, \quad X^{\frac{9}{85}} \ll H \ll X^{\frac{19}{110}};$$
$$X^{\frac{53}{110}} \ll M \ll X^{\frac{49}{100}}, \quad X^{\frac{9}{85}} \ll H \ll M^{-\frac{58}{9}} X^{\frac{59}{18}},$$

we apply Lemma 7 with condition 2).

In the regions:

$$X^{\frac{19}{44}} \ll M \ll X^{\frac{133}{290}}, \quad M^{-\frac{6}{5}} X^{\frac{19}{25}} \ll H \ll M^{\frac{1}{7}} X^{\frac{13}{70}};$$
$$X^{\frac{133}{290}} \ll M \ll X^{\frac{19}{40}}, \quad M^{\frac{1}{19}} X^{\frac{1}{5}} \ll H \ll M^{\frac{1}{7}} X^{\frac{13}{70}};$$
$$X^{\frac{19}{40}} \ll M \ll X^{\frac{339}{700}}, \quad M^{-1} X^{\frac{7}{10}} \ll H \ll M^{-1} X^{\frac{51}{70}};$$
$$X^{\frac{339}{700}} \ll M \ll X^{\frac{49}{100}}, \quad M^{-1} X^{\frac{7}{10}} \ll H \ll M^{-6} X^{\frac{63}{20}},$$

we apply Lemma 7 with condition 3).

Putting together the above regions, we get Lemma 8.

Lemma 9. *Under the assumption of Lemma* 5, *suppose that* M *and* H *lie in one of the following regions*:

(i) $X^{\frac{1843}{5280}} \ll M \ll X^{\frac{1188}{2975}}, \quad M^{-\frac{70}{59}} X^{\frac{171}{295}} \ll H \ll M^{-\frac{58}{49}} X^{\frac{57}{98}};$

(ii) $X^{\frac{1188}{2975}} \ll M \ll X^{\frac{3963}{9860}}, \quad X^{\frac{9}{85}} \ll H \ll M^{-\frac{58}{49}} X^{\frac{57}{98}};$

(iii) $X^{\frac{1063}{2640}} \ll M \ll X^{\frac{4331}{9860}}, \quad M^{-\frac{29}{19}} X^{\frac{59}{76}} \ll H \ll M^{-\frac{35}{23}} X^{\frac{179}{230}}.$

Then (7) *holds for* $T_0 \le T \le X$.

Proof. First we show that (8) holds for $T = 1/\eta = 2X^{\frac{19}{20}-\varepsilon}$, providing M and H satisfy the following conditions:

$$MH \ll X^{\frac{25}{44}}, \quad M^{35}H^{23} \ll X^{\frac{179}{10}}, \quad X^{\frac{19}{22}} \ll M^2 H,$$
$$M^2 H^{13} \ll X^{\frac{31}{10}}, \quad X^{\frac{171}{5}} \ll M^{70} H^{59}.$$

We apply the mean value estimate and Halász method to $M^2(s)H(s)$, $H^6(s)$ and $K^2(s)$ to get

$$I \ll U^2 V^2 W^2 x^{-1} F \log^c x,$$

where

$$F = \min\{V^{-4}W^{-2}(M^2H + T), V^{-4}W^{-2}M^2H + V^{-12}W^{-6}M^2HT, W^{-12}(H^6+T), W^{-12}H^6 + W^{-36}H^6T, U^{-4}(K^2+T), U^{-4}K^2 + U^{-12}K^2T\}.$$

Consider the following cases:

(a) $F \le 2V^{-4}W^{-2}M^2H, 2W^{-12}H^6, 2U^{-4}K^2$. Then

$$U^2V^2W^2F \ll U^2V^2W^2 \min\{V^{-4}W^{-2}M^2H, W^{-12}H^6, U^{-4}K^2\}$$
$$\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(U^{-4}K^2)^{\frac{1}{2}}$$
$$= WH^{\frac{1}{2}}MK \ll x \log^{-11B} x.$$

(b) $F \le 2V^{-4}W^{-2}M^2H, 2W^{-12}H^6, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\quad \ll U^2V^2W^2 \min\{V^{-4}W^{-2}M^2H, W^{-12}H^6, U^{-4}T, U^{-12}K^2T\}\\
&\quad \le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(W^{-12}H^6)^{\frac{1}{12}}(U^{-4}T)^{\frac{3}{8}}(U^{-12}K^2T)^{\frac{1}{24}}\\
&\quad = T^{\frac{5}{12}}MHK^{\frac{1}{12}} \ll x^{1-\varepsilon_1}.
\end{aligned}$$

(c) $F \le 2V^{-4}W^{-2}M^2H, F > 2W^{-12}H^6, F \le 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-4}W^{-2}M^2H, W^{-12}T, W^{-36}H^6T, U^{-4}K^2\}\\
&\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(U^{-4}K^2)^{\frac{1}{2}}\\
&= WH^{\frac{1}{2}}MK \ll x\log^{-11B} x.
\end{aligned}$$

(d) $F \le 2V^{-4}W^{-2}M^2H, F > 2W^{-12}H^6, 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\ll U^2V^2W^2 \min\{V^{-4}W^{-2}M^2H, W^{-12}T, W^{-36}H^6T, U^{-4}T, U^{-12}K^2T\}\\
&\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(W^{-36}H^6T)^{\frac{1}{36}}(U^{-4}T)^{\frac{11}{24}}(U^{-12}K^2T)^{\frac{1}{72}}\\
&= T^{\frac{1}{2}}MH^{\frac{2}{3}}K^{\frac{1}{36}} \ll x^{1-\varepsilon_1}.
\end{aligned}$$

(e) $F > 2V^{-4}W^{-2}M^2H, F \le 2W^{-12}H^6, 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\ll U^2V^2W^2 \min\{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-12}H^6, U^{-4}K^2\}\\
&\le U^2V^2W^2(V^{-4}W^{-2}T)^{\frac{3}{8}}(V^{-12}W^{-6}M^2HT)^{\frac{1}{24}}(W^{-12}H^6)^{\frac{1}{12}}(U^{-4}K^2)^{\frac{1}{2}}\\
&= T^{\frac{5}{12}}M^{\frac{1}{12}}H^{\frac{13}{24}}K \ll x^{1-\varepsilon_1}.
\end{aligned}$$

(f) $F > 2V^{-4}W^{-2}M^2H, F \le 2W^{-12}H^6, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\quad \ll U^2V^2W^2 \min\{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-12}H^6,\\
&\qquad\qquad U^{-4}T, U^{-12}K^2T\}\\
&\quad \le U^2V^2W^2(V^{-4}W^{-2}T)^{\frac{3}{8}}(V^{-12}W^{-6}M^2HT)^{\frac{1}{24}}(W^{-12}H^6)^{\frac{1}{12}}(U^{-4}T)^{\frac{1}{2}}\\
&\quad = T^{\frac{11}{12}}M^{\frac{1}{12}}H^{\frac{13}{24}} \ll x^{1-\varepsilon_1}.
\end{aligned}$$

(g) $F > 2V^{-4}W^{-2}M^2H, 2W^{-12}H^6, F \le 2U^{-4}K^2$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\ll U^2V^2W^2\min\{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-12}T,\\
&\qquad W^{-36}H^6T, U^{-4}K^2\}\\
&\le U^2V^2W^2(V^{-4}W^{-2}T)^{\frac{11}{24}}(V^{-12}W^{-6}M^2HT)^{\frac{1}{72}}(W^{-36}H^6T)^{\frac{1}{36}}(U^{-4}K^2)^{\frac{1}{2}}\\
&= T^{\frac{1}{2}}M^{\frac{1}{36}}H^{\frac{13}{72}}K \ll x^{1-\varepsilon_1}.
\end{aligned}$$

(h) $F > 2V^{-4}W^{-2}M^2H, 2W^{-12}H^6, 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}W^{-2}, V^{-12}W^{-6}M^2H, W^{-12}, W^{-36}H^6,\\
&\qquad U^{-4}, U^{-12}K^2\}T\\
&\le U^2V^2W^2(V^{-4}W^{-2})^{\frac{1}{2}}(W^{-12})^{\frac{1}{12}}(U^{-4})^{\frac{3}{8}}(U^{-12}K^2)^{\frac{1}{24}}T\\
&= TK^{\frac{1}{12}} \ll x^{1-\varepsilon_1},
\end{aligned}$$

since $X^{\frac{2}{5}} \ll MH$ (the latter follows from $X^{\frac{171}{5}} \ll M^{70}H^{59}$).

In every region, our conditions are satisfied. So the proof of Lemma 9 is complete.

LEMMA 10. *Under the assumption of Lemma* 5, *suppose that* M *and* H *lie in one of the following regions*:

(i) $X^{\frac{37}{100}} \ll M \ll X^{\frac{5541}{13940}}, \quad M^{-\frac{82}{69}}X^{\frac{133}{230}} \ll H \ll M^{-\frac{70}{59}}X^{\frac{171}{295}}$;

(ii) $X^{\frac{5541}{13940}} \ll M \ll X^{\frac{1188}{2975}}, \quad X^{\frac{9}{85}} \ll H \ll M^{-\frac{70}{59}}X^{\frac{171}{295}}$;

(iii) $X^{\frac{19}{45}} \ll M \ll X^{\frac{4331}{9860}}, \quad M^{-\frac{35}{23}}X^{\frac{179}{230}} \ll H \ll M^{-\frac{41}{27}}X^{\frac{421}{540}}$.

Then (7) *holds for* $T_0 \le T \le X$.

Proof. First we show that (8) holds for $T = 1/\eta = 2X^{\frac{19}{20}-\varepsilon}$, providing M and H satisfy the following conditions:

$$MH \ll X^{\frac{73}{130}}, \quad M^{41}H^{27} \ll X^{\frac{421}{20}}, \quad X^{\frac{57}{65}} \ll M^2H,$$
$$M^2H^{15} \ll X^{\frac{33}{10}}, \quad X^{\frac{399}{10}} \ll M^{82}H^{69}.$$

We apply the mean value estimate and Halász method to $M^2(s)H(s)$, $H^7(s)$ and $K^2(s)$ to get

$$I \ll U^2V^2W^2x^{-1}F\log^c x,$$

where

$$\begin{aligned}
F = \min\{&V^{-4}W^{-2}(M^2H+T), V^{-4}W^{-2}M^2H + V^{-12}W^{-6}M^2HT,\\
&W^{-14}(H^7+T), W^{-14}H^7 + W^{-42}H^7T, U^{-4}(K^2+T),\\
&U^{-4}K^2 + U^{-12}K^2T\}.
\end{aligned}$$

Consider the following cases:

(a) $F \le 2V^{-4}W^{-2}M^2H, 2W^{-14}H^7, 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-4}W^{-2}M^2H, W^{-14}H^7, U^{-4}K^2\} \\ &\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(U^{-4}K^2)^{\frac{1}{2}} \\ &= WH^{\frac{1}{2}}MK \ll x\log^{-11B} x. \end{aligned}$$

(b) $F \le 2V^{-4}W^{-2}M^2H, 2W^{-14}H^7, F > 2U^{-4}K^2$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\quad\ll U^2V^2W^2 \min\{V^{-4}W^{-2}M^2H, W^{-14}H^7, U^{-4}T, U^{-12}K^2T\} \\ &\quad\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(W^{-14}H^7)^{\frac{1}{14}}(U^{-4}T)^{\frac{11}{28}}(U^{-12}K^2T)^{\frac{1}{28}} \\ &\quad= T^{\frac{3}{7}}MHK^{\frac{1}{14}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(c) $F \le 2V^{-4}W^{-2}M^2H, F > 2W^{-14}H^7, F \le 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-4}W^{-2}M^2H, W^{-14}T, W^{-42}H^7T, U^{-4}K^2\} \\ &\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(U^{-4}K^2)^{\frac{1}{2}} \\ &= WH^{\frac{1}{2}}MK \ll x\log^{-11B} x. \end{aligned}$$

(d) $F \le 2V^{-4}W^{-2}M^2H, F > 2W^{-14}H^7, 2U^{-4}K^2$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\ll U^2V^2W^2 \min\{V^{-4}W^{-2}M^2H, W^{-14}T, W^{-42}H^7T, U^{-4}T, U^{-12}K^2T\} \\ &\le U^2V^2W^2(V^{-4}W^{-2}M^2H)^{\frac{1}{2}}(W^{-42}H^7T)^{\frac{1}{42}}(U^{-4}T)^{\frac{13}{28}}(U^{-12}K^2T)^{\frac{1}{84}} \\ &= T^{\frac{1}{2}}MH^{\frac{2}{3}}K^{\frac{1}{42}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(e) $F > 2V^{-4}W^{-2}M^2H, F \le 2W^{-14}H^7, 2U^{-4}K^2$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\ll U^2V^2W^2 \min\{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-14}H^7, U^{-4}K^2\} \\ &\le U^2V^2W^2(V^{-4}W^{-2}T)^{\frac{11}{28}}(V^{-12}W^{-6}M^2HT)^{\frac{1}{28}}(W^{-14}H^7)^{\frac{1}{14}}(U^{-4}K^2)^{\frac{1}{2}} \\ &= T^{\frac{3}{7}}M^{\frac{1}{14}}H^{\frac{15}{28}}K \ll x^{1-\varepsilon_1}. \end{aligned}$$

(f) $F > 2V^{-4}W^{-2}M^2H, F \le 2W^{-14}H^7, F > 2U^{-4}K^2$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\quad\ll U^2V^2W^2 \min\{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-14}H^7, \\ &\qquad\qquad U^{-4}T, U^{-12}K^2T\} \\ &\quad\le U^2V^2W^2(V^{-4}W^{-2}T)^{\frac{11}{28}}(V^{-12}W^{-6}M^2HT)^{\frac{1}{28}}(W^{-14}H^7)^{\frac{1}{14}}(U^{-4}T)^{\frac{1}{2}} \\ &\quad= T^{\frac{13}{14}}M^{\frac{1}{14}}H^{\frac{15}{28}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(g) $F > 2V^{-4}W^{-2}M^2H, 2W^{-14}H^7, F \le 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-14}T, \\ &\qquad W^{-42}H^7T, U^{-4}K^2\} \\ &\le U^2V^2W^2(V^{-4}W^{-2}T)^{\frac{13}{28}}(V^{-12}W^{-6}M^2HT)^{\frac{1}{84}} \\ &\quad \times (W^{-42}H^7T)^{\frac{1}{42}}(U^{-4}K^2)^{\frac{1}{2}} \\ &= T^{\frac{1}{2}}M^{\frac{1}{42}}H^{\frac{5}{28}}K \ll x^{1-\varepsilon_1}. \end{aligned}$$

(h) $F > 2V^{-4}W^{-2}M^2H, 2W^{-14}H^7, 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}W^{-2}, V^{-12}W^{-6}M^2H, W^{-14}, W^{-42}H^7, \\ &\qquad U^{-4}, U^{-12}K^2\}T \\ &\le U^2V^2W^2(V^{-4}W^{-2})^{\frac{1}{2}}(W^{-14})^{\frac{1}{14}}(U^{-4})^{\frac{11}{28}}(U^{-12}K^2)^{\frac{1}{28}}T \\ &= TK^{\frac{1}{14}} \ll x^{1-\varepsilon_1}, \end{aligned}$$

since $X^{\frac{3}{10}} \ll MH$ (the latter follows from $X^{\frac{57}{65}} \ll M^2H$).

In every region, our conditions are satisfied, so the proof of Lemma 10 is complete.

LEMMA 11. *Under the assumption of Lemma* 5, *suppose that M and H lie in one of the following regions*:

(i) $X^{\frac{1843}{5280}} \ll M \ll X^{\frac{37}{100}}$, $M^{-\frac{70}{59}}X^{\frac{171}{295}} \ll H \ll M^{-\frac{58}{49}}X^{\frac{57}{98}}$;

(ii) $X^{\frac{37}{100}} \ll M \ll X^{\frac{5541}{13940}}$, $M^{-\frac{82}{69}}X^{\frac{133}{230}} \ll H \ll M^{-\frac{58}{49}}X^{\frac{57}{98}}$;

(iii) $X^{\frac{5541}{13940}} \ll M \ll X^{\frac{3963}{9860}}$, $X^{\frac{9}{85}} \ll H \ll M^{-\frac{58}{49}}X^{\frac{57}{98}}$;

(iv) $X^{\frac{1063}{2640}} \ll M \ll X^{\frac{19}{45}}$, $M^{-\frac{29}{19}}X^{\frac{59}{76}} \ll H \ll M^{-\frac{35}{23}}X^{\frac{179}{230}}$;

(v) $X^{\frac{19}{45}} \ll M \ll X^{\frac{4331}{9860}}$, $M^{-\frac{29}{19}}X^{\frac{59}{76}} \ll H \ll M^{-\frac{41}{27}}X^{\frac{421}{540}}$.

Then (7) *holds for $T_0 \le T \le X$.*

Proof. Putting together regions in Lemmas 9 and 10, we can get Lemma 11.

LEMMA 12. *Under the assumption of Lemma* 5, *suppose that M and H lie in one of the following regions*:

(i) $X^{\frac{4331}{9860}} \ll M \ll X^{\frac{2691}{5950}}$, $M^{-\frac{35}{12}}X^{\frac{57}{40}} \ll H \ll M^{-\frac{29}{10}}X^{\frac{57}{40}}$;

(ii) $X^{\frac{2691}{5950}} \ll M \ll X^{\frac{897}{1972}}$, $X^{\frac{9}{85}} \ll H \ll M^{-\frac{29}{10}}X^{\frac{57}{40}}$;

(iii) $X^{\frac{233}{480}} \ll M \ll X^{\frac{49}{100}}$, $M^{-\frac{58}{9}}X^{\frac{59}{18}} \ll H \ll M^{-\frac{70}{11}}X^{\frac{179}{55}}$;

(iv) $X^{\frac{49}{100}} \ll M \ll X^{\frac{2944}{5950}}$, $X^{\frac{9}{85}} \ll H \ll M^{-\frac{70}{11}}X^{\frac{179}{55}}$.

Then (7) *holds for $T_0 \le T \le X$.*

P r o o f. First we show that (8) holds for $T = 1/\eta = 2X^{\frac{19}{20}-\varepsilon}$, providing that M and H satisfy the following conditions:

$$M^2H \ll X^{\frac{25}{22}}, \quad M^{70}H^{11} \ll X^{\frac{179}{5}}, \quad X^{\frac{19}{44}} \ll M,$$
$$MH^6 \ll X^{\frac{31}{20}}, \quad X^{\frac{171}{10}} \ll M^{35}H^{12}.$$

We apply the mean value estimate and Halász method to $M^2(s)$, $H^6(s)$ and $K^2(s)H(s)$ to get

$$I \ll U^2V^2W^2x^{-1}F\log^c x,$$

where

$$F = \min\{V^{-4}(M^2+T), V^{-4}M^2 + V^{-12}M^2T, W^{-12}(H^6+T), W^{-12}H^6 + W^{-36}H^6T, U^{-4}W^{-2}(K^2H+T), U^{-4}W^{-2}K^2H + U^{-12}W^{-6}K^2HT\}.$$

Consider the following cases:

(a) $F \le 2V^{-4}M^2, 2W^{-12}H^6, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-12}H^6, U^{-4}W^{-2}K^2H\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac12}(U^{-4}W^{-2}K^2H)^{\frac12} \\ &= WH^{\frac12}MK \ll x\log^{-11B} x. \end{aligned}$$

(b) $F \le 2V^{-4}M^2, 2W^{-12}H^6, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} &U^2V^2W^2F \\ &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-12}H^6, U^{-4}W^{-2}T, U^{-12}W^{-6}K^2HT\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac12}(W^{-12}H^6)^{\frac{1}{12}}(U^{-4}W^{-2}T)^{\frac38}(U^{-12}W^{-6}K^2HT)^{\frac{1}{24}} \\ &= T^{\frac{5}{12}}MH^{\frac{13}{24}}K^{\frac{1}{12}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(c) $F \le 2V^{-4}M^2, F > 2W^{-12}H^6, F \le 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-12}T, W^{-36}H^6T, U^{-4}W^{-2}K^2H\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac12}(U^{-4}W^{-2}K^2H)^{\frac12} \\ &= WH^{\frac12}MK \ll x\log^{-11B} x. \end{aligned}$$

(d) $F \le 2V^{-4}M^2, F > 2W^{-12}H^6, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}M^2, W^{-12}T, W^{-36}H^6T, \\ &\qquad U^{-4}W^{-2}T, U^{-12}W^{-6}K^2HT\} \\ &\le U^2V^2W^2(V^{-4}M^2)^{\frac12}(W^{-36}H^6T)^{\frac{1}{36}}(U^{-4}W^{-2}T)^{\frac{11}{24}} \\ &\quad \times (U^{-12}W^{-6}K^2HT)^{\frac{1}{72}} \\ &= T^{\frac12}MH^{\frac{13}{72}}K^{\frac{1}{36}} \ll x^{1-\varepsilon_1}. \end{aligned}$$

(e) $F > 2V^{-4}M^2, F \le 2W^{-12}H^6, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\quad \ll U^2V^2W^2\min\{V^{-4}T, V^{-12}M^2T, W^{-12}H^6, U^{-4}W^{-2}K^2H\}\\
&\quad \le U^2V^2W^2(V^{-4}T)^{\frac{3}{8}}(V^{-12}M^2T)^{\frac{1}{24}}(W^{-12}H^6)^{\frac{1}{12}}(U^{-4}W^{-2}K^2H)^{\frac{1}{2}}\\
&\quad = T^{\frac{5}{12}}M^{\frac{1}{12}}HK \ll x^{1-\varepsilon_1}.
\end{aligned}$$

(f) $F > 2V^{-4}M^2, F \le 2W^{-12}H^6, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}T, V^{-12}M^2T, W^{-12}H^6, U^{-4}W^{-2}T,\\
&\qquad U^{-12}W^{-6}K^2HT\}\\
&\le U^2V^2W^2(V^{-4}T)^{\frac{3}{8}}(V^{-12}M^2T)^{\frac{1}{24}}(W^{-12}H^6)^{\frac{1}{12}}(U^{-4}W^{-2}T)^{\frac{1}{2}}\\
&= T^{\frac{11}{12}}M^{\frac{1}{12}}H^{\frac{1}{2}} \ll x^{1-\varepsilon_1}.
\end{aligned}$$

(g) $F > 2V^{-4}M^2, 2W^{-12}H^6, F \le 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}
&U^2V^2W^2F\\
&\ll U^2V^2W^2\min\{V^{-4}T, V^{-12}M^2T, W^{-12}T, W^{-36}H^6T, U^{-4}W^{-2}K^2H\}\\
&\le U^2V^2W^2(V^{-4}T)^{\frac{11}{24}}(V^{-12}M^2T)^{\frac{1}{72}}(W^{-36}H^6T)^{\frac{1}{36}}(U^{-4}W^{-2}K^2H)^{\frac{1}{2}}\\
&= T^{\frac{1}{2}}M^{\frac{1}{36}}H^{\frac{2}{3}}K \ll x^{1-\varepsilon_1}.
\end{aligned}$$

(h) $F > 2V^{-4}M^2, 2W^{-12}H^6, 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2\min\{V^{-4}, V^{-12}M^2, W^{-12}, W^{-36}H^6, U^{-4}W^{-2},\\
&\qquad U^{-12}W^{-6}K^2H\}T\\
&\le U^2V^2W^2(V^{-4})^{\frac{3}{8}}(V^{-12}M^2)^{\frac{1}{24}}(W^{-12})^{\frac{1}{12}}(U^{-4}W^{-2})^{\frac{1}{2}}T\\
&= TM^{\frac{1}{12}} \ll x^{1-\varepsilon_1},
\end{aligned}$$

since $M \ll X^{\frac{3}{5}}$ (the latter follows from $M^{70}H^{11} \ll X^{\frac{179}{5}}$).

In every region, our conditions are satisfied. So the proof of Lemma 12 is complete.

4. Mean value estimate (III)

LEMMA 13. *Suppose that $PQRK = X$ and that $P(s)$, $Q(s)$, $R(s)$ and $K(s)$ are Dirichlet polynomials. Define $G(s) = P(s)Q(s)R(s)K(s)$. Let $b = 1 + 1/\log X$ and $T_0 = \log^{\frac{B}{\varepsilon}} X$. Assume further that for $T_0 \le |t| \le 2X$, $P(b+it)Q(b+it) \ll \log^{-\frac{B}{\varepsilon}} x$ and $R(b+it) \ll \log^{-\frac{B}{\varepsilon}} x$. Moreover, assume that*

$$X^{\frac{9}{85}} \ll R \ll Q$$

and that P *and* Q *lie in one of the following regions*:

(i) $X^{\frac{133}{580}} \ll P \ll X^{\frac{19}{80}}, \quad P^{-1}X^{\frac{133}{290}} \ll Q \ll P;$

(ii) $X^{\frac{19}{80}} \ll P \ll X^{\frac{53}{220}}, \quad P^{-1}X^{\frac{133}{290}} \ll Q \ll P^{-1}X^{\frac{19}{40}};$

(iii) $X^{\frac{53}{220}} \ll P \ll X^{\frac{151}{580}}, \quad P^{-1}X^{\frac{133}{290}} \ll Q \ll P^{-1}X^{\frac{53}{110}};$

(iv) $X^{\frac{151}{580}} \ll P \ll X^{\frac{17}{55}}, \quad P^{-1}X^{\frac{897}{1972}} \ll Q \ll P^{-1}X^{\frac{53}{110}};$

(v) $X^{\frac{17}{55}} \ll P \ll X^{\frac{3441}{9860}}, \quad P^{-1}X^{\frac{897}{1972}} \ll Q \ll P^{-\frac{58}{67}}X^{\frac{59}{134}};$

(vi) $X^{\frac{3441}{9860}} \ll P \ll X^{\frac{37}{100}}, \quad X^{\frac{9}{85}} \ll Q \ll P^{-\frac{58}{67}}X^{\frac{59}{134}};$

(vii) $X^{\frac{37}{100}} \ll P \ll X^{\frac{653}{1700}}, \quad X^{\frac{9}{85}} \ll Q \ll P^{-1}X^{\frac{49}{100}}.$

Then (7) *holds for* $T_0 \le T \le X$.

Proof. Let $m = pq$, $h = r$.

(a) On applying Lemma 8 with region (iv), we see that (7) holds under the conditions

$$P^{-1}X^{\frac{897}{1972}} \ll Q \ll P^{-1}X^{\frac{133}{290}}, \quad X^{\frac{9}{85}} \ll R \ll Q \ll (PQ)^{-\frac{1}{5}}X^{\frac{29}{100}},$$

which can be written as

$$P^{-1}X^{\frac{897}{1972}} \ll Q \ll P^{-1}X^{\frac{133}{290}}, \quad X^{\frac{9}{85}} \ll Q \ll P^{-\frac{1}{6}}X^{\frac{29}{120}}, \quad X^{\frac{9}{85}} \ll R \ll Q.$$

In the regions:

$$X^{\frac{151}{580}} \ll P \ll X^{\frac{3441}{9860}}, \quad P^{-1}X^{\frac{897}{1972}} \ll Q \ll P^{-1}X^{\frac{133}{290}};$$

$$X^{\frac{3441}{9860}} \ll P \ll X^{\frac{1739}{4930}}, \quad X^{\frac{9}{85}} \ll Q \ll P^{-1}X^{\frac{133}{290}},$$

the above conditions on P and Q are satisfied.

(b) On applying Lemma 8 with region (vi), we see that (7) holds under the conditions

$$P^{-1}X^{\frac{133}{290}} \ll Q \ll P^{-1}X^{\frac{19}{40}}, \quad X^{\frac{9}{85}} \ll R \ll Q \ll (PQ)^{\frac{1}{7}}X^{\frac{13}{70}},$$

which can be written as

$$P^{-1}X^{\frac{133}{290}} \ll Q \ll P^{-1}X^{\frac{19}{40}}, \quad X^{\frac{9}{85}} \ll Q \ll P^{\frac{1}{6}}X^{\frac{13}{60}}, \quad X^{\frac{9}{85}} \ll R \ll Q.$$

In the regions:

$$X^{\frac{133}{580}} \ll P \ll X^{\frac{19}{80}}, \quad P^{-1}X^{\frac{133}{290}} \ll Q \ll P;$$

$$X^{\frac{19}{80}} \ll P \ll X^{\frac{1739}{4930}}, \quad P^{-1}X^{\frac{133}{290}} \ll Q \ll P^{-1}X^{\frac{19}{40}};$$

$$X^{\frac{1739}{4930}} \ll P \ll X^{\frac{251}{680}}, \quad X^{\frac{9}{85}} \ll Q \ll P^{-1}X^{\frac{19}{40}},$$

the above conditions on P and Q are satisfied.

(c) On applying Lemma 8 with region (vii), we see that (7) holds under the conditions

$$P^{-1}X^{\frac{19}{40}} \ll Q \ll P^{-1}X^{\frac{53}{110}}, \quad X^{\frac{9}{85}} \ll R \ll Q \ll (PQ)^{-1}X^{\frac{51}{70}},$$

which can be written as

$$P^{-1}X^{\frac{19}{40}} \ll Q \ll P^{-1}X^{\frac{53}{110}}, \quad X^{\frac{9}{85}} \ll Q \ll P^{-\frac{1}{2}}X^{\frac{51}{140}}, \quad X^{\frac{9}{85}} \ll R \ll Q.$$

In the regions:

$$X^{\frac{53}{220}} \ll P \ll X^{\frac{251}{680}}, \quad P^{-1}X^{\frac{19}{40}} \ll Q \ll P^{-1}X^{\frac{53}{110}};$$
$$X^{\frac{251}{680}} \ll P \ll X^{\frac{703}{1870}}, \quad X^{\frac{9}{85}} \ll Q \ll P^{-1}X^{\frac{53}{110}},$$

the above conditions on P and Q are satisfied.

(d) On applying Lemma 8 with regions (viii) and (x), we see that (7) holds under the conditions

$$P^{-1}X^{\frac{53}{110}} \ll Q \ll P^{-1}X^{\frac{49}{100}}, \quad X^{\frac{9}{85}} \ll R \ll Q \ll (PQ)^{-\frac{58}{9}}X^{\frac{59}{18}},$$

which can be written as

$$P^{-1}X^{\frac{53}{110}} \ll Q \ll P^{-1}X^{\frac{49}{100}}, \quad X^{\frac{9}{85}} \ll Q \ll P^{-\frac{58}{67}}X^{\frac{59}{134}}, \quad X^{\frac{9}{85}} \ll R \ll Q.$$

In the regions:

$$X^{\frac{17}{55}} \ll P \ll X^{\frac{37}{100}}, \quad P^{-1}X^{\frac{53}{110}} \ll Q \ll P^{-\frac{58}{67}}X^{\frac{59}{134}};$$
$$X^{\frac{37}{100}} \ll P \ll X^{\frac{703}{1870}}, \quad P^{-1}X^{\frac{53}{110}} \ll Q \ll P^{-1}X^{\frac{49}{100}};$$
$$X^{\frac{703}{1870}} \ll P \ll X^{\frac{653}{1700}}, \quad X^{\frac{9}{85}} \ll Q \ll P^{-1}X^{\frac{49}{100}},$$

the above conditions on P and Q are satisfied.

Putting together the above regions, we get Lemma 13.

LEMMA 14. *Suppose that $PQRL = X$ and that $P(s)$, $Q(s)$ and $R(s)$ are Dirichlet polynomials*,

$$L(s) = \sum_{l \sim L} \frac{1}{l^s},$$

and $F(s) = P(s)Q(s)R(s)L(s)$. Let $b = 1 + 1/\log X$ and $T_1 = \sqrt{L}$. Assume further that for $T_1 \le |t| \le 2X$, $P(b+it)Q(b+it) \ll \log^{-\frac{B}{\varepsilon}} x$ and $R(b+it) \ll \log^{-\frac{B}{\varepsilon}} x$. Moreover, assume that

$$X^{\frac{9}{85}} \ll R \ll Q$$

and that P and Q lie in one of the following regions:

(i) $X^{\frac{9}{85}} \ll P \ll X^{\frac{21}{100}}, \quad X^{\frac{9}{85}} \ll Q \ll P;$

(ii) $X^{\frac{21}{100}} \ll P \ll X^{\frac{13}{60}}, \quad X^{\frac{9}{85}} \ll Q \ll P^{-\frac{2}{3}}X^{\frac{7}{20}};$

(iii) $X^{\frac{13}{60}} \ll P \ll X^{\frac{12613}{49300}}, \quad X^{\frac{9}{85}} \ll Q \ll P^{-\frac{1}{6}}X^{\frac{29}{120}};$

(iv) $X^{\frac{12613}{49300}} \ll P \ll X^{\frac{3441}{9860}}, \quad X^{\frac{9}{85}} \ll Q \ll P^{-1}X^{\frac{897}{1972}}.$

Then for $T_1 \le T \le X$, *we have*

$$\min^2\left(\eta, \frac{1}{T}\right) \int_T^{2T} |F(b+it)|^2\, dt \ll \eta^2 \log^{-10B} x. \tag{9}$$

Proof. Let $m = pq$ and $n = r$. An application of Lemma 3 yields that (9) holds under one of the following conditions:

(a) $M \ll X^{\frac{9}{32}}, \qquad N \ll M^{\frac{2}{3}} X^{\frac{1}{10}};$

(b) $X^{\frac{9}{32}} \ll M \ll X^{\frac{53}{160}}, \qquad N \ll X^{\frac{23}{80}};$

(c) $X^{\frac{53}{160}} \ll M \ll X^{\frac{13}{32}}, \qquad N \ll M^{-\frac{2}{3}} X^{\frac{61}{120}};$

(d) $X^{\frac{13}{32}} \ll M \ll X^{\frac{19}{45}}, \qquad N \ll M^{-2} X^{\frac{21}{20}};$

(e) $X^{\frac{19}{45}} \ll M \ll X^{\frac{897}{1972}}, \qquad N \ll M^{-2} X^{\frac{21}{20}}.$

Using the same discussion as in Lemma 8 with regions (i) and (ii), we deduce that (9) holds under the condition

(f) $X^{\frac{19}{45}} \ll M \ll X^{\frac{897}{1972}}, \qquad M^{-2} X^{\frac{21}{20}} \ll N \ll M^{-\frac{1}{5}} X^{\frac{29}{100}}.$

In the regions:

$$X^{\frac{9}{85}} \ll P \ll X^{\frac{9}{64}}, \qquad X^{\frac{9}{85}} \ll Q \ll P;$$
$$X^{\frac{9}{64}} \ll P \ll X^{\frac{477}{2720}}, \qquad X^{\frac{9}{85}} \ll Q \ll P^{-1} X^{\frac{9}{32}},$$

condition (a) is satisfied.

In the regions:

$$X^{\frac{9}{64}} \ll P \ll X^{\frac{53}{320}}, \qquad P^{-1} X^{\frac{9}{32}} \ll Q \ll P;$$
$$X^{\frac{53}{320}} \ll P \ll X^{\frac{477}{2720}}, \qquad P^{-1} X^{\frac{9}{32}} \ll Q \ll P^{-1} X^{\frac{53}{160}};$$
$$X^{\frac{477}{2720}} \ll P \ll X^{\frac{613}{2720}}, \qquad X^{\frac{9}{85}} \ll Q \ll P^{-1} X^{\frac{53}{160}},$$

condition (b) is satisfied.

In the regions:

$$X^{\frac{53}{320}} \ll P \ll X^{\frac{13}{64}}, \qquad P^{-1} X^{\frac{53}{160}} \ll Q \ll P;$$
$$X^{\frac{13}{64}} \ll P \ll X^{\frac{613}{2720}}, \qquad P^{-1} X^{\frac{53}{160}} \ll Q \ll P^{-1} X^{\frac{13}{32}};$$
$$X^{\frac{613}{2720}} \ll P \ll X^{\frac{817}{2720}}, \qquad X^{\frac{9}{85}} \ll Q \ll P^{-1} X^{\frac{13}{32}},$$

condition (c) is satisfied.

In the regions:

$$X^{\frac{13}{64}} \ll P \ll X^{\frac{21}{100}}, \qquad P^{-1}X^{\frac{13}{32}} \ll Q \ll P;$$
$$X^{\frac{21}{100}} \ll P \ll X^{\frac{13}{60}}, \qquad P^{-1}X^{\frac{13}{32}} \ll Q \ll P^{-\frac{2}{3}}X^{\frac{7}{20}};$$
$$X^{\frac{13}{60}} \ll P \ll X^{\frac{817}{2720}}, \qquad P^{-1}X^{\frac{13}{32}} \ll Q \ll P^{-1}X^{\frac{19}{45}};$$
$$X^{\frac{817}{2720}} \ll P \ll X^{\frac{242}{765}}, \qquad X^{\frac{9}{85}} \ll Q \ll P^{-1}X^{\frac{19}{45}},$$

condition (d) is satisfied.

In the regions:

$$X^{\frac{13}{60}} \ll P \ll X^{\frac{12613}{49300}}, \qquad P^{-1}X^{\frac{19}{45}} \ll Q \ll P^{-\frac{1}{6}}X^{\frac{29}{120}};$$
$$X^{\frac{12613}{49300}} \ll P \ll X^{\frac{242}{765}}, \qquad P^{-1}X^{\frac{19}{45}} \ll Q \ll P^{-1}X^{\frac{897}{1972}};$$
$$X^{\frac{242}{765}} \ll P \ll X^{\frac{3441}{9860}}, \qquad X^{\frac{9}{85}} \ll Q \ll P^{-1}X^{\frac{897}{1972}},$$

conditions (e) and (f) are satisfied.

Putting together the above regions, we get Lemma 14.

LEMMA 15. *Under the assumption of Lemma* 14, *assume also that*

$$X^{\frac{9}{85}} \ll R \ll Q$$

and that P and Q lie in one of the following regions:

(i) $$X^{\frac{297}{800}} \ll P \ll X^{\frac{653}{1700}}, \qquad P^{-1}X^{\frac{49}{100}} \ll Q \ll X^{\frac{19}{160}};$$
(ii) $$X^{\frac{653}{1700}} \ll P \ll X^{\frac{5073}{13120}}, \qquad X^{\frac{9}{85}} \ll Q \ll X^{\frac{19}{160}};$$
(iii) $$X^{\frac{5073}{13120}} \ll P \ll X^{\frac{5541}{13940}}, \qquad X^{\frac{9}{85}} \ll Q \ll P^{-\frac{82}{69}}X^{\frac{133}{230}}.$$

Then (9) *holds for $T_1 \le T \le X$.*

Proof. Let

$$m = pq \quad \text{and} \quad n = r.$$

An application of Lemma 4 yields that (9) holds under the conditions

$$Q \ll P^{-1}X^{\frac{21}{40}}, \qquad X^{\frac{9}{85}} \ll R \ll Q \ll X^{\frac{19}{160}}.$$

In the regions:

$$X^{\frac{297}{800}} \ll P \ll X^{\frac{653}{1700}}, \qquad P^{-1}X^{\frac{49}{100}} \ll Q \ll X^{\frac{19}{160}};$$
$$X^{\frac{653}{1700}} \ll P \ll X^{\frac{5073}{13120}}, \qquad X^{\frac{9}{85}} \ll Q \ll X^{\frac{19}{160}};$$
$$X^{\frac{5073}{13120}} \ll P \ll X^{\frac{5541}{13940}}, \qquad X^{\frac{9}{85}} \ll Q \ll P^{-\frac{82}{69}}X^{\frac{133}{230}},$$

the above conditions on P and Q are satisfied.

So the proof of Lemma 15 is complete.

5. The remainder term in the sieve method

LEMMA 16. *Suppose that* $M \ll X^{\frac{21}{40}}$, $H \ll X^{\frac{19}{160}}$ *and that* $a(m) = O(1)$, $b(h) = O(1)$. *Then for real numbers* $x \in (X, 2X)$, *except for a set the measure of which is* $O(X \log^{-B} X)$, *we have*

$$\Sigma = \sum_{\substack{m\sim M\\ h\sim H}} a(m)b(h)\bigg(\sum_{x<mhl\le x+\eta x} 1 - \frac{\eta x}{mh}\bigg) = O(\eta x \log^{-B} x).$$

Proof. If $MH \le X^{\frac{1}{20}}$, the conclusion is obvious. In the following we suppose

$$MH > X^{\frac{1}{20}}. \tag{10}$$

Let $b = 1 + 1/\log X$ and $MHL = X$. Suppose that $M(s)$ and $H(s)$ are Dirichlet polynomials, $L(s) = \sum_{l\sim L} 1/l^s$ and $F(s) = M(s)H(s)L(s)$. Perron's formula yields

$$\sum_{\substack{x<mhl\le x+\eta x\\ m\sim M,\, h\sim H}} a(m)b(h) = \frac{1}{2\pi i}\int_{b-iX}^{b+iX} F(s)\frac{(1+\eta)^s - 1}{s} x^s\, ds + O(x^{\varepsilon}).$$

If $s = b + it$ and $|t| \le c_1 L$, by Theorem 1 on page 442 of [18], we have

$$\sum_{c_1L<l\le c_2L} \frac{1}{l^s} = \frac{(c_2L)^{1-s} - (c_1L)^{1-s}}{1-s} + O\bigg(\frac{1}{L}\bigg).$$

Moreover,

$$\frac{(1+\eta)^s - 1}{s} x^s = \eta x^s + O(|s|\eta^2 x), \qquad \frac{(1+\eta)^s - 1}{s} x^s \ll \eta x.$$

Let $T_1 = \sqrt{L}$. Then

$$\frac{1}{2\pi i}\int_{b-iT_1}^{b+iT_1} F(s)\frac{(1+\eta)^s - 1}{s} x^s\, ds$$
$$= \frac{\eta}{2\pi i}\int_{b-iT_1}^{b+iT_1} M(s)H(s)\frac{(c_2L)^{1-s} - (c_1L)^{1-s}}{1-s} x^s\, ds + O(S_1) + O(S_2),$$

where

$$S_1 = \frac{\eta x}{L}\int_{-T_1}^{T_1} |M(b+it)H(b+it)|\, dt,$$

$$S_2 = \eta^2 x \int_{-T_1}^{T_1} |M(b+it)H(b+it)|\, dt.$$

A trivial estimate yields

$$S_1 \ll \frac{\eta x}{\sqrt{L}} \ll \eta x^{1-\varepsilon} \quad \text{and} \quad S_2 \ll \eta^2 x\sqrt{L} \ll \eta x^{1-\varepsilon}.$$

By Perron's formula again,

$$\frac{\eta}{2\pi i}\int_{b-iT_1}^{b+iT_1} M(s)H(s)\frac{(c_2L)^{1-s}-(c_1L)^{1-s}}{1-s}x^s\,ds$$
$$= \eta x \sum_{\substack{m\sim M\\ h\sim H}} \frac{a(m)b(h)}{mh} + O\biggl(\frac{\eta x^{1+\varepsilon}}{T_1}\biggr) + O\biggl(\frac{\eta x^{1+\varepsilon}}{MH}\biggr)$$
$$= \eta x \sum_{\substack{m\sim M\\ h\sim H}} \frac{a(m)b(h)}{mh} + O(\eta x^{1-\varepsilon}).$$

Now we have

$$(11)\qquad \Sigma = \sum_{\substack{x<mhl\le x+\eta x\\ m\sim M,\, h\sim H}} a(m)b(h) - \eta x \sum_{\substack{m\sim M\\ h\sim H}} \frac{a(m)b(h)}{mh}$$
$$= \frac{1}{2\pi i}\int_{b-iX}^{b-iT_1} F(s)\varrho(s)x^s\,ds + \frac{1}{2\pi i}\int_{b+iT_1}^{b+iX} F(s)\varrho(s)x^s\,ds + O(\eta x^{1-\varepsilon}),$$

where

$$\varrho(s) = \frac{(1+\eta)^s-1}{s}.$$

If $s=b+it$ and $|\mathrm{Im}(s)|\sim T$, then $\varrho(s)\ll\min(\eta,1/T)$. We proceed to estimate

$$\Psi = \int_X^{2X} dx\biggl|\int_{b+iT_1}^{b+iX} F(s)\varrho(s)x^s\,ds\biggr|^2.$$

We have

$$\Psi \ll \log^2 x \max_{T_1\le T\le X} \int_X^{2X} dx\biggl|\int_{b+iT}^{b+2iT} F(s)\varrho(s)x^s\,ds\biggr|^2.$$

Here

$$\int_X^{2X} dx\biggl|\int_{b+iT}^{b+2iT} F(s)\varrho(s)x^s\,ds\biggr|^2$$

$$= \int_X^{2X} dx \int_{b+iT}^{b+2iT} ds_1 \int_{b+iT}^{b+2iT} F(s_1)\overline{F(s_2)}\varrho(s_1)\overline{\varrho(s_2)}x^{s_1+\bar{s}_2}\,\overline{ds_2}$$

$$\ll \min{}^2\left(\eta, \frac{1}{T}\right) \int_{b+iT}^{b+2iT} |ds_1| \int_{b+iT}^{b+2iT} |F(s_1)F(s_2)| \left| \int_X^{2X} x^{s_1+\bar{s}_2}\,dx \right| |ds_2|$$

$$\ll x^3 \min{}^2\left(\eta, \frac{1}{T}\right) \int_{b+iT}^{b+2iT} |ds_1| \int_{b+iT}^{b+2iT} \frac{|F(s_1)|^2 + |F(s_2)|^2}{|1+s_1+\bar{s}_2|} |ds_2|$$

$$\ll x^3 \log x \min{}^2\left(\eta, \frac{1}{T}\right) \int_T^{2T} |F(b+it)|^2\,dt.$$

By Lemma 4,

$$\Psi \ll x^3 \log^3 x \max_{T_1 \le T \le X} \min{}^2\left(\eta, \frac{1}{T}\right) \int_T^{2T} |F(b+it)|^2\,dt \ll \eta^2 x^{3-\varepsilon_1}.$$

Hence, the measure of the set of x satisfying

$$\left| \int_{b+iT_1}^{b+iX} F(s)\varrho(s)x^s\,ds \right| \ge \eta x \log^{-B} x$$

is $O(X \log^{-B} X)$.

In the same way, we can deal with the integral

$$\int_{b-iX}^{b-iT_1} F(s)\varrho(s)x^s\,ds.$$

So the proof of Lemma 16 is complete.

LEMMA 17. *Suppose that* $a(p)$, $b(q)$, $c(r) = O(1)$ *and that*

$$X^{\frac{9}{85}} \ll R \ll Q.$$

Moreover, assume that P *and* Q *lie in one of the following regions*:

(i) $X^{\frac{9}{85}} \ll P \ll X^{\frac{21}{100}}$, $X^{\frac{9}{85}} \ll Q \ll P$;

(ii) $X^{\frac{21}{100}} \ll P \ll X^{\frac{13}{60}}$, $X^{\frac{9}{85}} \ll Q \ll P^{-\frac{2}{3}} X^{\frac{7}{20}}$;

(iii) $X^{\frac{13}{60}} \ll P \ll X^{\frac{12613}{49300}}$, $X^{\frac{9}{85}} \ll Q \ll P^{-\frac{1}{6}} X^{\frac{29}{120}}$;

(iv) $X^{\frac{12613}{49300}} \ll P \ll X^{\frac{3441}{9860}}$, $X^{\frac{9}{85}} \ll Q \ll P^{-1} X^{\frac{897}{1972}}$;

(v) $X^{\frac{297}{800}} \ll P \ll X^{\frac{653}{1700}}$, $P^{-1} X^{\frac{49}{100}} \ll Q \ll X^{\frac{19}{160}}$;

(vi) $X^{\frac{653}{1700}} \ll P \ll X^{\frac{5073}{13120}}$, $X^{\frac{9}{85}} \ll Q \ll X^{\frac{19}{160}}$;

(vii) $X^{\frac{5073}{13120}} \ll P \ll X^{\frac{5541}{13940}}$, $X^{\frac{9}{85}} \ll Q \ll P^{-\frac{82}{69}} X^{\frac{133}{230}}$.

Let $b = 1 + 1/\log X$, $PQRL = X$ *and* $T_1 = \sqrt{L}$. *Assume further that for* $T_1 \le |t| \le 2X$,

$$P(b+it)Q(b+it) \ll \log^{-\frac{B}{\varepsilon}} x \quad \text{and} \quad R(b+it) \ll \log^{-\frac{B}{\varepsilon}} x.$$

Then for real numbers $x \in (X, 2X)$, *except for a set the measure of which is* $O(X \log^{-B} X)$, *we have*

$$\sum_{p\sim P}\sum_{q\sim Q}\sum_{r\sim R} a(p)b(q)c(r)\Big(\sum_{x<pqrl\le x+\eta x} 1 - \frac{\eta x}{pqr}\Big) = O(\eta x \log^{-B} x).$$

Proof. Using Lemmas 14, 15 and the discussion in Lemma 16, we get the assertion.

6. Asymptotic formula

LEMMA 18. *Suppose that* $X^{\frac{76}{85}} \ll M \ll X^{1-\delta}$ *and that* $0 \le a(m) = O(1)$. *If* m *has a prime factor* $< X^{\delta}$, *then* $a(m) = 0$. *Then for real numbers* $x \in (X, 2X)$, *except for a set the measure of which is* $O(X \log^{-B} X)$, *we have*

$$\Sigma = \sum_{\substack{x<mp\le x+\eta x\\ m\sim M}} a(m) = \eta\Big(1 + O\Big(\frac{1}{\log x}\Big)\Big)\sum_{m\sim M} a(m) \sum_{\frac{x}{m}<p\le\frac{2x}{m}} 1 + O(\eta x \log^{-B} x).$$

Proof. Let $b = 1 + 1/\log X$ and $MH = X$. Suppose that $M(s)$ is a Dirichlet polynomial, $H(s) = \sum_{h\sim H} \Lambda(h)/h^s$ and $G(s) = M(s)H(s)$. Perron's formula yields

$$\Sigma_1 = \sum_{\substack{x<mh\le x+\eta x\\ m\sim M}} a(m)\Lambda(h) = \frac{1}{2\pi i}\int_{b-iX}^{b+iX} G(s)\frac{(1+\eta)^s - 1}{s} x^s\,ds + O(x^{\varepsilon}).$$

Let $T_0 = \log^{\frac{B}{\varepsilon}} X$. By (1) and the discussion in Lemma 16,

$$\frac{1}{2\pi i}\int_{b-iT_0}^{b+iT_0} G(s)\frac{(1+\eta)^s - 1}{s} x^s\,ds = \frac{\eta}{2\pi i}\int_{b-iT_0}^{b+iT_0} M(s)\frac{(c_2H)^{1-s} - (c_1H)^{1-s}}{1-s} x^s\,ds + O(S_1) + O(S_2),$$

where

$$S_1 = \eta x \log^{-\frac{2B}{\varepsilon}} x \int_{-T_0}^{T_0} |M(b+it)|\,dt, \qquad S_2 = \eta^2 x \int_{-T_0}^{T_0} |M(b+it)|\,dt.$$

A trivial estimate yields

$$S_1 \ll \eta x \log^{-2B} x \quad \text{and} \quad S_2 \ll \eta x \log^{-2B} x.$$

By Perron's formula again,

$$\begin{aligned}\frac{\eta}{2\pi i} \int_{b-iT_0}^{b+iT_0} M(s) \frac{(c_2H)^{1-s} - (c_1H)^{1-s}}{1-s} x^s \, ds \\ &= \eta x \sum_{m\sim M} \frac{a(m)}{m} + O\left(\frac{\eta x \log^2 x}{T_0}\right) + O\left(\frac{\eta x \log^2 x}{M}\right) \\ &= \eta x \sum_{m\sim M} \frac{a(m)}{m} + O(\eta x \log^{-2B} x).\end{aligned}$$

Hence,

$$\begin{aligned}\Sigma_1 = \eta x \sum_{m\sim M} \frac{a(m)}{m} &+ O(\eta x \log^{-2B} x) \\ &+ \frac{1}{2\pi i} \int_{b-iX}^{b-iT_0} G(s)\varrho(s) x^s \, ds + \frac{1}{2\pi i} \int_{b+iT_0}^{b+iX} G(s)\varrho(s) x^s \, ds,\end{aligned}$$

where

$$\varrho(s) = \frac{(1+\eta)^s - 1}{s}.$$

By Lemma 1 and the discussion in Lemma 16, we have

$$\frac{1}{2\pi i} \int_{b+iT_0}^{b+iX} G(s)\varrho(s) x^s \, ds = O(\eta x \log^{-2B} x)$$

and

$$\frac{1}{2\pi i} \int_{b-iX}^{b-iT_0} G(s)\varrho(s) x^s \, ds = O(\eta x \log^{-2B} x)$$

for real numbers $x \in (X, 2X)$, except for a set the measure of which is $O(X \log^{-B} X)$. So,

$$\begin{aligned}\Sigma_1 &= \eta x \sum_{m\sim M} \frac{a(m)}{m} + O(\eta x \log^{-2B} x), \\ \Sigma &= \eta x \left(1 + O\left(\frac{1}{\log x}\right)\right) \sum_{m\sim M} \frac{a(m)}{m \log(x/m)} + O(\eta x \log^{-B} x) \\ &= \eta \left(1 + O\left(\frac{1}{\log x}\right)\right) \sum_{m\sim M} a(m) \sum_{\frac{x}{m} < p \le \frac{2x}{m}} 1 + O(\eta x \log^{-B} x).\end{aligned}$$

The proof of Lemma 18 is complete.

LEMMA 19. *Suppose that* $X^{\frac{9}{85}} \ll HK \ll X^{\frac{76}{85}}$, $MHK = X$ *and that* $0 \le b(h) = O(1)$, $0 \le g(k) = O(1)$. *If* h *has a prime factor* $< X^{\delta}$, *then* $b(h) = 0$, *and similarly for* $g(k)$. *Suppose that* $H(s)$ *and* $K(s)$ *are Dirichlet polynomials*, $M(s) = \sum_{m\sim M} \Lambda(m)/m^s$ *and* $G(s) = M(s)H(s)K(s)$. *Let* $b = 1 + 1/\log X$ *and* $T_0 = \log^{\frac{B}{\varepsilon}} X$.

If (7) *holds for* $T_0 \le T \le X$, *then*

$$\sum_{\substack{x<hkp\le x+\eta x\\ h\sim H,\, k\sim K}} b(h)g(k)$$
$$= \eta\left(1 + O\left(\frac{1}{\log x}\right)\right) \sum_{h\sim H,\, k\sim K} b(h)g(k) \sum_{\frac{x}{hk}<p\le\frac{2x}{hk}} 1 + O(\eta x \log^{-B} x).$$

7. Buchstab's function. We define $w(u)$ as the continuous solution of the equations

$$\begin{cases} w(u) = 1/u, & 1 \le u \le 2,\\ (uw(u))' = w(u-1), & u > 2. \end{cases} \tag{12}$$

$w(u)$ is called *Buchstab's function* and plays an important role in finding asymptotic formulas in the sieve method. In particular,

$$w(u) = \frac{1+\log(u-1)}{u}, \quad 2 \le u \le 3,$$
$$w(u) = \frac{1+\log(u-1)}{u} + \frac{1}{u}\int_2^{u-1} \frac{\log(t-1)}{t}\,dt, \quad 3 \le u \le 4,$$
$$w(u) = \frac{1+\log(u-1)}{u} + \frac{1}{u}\int_2^{u-1} \frac{\log(t-1)}{t}\,dt + \frac{1}{u}\int_3^{u-1}\frac{dt}{t}\int_2^{t-1}\frac{\log(s-1)}{s}\,ds, \quad 4 \le u \le 5.$$

LEMMA 20. *We have the following bounds*:

(i) $w(u) \ge 0.5607$ *for* $u \ge 2.47$,
(ii) $w(u) \le 0.5644$ *for* $u \ge 3$,
(iii) $0.5612 \le w(u) \le 0.5617$ *for* $u \ge 4$.

Proof. It is easy to see that $0.5 \le w(u) \le 1$ for $1 \le u \le 2$. Then we employ induction.

Suppose that $0.5 \le w(u) \le 1$ for $1 \le k \le u \le k+1$. If $k+1 \le u \le k+2$, then (12) yields

$$uw(u) = (k+1)w(k+1) + \int_k^{u-1} w(t)\,dt. \tag{13}$$

Hence, $0.5 \le w(u) \le 1$ for $k+1 \le u \le k+2$. By induction, we obtain $0.5 \le w(u) \le 1$ for $u \ge 1$.

If $u > 2$, (12) yields

$$w'(u) = \frac{w(u-1) - w(u)}{u}. \tag{14}$$

If $2 \le u \le 3$, by calculation, we have

$$\max_{0 \le k \le 10^4} w(2 + 10^{-4}k) \le 0.56716.$$

From (14) and $0.5 \le w(u) \le 1$ for $u \ge 1$, it follows that $|w'(u)| \le \frac{1}{4}$ if $u > 2$. Using Lagrange's mean value theorem, we have $w(u) \le 0.5672$ for $2 \le u \le 3$. By induction, we obtain $0.5 \le w(u) \le 0.5672$ for $u \ge 2$.

If $3 \le u \le 4$, by calculation, we have

$$\max_{0 \le k \le 10^4} w(3 + 10^{-4}k) \le 0.56439,$$
$$\min_{0 \le k \le 10^4} w(3 + 10^{-4}k) \ge 0.56081.$$

From (14) and $0.5 \le w(u) \le 0.5672$ for $u \ge 2$, it follows that $|w'(u)| \le 0.0224$ if $u > 3$. The above discussion implies that $0.5607 \le w(u) \le 0.5644$ for $u \ge 3$. By the same discussion we can also get $0.5607 \le w(u)$ for $u \ge 2.47$.

If $4 \le u \le 5$, by calculation, we have

$$\max_{0 \le k \le 10^4} w(4 + 10^{-4}k) \le 0.5616, \qquad \min_{0 \le k \le 10^4} w(4 + 10^{-4}k) \ge 0.5613.$$

The above discussion and the fact that $|w'(u)| \le 0.0224$ for $u > 3$ imply that $0.5612 \le w(u) \le 0.5617$ for $u \ge 4$.

Gathering together the above discussion, we get Lemma 20.

LEMMA 21. *Suppose that $\mathcal{E} = \{n : t < n \le 2t\}$ and $z \le t$. Let*

$$P(z) = \prod_{p<z} p.$$

Then for sufficiently large t and z, we have

$$S(\mathcal{E}, z) = \sum_{\substack{t<n\le 2t \\ (n,P(z))=1}} 1 = \left(w\left(\frac{\log t}{\log z}\right) + O(\varepsilon)\right)\frac{t}{\log z}.$$

Proof. See Lemma 5 of [10]. If $(2t)^{\frac{1}{2}} < z \le t$, it is the prime number theorem.

8. Sieve method. We proceed to show that

$$\pi(x+\eta x)-\pi(x)\ge 0.011\eta x\log^{-1}x \tag{15}$$

for real numbers $x\in(X,2X)$, except for a set the measure of which is $O(X\log^{-B}X)$.

Let

$$\mathcal{A}=\{n: x<n\le x+\eta x\}, \tag{16}$$
$$P(z)=\prod_{p<z}p,\qquad S(\mathcal{A},z)=\sum_{\substack{a\in\mathcal{A}\\(a,P(z))=1}}1.$$

Then

$$\pi(x+\eta x)-\pi(x)=S(\mathcal{A},(2X)^{\frac12}). \tag{17}$$

Buchstab's identity yields

$$\begin{aligned}S(\mathcal{A},(2X)^{\frac12})&=S(\mathcal{A},X^{\frac{9}{85}})-\sum_{X^{\frac{9}{85}}<p\le(2X)^{\frac12}}S(\mathcal{A}_p,p)\\&=S(\mathcal{A},X^{\frac{9}{85}})-\sum_{X^{\frac{9}{85}}<p\le(2X)^{\frac12}}S(\mathcal{A}_p,X^{\frac{9}{85}})\\&\quad+\sum_{X^{\frac{9}{85}}<p\le(2X)^{\frac12}}\ \sum_{X^{\frac{9}{85}}<q<\min(p,(2X/p)^{\frac12})}S(\mathcal{A}_{pq},q).\end{aligned} \tag{18}$$

The following lemmas always concern real numbers $x\in(X,2X)$, except for a set the measure of which is $O(X\log^{-B}X)$.

Let

$$\mathcal{B}=\{n: x<n\le 2x\}. \tag{19}$$

LEMMA 22.

$$S(\mathcal{A},X^{\frac{9}{85}})\ge 5.300221\eta x\log^{-1}x.$$

Proof. We have

$$S(\mathcal{A},X^{\frac{9}{85}})=S(\mathcal{A},X^{\delta})-\sum_{X^{\delta}<p\le X^{\frac{9}{85}}}S(\mathcal{A}_p,p). \tag{20}$$

Let

$$\widetilde{r}(\mathcal{A},d)=|\mathcal{A}_d|-\frac{\eta x}{d},\qquad W(z)=\prod_{p<z}\left(1-\frac1p\right)=(1+O(\varepsilon))\frac{e^{-\gamma}}{\log z},$$

where γ is Euler's constant.

Let $z = X^{\delta}$ and $D = X^{\frac{21}{40}}$. Applying Iwaniec's sieve method (see Theorem 1 of [8]), we have

$$S(\mathcal{A}, X^{\delta}) \geq \frac{\eta x}{\log z} f\left(\frac{\log D}{\log z}\right) - O(\varepsilon \eta x \log^{-1} x) - R^{-},$$

where

$$R^{-} = \sum_{m \leq X^{\frac{21}{40}}} a(m) \widetilde{r}(\mathcal{A}, m).$$

Lemma 16 yields $R^{-} = O(\eta x \log^{-5} x)$. By Theorem 8 on page 181 of [17], we have

$$f\left(\frac{\log D}{\log z}\right) = e^{-\gamma} + O(\varepsilon^2),$$

where γ is Euler's constant. Thus,

$$S(\mathcal{A}, X^{\delta}) \geq \frac{e^{-\gamma}}{\delta} \eta x \log^{-1} x + O(\varepsilon \eta x \log^{-1} x).$$

In the same way,

$$\begin{aligned} S(\mathcal{A}, X^{\delta}) &\leq \frac{\eta x}{\log z} F\left(\frac{\log D}{\log z}\right) + O(\varepsilon \eta x \log^{-1} x) + \sum_{m \leq X^{\frac{21}{40}}} b(m) \widetilde{r}(\mathcal{A}, m) \\ &\leq \frac{e^{-\gamma}}{\delta} \eta x \log^{-1} x + O(\varepsilon \eta x \log^{-1} x). \end{aligned}$$

So, we have the asymptotic formula

$$S(\mathcal{A}, X^{\delta}) = \frac{e^{-\gamma}}{\delta} \eta x \log^{-1} x + O(\varepsilon \eta x \log^{-1} x). \tag{21}$$

Now,

$$\sum_{X^{\delta} < p \leq X^{\frac{9}{85}}} S(\mathcal{A}_p, p) = \sum_{x < pq \leq x + \eta x} 1,$$

where $X^{\delta} < p \leq X^{\frac{9}{85}}$ and the least prime factor of q is greater than p.

Using Lemmas 18 and 21 and the prime number theorem, we have

$$\begin{aligned} &\sum_{X^{\delta} < p \leq X^{\frac{9}{85}}} S(\mathcal{A}_p, p) \\ &\quad = \eta \sum_{X^{\delta} < p \leq X^{\frac{9}{85}}} S(\mathcal{B}_p, p) + O(\varepsilon \eta x \log^{-1} x) \\ &\quad = \eta x \sum_{X^{\delta} < p \leq X^{\frac{9}{85}}} \frac{1}{p \log p} w\left(\frac{\log(x/p)}{\log p}\right) + O(\delta \eta x \log^{-1} x) \end{aligned} \tag{22}$$

$$= \eta x \log^{-1} x \int_{\delta}^{\frac{9}{85}} \frac{1}{t^2} w\left(\frac{1-t}{t}\right) dt + O(\delta\eta x \log^{-1} x)$$

$$= \eta x \log^{-1} x \int_{\frac{85}{9}}^{\frac{1}{\delta}} w(u-1)\, du + O(\delta\eta x \log^{-1} x)$$

$$= \eta x \log^{-1} x \left\{ \frac{1}{\delta} w\left(\frac{1}{\delta}\right) - \frac{85}{9} w\left(\frac{85}{9}\right) \right\} + O(\delta\eta x \log^{-1} x)$$

$$= \frac{e^{-\gamma}}{\delta} \eta x \log^{-1} x - \frac{85}{9} w\left(\frac{85}{9}\right) \eta x \log^{-1} x + O(\delta\eta x \log^{-1} x),$$

since $w(1/\delta) = e^{-\gamma} + O(\varepsilon^2)$ (see Lemma 12 on page 179 of [17]). Hence,

$$S(\mathcal{A}, X^{\frac{9}{85}}) = \frac{85}{9} w\left(\frac{85}{9}\right) \eta x \log^{-1} x + O(\delta\eta x \log^{-1} x). \tag{23}$$

By Lemma 20, we get

$$S(\mathcal{A}, X^{\frac{9}{85}}) \geq \frac{85}{9} \cdot 0.5612 \eta x \log^{-1} x + O(\delta\eta x \log^{-1} x) \geq 5.300221 \eta x \log^{-1} x.$$

So the proof of Lemma 22 is complete.

Lemma 23.

$$\sum_{X^{\frac{9}{85}} < p \leq (2X)^{\frac{1}{2}}} S(\mathcal{A}_p, X^{\frac{9}{85}}) \leq 8.234757 \eta x \log^{-1} x.$$

Proof. Buchstab's identity yields

$$\sum_{X^{\frac{9}{85}} < p \leq (2X)^{\frac{1}{2}}} S(\mathcal{A}_p, X^{\frac{9}{85}})$$

$$= \sum_{X^{\frac{9}{85}} < p \leq (2X)^{\frac{1}{2}}} S(\mathcal{A}_p, X^{\delta}) - \sum_{X^{\frac{9}{85}} < p \leq (2X)^{\frac{1}{2}}} \sum_{X^{\delta} < q < X^{\frac{9}{85}}} S(\mathcal{A}_{pq}, q).$$

Using Lemma 16, in the same way as in Lemma 22, we have

$$\sum_{X^{\frac{9}{85}} < p \leq (2X)^{\frac{1}{2}}} S(\mathcal{A}_p, X^{\delta})$$

$$= \sum_{X^{\frac{9}{85}} < p \leq (2X)^{\frac{1}{2}}} \frac{e^{-\gamma}}{\delta} \cdot \frac{1}{p} \cdot \eta x \log^{-1} x + O(\varepsilon\eta x \log^{-1} x)$$

$$= \frac{e^{-\gamma}}{\delta} \cdot \eta x \log^{-1} x \int_{\frac{9}{85}}^{\frac{1}{2}} \frac{dt}{t} + O(\varepsilon\eta x \log^{-1} x).$$

Using Lemmas 18 and 21, in the same way as in Lemma 22, we have

$$\sum_{X^{\frac{9}{85}}<p\le(2X)^{\frac12}}\ \sum_{X^{\delta}<q<X^{\frac{9}{85}}} S(\mathcal{A}_{pq},q)$$
$$=\eta\sum_{X^{\frac{9}{85}}<p\le(2X)^{\frac12}}\ \sum_{X^{\delta}<q<X^{\frac{9}{85}}} S(\mathcal{B}_{pq},q)+O(\varepsilon\eta x\log^{-1}x)$$
$$=\eta x\sum_{X^{\frac{9}{85}}<p\le(2X)^{\frac12}}\ \sum_{X^{\delta}<q<X^{\frac{9}{85}}}\frac{1}{pq\log q}w\left(\frac{\log(x/(pq))}{\log q}\right)$$
$$+O(\delta\eta x\log^{-1}x)$$
$$=\eta x\log^{-1}x\int_{\frac{9}{85}}^{\frac12}\frac{dt}{t}\int_{\delta}^{\frac{9}{85}}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du+O(\delta\eta x\log^{-1}x)$$
$$=\eta x\log^{-1}x\int_{\frac{9}{85}}^{\frac12}\frac{dt}{t(1-t)}\int_{\frac{85}{9}(1-t)}^{\frac{1}{\delta}(1-t)}w(r-1)\,dr+O(\delta\eta x\log^{-1}x)$$
$$=\frac{1}{\delta}\cdot\eta x\log^{-1}x\int_{\frac{9}{85}}^{\frac12}\frac{1}{t}w\left(\frac{1}{\delta}(1-t)\right)dt$$
$$-\frac{85}{9}\cdot\eta x\log^{-1}x\int_{\frac{9}{85}}^{\frac12}\frac{1}{t}w\left(\frac{85}{9}(1-t)\right)dt+O(\delta\eta x\log^{-1}x)$$
$$=\frac{e^{-\gamma}}{\delta}\cdot\eta x\log^{-1}x\int_{\frac{9}{85}}^{\frac12}\frac{dt}{t}-\frac{85}{9}\cdot\eta x\log^{-1}x\int_{\frac{9}{85}}^{\frac12}\frac{1}{t}w\left(\frac{85}{9}(1-t)\right)dt$$
$$+O(\delta\eta x\log^{-1}x).$$

Gathering together the above discussion and applying Lemma 20, we have

$$\sum_{X^{\frac{9}{85}}<p\le(2X)^{\frac12}} S(\mathcal{A}_p,X^{\frac{9}{85}})$$
$$=\frac{85}{9}\cdot\eta x\log^{-1}x\int_{\frac{9}{85}}^{\frac12}\frac{1}{t}w\left(\frac{85}{9}(1-t)\right)dt+O(\delta\eta x\log^{-1}x)$$

$$\leq 0.5617 \cdot \frac{85}{9} \cdot \log\left(\frac{85}{18}\right)\eta x \log^{-1} x + O(\delta\eta x \log^{-1} x)$$
$$\leq 8.234757\eta x \log^{-1} x.$$

So the proof of Lemma 23 is complete.

We now set

$$\Omega = \sum_{X^{\frac{9}{85}} < p \leq (2X)^{\frac{1}{2}}} \sum_{X^{\frac{9}{85}} < q < \min(p, (\frac{2X}{p})^{\frac{1}{2}})} S(\mathcal{A}_{pq}, q) \tag{24}$$
$$\geq \sum_{i=1}^{94} \sum_{(p,q)\in D_i} S(\mathcal{A}_{pq}, q) = \sum_{i=1}^{94} \Omega_i,$$

where

$$D_1 = \{(p,q) : X^{\frac{9}{85}} < p \leq X^{\frac{21}{100}},\ X^{\frac{9}{85}} < q < p\},$$
$$D_2 = \{(p,q) : X^{\frac{21}{100}} < p \leq X^{\frac{13}{60}},\ X^{\frac{9}{85}} < q < p^{-\frac{2}{3}} X^{\frac{7}{20}}\},$$
$$D_3 = \{(p,q) : X^{\frac{13}{60}} < p \leq X^{\frac{133}{580}},\ X^{\frac{9}{85}} < q < p^{-\frac{1}{6}} X^{\frac{29}{120}}\},$$
$$D_4 = \{(p,q) : X^{\frac{133}{580}} < p \leq X^{\frac{19}{80}},\ X^{\frac{9}{85}} < q < p^{-\frac{1}{6}} X^{\frac{29}{120}}\},$$
$$D_5 = \{(p,q) : X^{\frac{133}{580}} < p \leq X^{\frac{19}{80}},\ p^{-1} X^{\frac{133}{290}} < q < p\},$$
$$D_6 = \{(p,q) : X^{\frac{19}{80}} < p \leq X^{\frac{53}{220}},\ X^{\frac{9}{85}} < q < p^{-\frac{1}{6}} X^{\frac{29}{120}}\},$$
$$D_7 = \{(p,q) : X^{\frac{19}{80}} < p \leq X^{\frac{53}{220}},\ p^{-1} X^{\frac{133}{290}} < q < p^{-1} X^{\frac{19}{40}}\},$$
$$D_8 = \{(p,q) : X^{\frac{53}{220}} < p \leq X^{\frac{12613}{49300}},\ X^{\frac{9}{85}} < q < p^{-\frac{1}{6}} X^{\frac{29}{120}}\},$$
$$D_9 = \{(p,q) : X^{\frac{53}{220}} < p \leq X^{\frac{12613}{49300}},\ p^{-1} X^{\frac{133}{290}} < q < p^{-1} X^{\frac{53}{110}}\},$$
$$D_{10} = \{(p,q) : X^{\frac{12613}{49300}} < p \leq X^{\frac{151}{580}},\ X^{\frac{9}{85}} < q < p^{-1} X^{\frac{897}{1972}}\},$$
$$D_{11} = \{(p,q) : X^{\frac{12613}{49300}} < p \leq X^{\frac{151}{580}},\ p^{-1} X^{\frac{133}{290}} < q < p^{-1} X^{\frac{53}{110}}\},$$
$$D_{12} = \{(p,q) : X^{\frac{151}{580}} < p \leq X^{\frac{19}{70}},\ X^{\frac{9}{85}} < q < p^{-1} X^{\frac{897}{1972}}\},$$
$$D_{13} = \{(p,q) : X^{\frac{151}{580}} < p \leq X^{\frac{19}{70}},\ p^{-1} X^{\frac{897}{1972}} < q < p^{-1} X^{\frac{53}{110}}\},$$
$$D_{14} = \{(p,q) : X^{\frac{19}{70}} < p \leq X^{\frac{3}{10}},\ X^{\frac{9}{85}} < q < p^{-1} X^{\frac{897}{1972}}\},$$
$$D_{15} = \{(p,q) : X^{\frac{19}{70}} < p \leq X^{\frac{3}{10}},\ p^{-1} X^{\frac{897}{1972}} < q < p^{-1} X^{\frac{53}{110}}\},$$
$$D_{16} = \{(p,q) : X^{\frac{19}{70}} < p \leq X^{\frac{3}{10}},\ p^{-\frac{6}{5}} X^{\frac{57}{100}} < q < p^{-\frac{1}{8}} X^{\frac{23}{80}}\},$$
$$D_{17} = \{(p,q) : X^{\frac{3}{10}} < p \leq X^{\frac{17}{55}},\ X^{\frac{9}{85}} < q < p^{-1} X^{\frac{897}{1972}}\},$$

$$D_{18} = \{(p,q) : X^{\frac{3}{10}} < p \le X^{\frac{17}{55}},\ p^{-1}X^{\frac{897}{1972}} < q < p^{-1}X^{\frac{53}{110}}\},$$

$$D_{19} = \{(p,q) : X^{\frac{3}{10}} < p \le X^{\frac{17}{55}},\ p^{-\frac{12}{11}}X^{\frac{57}{110}} < q < p^{-\frac{1}{8}}X^{\frac{23}{80}}\},$$

$$D_{20} = \{(p,q) : X^{\frac{17}{55}} < p \le X^{\frac{19}{60}},\ X^{\frac{9}{85}} < q < p^{-1}X^{\frac{897}{1972}}\},$$

$$D_{21} = \{(p,q) : X^{\frac{17}{55}} < p \le X^{\frac{19}{60}},\ p^{-1}X^{\frac{897}{1972}} < q < p^{-\frac{58}{67}}X^{\frac{59}{134}}\},$$

$$D_{22} = \{(p,q) : X^{\frac{17}{55}} < p \le X^{\frac{19}{60}},\ p^{-\frac{12}{11}}X^{\frac{57}{110}} < q < p^{-\frac{1}{8}}X^{\frac{23}{80}}\},$$

$$D_{23} = \{(p,q) : X^{\frac{19}{60}} < p \le X^{\frac{247}{770}},\ X^{\frac{9}{85}} < q < p^{-1}X^{\frac{897}{1972}}\},$$

$$D_{24} = \{(p,q) : X^{\frac{19}{60}} < p \le X^{\frac{247}{770}},\ p^{-1}X^{\frac{897}{1972}} < q < p^{-\frac{58}{67}}X^{\frac{59}{134}}\},$$

$$D_{25} = \{(p,q) : X^{\frac{19}{60}} < p \le X^{\frac{247}{770}},\ X^{\frac{19}{110}} < q < p^{-\frac{1}{8}}X^{\frac{23}{80}}\},$$

$$D_{26} = \{(p,q) : X^{\frac{247}{770}} < p \le X^{\frac{359}{1100}},\ X^{\frac{9}{85}} < q < p^{-1}X^{\frac{897}{1972}}\},$$

$$D_{27} = \{(p,q) : X^{\frac{247}{770}} < p \le X^{\frac{359}{1100}},\ p^{-1}X^{\frac{897}{1972}} < q < p^{-\frac{58}{67}}X^{\frac{59}{134}}\},$$

$$D_{28} = \{(p,q) : X^{\frac{247}{770}} < p \le X^{\frac{359}{1100}},\ X^{\frac{19}{110}} < q < p^{-1}X^{\frac{25}{44}}\},$$

$$D_{29} = \{(p,q) : X^{\frac{359}{1100}} < p \le X^{\frac{481}{1450}},\ X^{\frac{9}{85}} < q < p^{-1}X^{\frac{897}{1972}}\},$$

$$D_{30} = \{(p,q) : X^{\frac{359}{1100}} < p \le X^{\frac{481}{1450}},\ p^{-1}X^{\frac{897}{1972}} < q < p^{-\frac{58}{67}}X^{\frac{59}{134}}\},$$

$$D_{31} = \{(p,q) : X^{\frac{359}{1100}} < p \le X^{\frac{481}{1450}},\ X^{\frac{19}{110}} < q < p^{-6}X^{\frac{11}{5}}\},$$

$$D_{32} = \{(p,q) : X^{\frac{481}{1450}} < p \le X^{\frac{443}{1305}},\ X^{\frac{9}{85}} < q < p^{-1}X^{\frac{897}{1972}}\},$$

$$D_{33} = \{(p,q) : X^{\frac{481}{1450}} < p \le X^{\frac{443}{1305}},\ p^{-1}X^{\frac{897}{1972}} < q < p^{-\frac{58}{67}}X^{\frac{59}{134}}\},$$

$$D_{34} = \{(p,q) : X^{\frac{481}{1450}} < p \le X^{\frac{443}{1305}},\ X^{\frac{19}{110}} < q < p^{-1}X^{\frac{157}{290}}\},$$

$$D_{35} = \{(p,q) : X^{\frac{443}{1305}} < p \le X^{\frac{19}{55}},\ X^{\frac{9}{85}} < q < p^{-1}X^{\frac{897}{1972}}\},$$

$$D_{36} = \{(p,q) : X^{\frac{443}{1305}} < p \le X^{\frac{19}{55}},\ p^{-1}X^{\frac{897}{1972}} < q < p^{-\frac{58}{67}}X^{\frac{59}{134}}\},$$

$$D_{37} = \{(p,q) : X^{\frac{443}{1305}} < p \le X^{\frac{19}{55}},\ X^{\frac{19}{110}} < q < p^{-\frac{2}{11}}X^{\frac{29}{110}}\},$$

$$D_{38} = \{(p,q) : X^{\frac{19}{55}} < p \le X^{\frac{3441}{9860}},\ X^{\frac{9}{85}} < q < p^{-1}X^{\frac{897}{1972}}\},$$

$$D_{39} = \{(p,q) : X^{\frac{19}{55}} < p \le X^{\frac{3441}{9860}},\ p^{-1}X^{\frac{897}{1972}} < q < p^{-\frac{58}{67}}X^{\frac{59}{134}}\},$$

$$D_{40} = \{(p,q) : X^{\frac{19}{55}} < p \le X^{\frac{3441}{9860}},\ p^{-\frac{58}{49}}X^{\frac{57}{98}} < q < p^{-\frac{2}{11}}X^{\frac{29}{110}}\},$$

$$D_{41} = \{(p,q) : X^{\frac{3441}{9860}} < p \le X^{\frac{1843}{5280}},\ X^{\frac{9}{85}} < q < p^{-\frac{58}{67}}X^{\frac{59}{134}}\},$$

$$D_{42} = \{(p,q) : X^{\frac{3441}{9860}} < p \le X^{\frac{1843}{5280}},\ p^{-\frac{58}{49}}X^{\frac{57}{98}} < q < p^{-\frac{2}{11}}X^{\frac{29}{110}}\},$$

$$D_{43}=\{(p,q): X^{\frac{1843}{5280}}<p\leq X^{\frac{37}{100}},\ X^{\frac{9}{85}}<q<p^{-\frac{58}{67}}X^{\frac{59}{134}}\},$$

$$D_{44}=\{(p,q): X^{\frac{1843}{5280}}<p\leq X^{\frac{37}{100}},\ p^{-\frac{70}{59}}X^{\frac{171}{295}}<q<p^{-\frac{58}{49}}X^{\frac{57}{98}}\},$$

$$D_{45}=\{(p,q): X^{\frac{1843}{5280}}<p\leq X^{\frac{37}{100}},\ p^{-\frac{58}{49}}X^{\frac{57}{98}}<q<p^{-\frac{2}{11}}X^{\frac{29}{110}}\},$$

$$D_{46}=\{(p,q): X^{\frac{37}{100}}<p\leq X^{\frac{297}{800}},\ X^{\frac{9}{85}}<q<p^{-1}X^{\frac{49}{100}}\},$$

$$D_{47}=\{(p,q): X^{\frac{37}{100}}<p\leq X^{\frac{297}{800}},\ p^{-\frac{82}{69}}X^{\frac{133}{230}}<q<p^{-\frac{58}{49}}X^{\frac{57}{98}}\},$$

$$D_{48}=\{(p,q): X^{\frac{37}{100}}<p\leq X^{\frac{297}{800}},\ p^{-\frac{58}{49}}X^{\frac{57}{98}}<q<p^{-\frac{2}{11}}X^{\frac{29}{110}}\},$$

$$D_{49}=\{(p,q): X^{\frac{297}{800}}<p\leq X^{\frac{2143}{5620}},\ X^{\frac{9}{85}}<q<p^{-1}X^{\frac{49}{100}}\},$$

$$D_{50}=\{(p,q): X^{\frac{297}{800}}<p\leq X^{\frac{2143}{5620}},\ p^{-1}X^{\frac{49}{100}}<q<X^{\frac{19}{160}}\},$$

$$D_{51}=\{(p,q): X^{\frac{297}{800}}<p\leq X^{\frac{2143}{5620}},\ p^{-\frac{82}{69}}X^{\frac{133}{230}}<q<p^{-\frac{58}{49}}X^{\frac{57}{98}}\},$$

$$D_{52}=\{(p,q): X^{\frac{297}{800}}<p\leq X^{\frac{2143}{5620}},\ p^{-\frac{58}{49}}X^{\frac{57}{98}}<q<p^{-\frac{2}{11}}X^{\frac{29}{110}}\},$$

$$D_{53}=\{(p,q): X^{\frac{2143}{5620}}<p\leq X^{\frac{653}{1700}},\ X^{\frac{9}{85}}<q<p^{-1}X^{\frac{49}{100}}\},$$

$$D_{54}=\{(p,q): X^{\frac{2143}{5620}}<p\leq X^{\frac{653}{1700}},\ p^{-1}X^{\frac{49}{100}}<q<X^{\frac{19}{160}}\},$$

$$D_{55}=\{(p,q): X^{\frac{2143}{5620}}<p\leq X^{\frac{653}{1700}},\ p^{-\frac{82}{69}}X^{\frac{133}{230}}<q<p^{-\frac{58}{49}}X^{\frac{57}{98}}\},$$

$$D_{56}=\{(p,q): X^{\frac{2143}{5620}}<p\leq X^{\frac{653}{1700}},\ p^{-\frac{58}{49}}X^{\frac{57}{98}}<q<p^{-\frac{29}{19}}X^{\frac{59}{76}}\},$$

$$D_{57}=\{(p,q): X^{\frac{653}{1700}}<p\leq X^{\frac{5073}{13120}},\ X^{\frac{9}{85}}<q<X^{\frac{19}{160}}\},$$

$$D_{58}=\{(p,q): X^{\frac{653}{1700}}<p\leq X^{\frac{5073}{13120}},\ p^{-\frac{82}{69}}X^{\frac{133}{230}}<q<p^{-\frac{58}{49}}X^{\frac{57}{98}}\},$$

$$D_{59}=\{(p,q): X^{\frac{653}{1700}}<p\leq X^{\frac{5073}{13120}},\ p^{-\frac{58}{49}}X^{\frac{57}{98}}<q<p^{-\frac{29}{19}}X^{\frac{59}{76}}\},$$

$$D_{60}=\{(p,q): X^{\frac{5073}{13120}}<p\leq X^{\frac{5541}{13940}},\ X^{\frac{9}{85}}<q<p^{-\frac{82}{69}}X^{\frac{133}{230}}\},$$

$$D_{61}=\{(p,q): X^{\frac{5073}{13120}}<p\leq X^{\frac{5541}{13940}},\ p^{-\frac{82}{69}}X^{\frac{133}{230}}<q<p^{-\frac{58}{49}}X^{\frac{57}{98}}\},$$

$$D_{62}=\{(p,q): X^{\frac{5073}{13120}}<p\leq X^{\frac{5541}{13940}},\ p^{-\frac{58}{49}}X^{\frac{57}{98}}<q<p^{-\frac{29}{19}}X^{\frac{59}{76}}\},$$

$$D_{63}=\{(p,q): X^{\frac{5541}{13940}}<p\leq X^{\frac{3963}{9860}},\ X^{\frac{9}{85}}<q<p^{-\frac{58}{49}}X^{\frac{57}{98}}\},$$

$$D_{64}=\{(p,q): X^{\frac{5541}{13940}}<p\leq X^{\frac{3963}{9860}},\ p^{-\frac{58}{49}}X^{\frac{57}{98}}<q<p^{-\frac{29}{19}}X^{\frac{59}{76}}\},$$

$$D_{65}=\{(p,q): X^{\frac{3963}{9860}}<p\leq X^{\frac{1063}{2640}},\ X^{\frac{9}{85}}<q<p^{-\frac{29}{19}}X^{\frac{59}{76}}\},$$

$$D_{66}=\{(p,q): X^{\frac{1063}{2640}}<p\leq X^{\frac{19}{45}},\ X^{\frac{9}{85}}<q<p^{-\frac{29}{19}}X^{\frac{59}{76}}\},$$

$$D_{67}=\{(p,q): X^{\frac{1063}{2640}}<p\leq X^{\frac{19}{45}},\ p^{-\frac{29}{19}}X^{\frac{59}{76}}<q<p^{-\frac{35}{23}}X^{\frac{179}{230}}\},$$

$$D_{68} = \{(p,q) : X^{\frac{19}{45}} < p \le X^{\frac{19}{44}},\ X^{\frac{9}{85}} < q < p^{-\frac{29}{19}} X^{\frac{59}{76}}\},$$

$$D_{69} = \{(p,q) : X^{\frac{19}{45}} < p \le X^{\frac{19}{44}},\ p^{-\frac{29}{19}} X^{\frac{59}{76}} < q < p^{-\frac{41}{27}} X^{\frac{421}{540}}\},$$

$$D_{70} = \{(p,q) : X^{\frac{19}{45}} < p \le X^{\frac{19}{44}},\ p^{-\frac{29}{10}} X^{\frac{57}{40}} < q < p^{-\frac{1}{5}} X^{\frac{29}{100}}\},$$

$$D_{71} = \{(p,q) : X^{\frac{19}{44}} < p \le X^{\frac{4331}{9860}},\ X^{\frac{9}{85}} < q < p^{-\frac{29}{19}} X^{\frac{59}{76}}\},$$

$$D_{72} = \{(p,q) : X^{\frac{19}{44}} < p \le X^{\frac{4331}{9860}},\ p^{-\frac{29}{19}} X^{\frac{59}{76}} < q < p^{-\frac{41}{27}} X^{\frac{421}{540}}\},$$

$$D_{73} = \{(p,q) : X^{\frac{19}{44}} < p \le X^{\frac{4331}{9860}},\ p^{-\frac{29}{10}} X^{\frac{57}{40}} < q < p^{-\frac{1}{5}} X^{\frac{29}{100}}\},$$

$$D_{74} = \{(p,q) : X^{\frac{19}{44}} < p \le X^{\frac{4331}{9860}},\ p^{-\frac{6}{5}} X^{\frac{19}{25}} < q < p^{\frac{1}{7}} X^{\frac{13}{70}}\},$$

$$D_{75} = \{(p,q) : X^{\frac{4331}{9860}} < p \le X^{\frac{2691}{5950}},\ p^{-\frac{35}{12}} X^{\frac{57}{40}} < q < p^{-\frac{29}{10}} X^{\frac{57}{40}}\},$$

$$D_{76} = \{(p,q) : X^{\frac{4331}{9860}} < p \le X^{\frac{2691}{5950}},\ p^{-\frac{29}{10}} X^{\frac{57}{40}} < q < p^{-\frac{1}{5}} X^{\frac{29}{100}}\},$$

$$D_{77} = \{(p,q) : X^{\frac{4331}{9860}} < p \le X^{\frac{2691}{5950}},\ p^{-\frac{6}{5}} X^{\frac{19}{25}} < q < p^{\frac{1}{7}} X^{\frac{13}{70}}\},$$

$$D_{78} = \{(p,q) : X^{\frac{2691}{5950}} < p \le X^{\frac{897}{1972}},\ X^{\frac{9}{85}} < q < p^{-\frac{29}{10}} X^{\frac{57}{40}}\},$$

$$D_{79} = \{(p,q) : X^{\frac{2691}{5950}} < p \le X^{\frac{897}{1972}},\ p^{-\frac{29}{10}} X^{\frac{57}{40}} < q < p^{-\frac{1}{5}} X^{\frac{29}{100}}\},$$

$$D_{80} = \{(p,q) : X^{\frac{2691}{5950}} < p \le X^{\frac{897}{1972}},\ p^{-\frac{6}{5}} X^{\frac{19}{25}} < q < p^{\frac{1}{7}} X^{\frac{13}{70}}\},$$

$$D_{81} = \{(p,q) : X^{\frac{897}{1972}} < p \le X^{\frac{133}{290}},\ X^{\frac{9}{85}} < q < p^{-\frac{1}{5}} X^{\frac{29}{100}}\},$$

$$D_{82} = \{(p,q) : X^{\frac{897}{1972}} < p \le X^{\frac{133}{290}},\ p^{-\frac{6}{5}} X^{\frac{19}{25}} < q < p^{\frac{1}{7}} X^{\frac{13}{70}}\},$$

$$D_{83} = \{(p,q) : X^{\frac{133}{290}} < p \le X^{\frac{19}{40}},\ X^{\frac{9}{85}} < q < p^{\frac{1}{7}} X^{\frac{13}{70}}\},$$

$$D_{84} = \{(p,q) : X^{\frac{19}{40}} < p \le X^{\frac{53}{110}},\ X^{\frac{9}{85}} < q < p^{-1} X^{\frac{51}{70}}\},$$

$$D_{85} = \{(p,q) : X^{\frac{53}{110}} < p \le X^{\frac{339}{700}},\ X^{\frac{9}{85}} < q < p^{-\frac{58}{9}} X^{\frac{59}{18}}\},$$

$$D_{86} = \{(p,q) : X^{\frac{53}{110}} < p \le X^{\frac{339}{700}},\ X^{\frac{19}{110}} < q < p^{-1} X^{\frac{51}{70}}\},$$

$$D_{87} = \{(p,q) : X^{\frac{339}{700}} < p \le X^{\frac{233}{480}},\ X^{\frac{9}{85}} < q < p^{-\frac{58}{9}} X^{\frac{59}{18}}\},$$

$$D_{88} = \{(p,q) : X^{\frac{339}{700}} < p \le X^{\frac{233}{480}},\ X^{\frac{19}{110}} < q < p^{-6} X^{\frac{63}{20}}\},$$

$$D_{89} = \{(p,q) : X^{\frac{233}{480}} < p \le X^{\frac{49}{100}},\ X^{\frac{9}{85}} < q < p^{-\frac{58}{9}} X^{\frac{59}{18}}\},$$

$$D_{90} = \{(p,q) : X^{\frac{233}{480}} < p \le X^{\frac{49}{100}},\ p^{-\frac{58}{9}} X^{\frac{59}{18}} < q < p^{-\frac{70}{11}} X^{\frac{179}{55}}\},$$

$$D_{91} = \{(p,q) : X^{\frac{233}{480}} < p \le X^{\frac{49}{100}},\ X^{\frac{19}{110}} < q < p^{-6} X^{\frac{63}{20}}\},$$

$$D_{92} = \{(p,q) : X^{\frac{49}{100}} < p \le X^{\frac{2944}{5950}},\ X^{\frac{9}{85}} < q < p^{-\frac{70}{11}} X^{\frac{179}{55}}\},$$

$$D_{93}=\{(p,q):X^{\frac{49}{100}}<p\le X^{\frac{2944}{5950}},\ X^{\frac{19}{110}}<q<p^{-1}X^{\frac{7}{10}}\},$$
$$D_{94}=\{(p,q):X^{\frac{2944}{5950}}<p\le X^{\frac{1}{2}},\ X^{\frac{19}{110}}<q<p^{-1}X^{\frac{7}{10}}\}.$$

9. Bounds of the sieve functions

Lemma 24.

$$\begin{aligned}&\Omega_{16}+\Omega_{19}+\Omega_{22}+\Omega_{25}+\Omega_{28}+\Omega_{31}+\Omega_{34}+\Omega_{37}+\Omega_{40}+\Omega_{42}+\Omega_{45}\\&\quad+\Omega_{48}+\Omega_{52}+\Omega_{56}+\Omega_{59}+\Omega_{62}+\Omega_{64}+\Omega_{65}+\Omega_{66}+\Omega_{68}+\Omega_{71}\\&\ \ge 0.394443\eta x\log^{-1}x.\end{aligned}$$

Proof. We have

$$\Omega_{16}=\sum_{X^{\frac{19}{70}}<p\le X^{\frac{3}{10}}}\ \sum_{p^{-\frac{6}{5}}X^{\frac{57}{100}}<q<p^{-\frac{1}{8}}X^{\frac{23}{80}}}S(\mathcal{A}_{pq},q)=\sum_{x<pqr\le x+\eta x}1,$$

where $X^{\frac{19}{70}}<p\le X^{\frac{3}{10}}$, $p^{-\frac{6}{5}}X^{\frac{57}{100}}<q<p^{-\frac{1}{8}}X^{\frac{23}{80}}$ and the least prime factor of r is greater than q.

Let $h=q$ and $k=r$. By Lemma 6 with region (i), (7) holds. Then Lemmas 19 and 21 yield

$$\begin{aligned}\Omega_{16}&=\eta\sum_{X^{\frac{19}{70}}<p\le X^{\frac{3}{10}}}\ \sum_{p^{-\frac{6}{5}}X^{\frac{57}{100}}<q<p^{-\frac{1}{8}}X^{\frac{23}{80}}}S(\mathcal{B}_{pq},q)+O(\varepsilon\eta x\log^{-1}x)\\
&=\eta x\sum_{X^{\frac{19}{70}}<p\le X^{\frac{3}{10}}}\ \sum_{p^{-\frac{6}{5}}X^{\frac{57}{100}}<q<p^{-\frac{1}{8}}X^{\frac{23}{80}}}\frac{1}{pq\log q}w\left(\frac{\log(x/(pq))}{\log q}\right)\\
&\quad+O(\varepsilon\eta x\log^{-1}x)\\
&=\eta x\log^{-1}x\int_{\frac{19}{70}}^{\frac{3}{10}}\frac{dt}{t}\int_{\frac{57}{100}-\frac{6}{5}t}^{\frac{23}{80}-\frac{t}{8}}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du+O(\varepsilon\eta x\log^{-1}x)\\
&=\eta x\log^{-1}x\Bigg(\int_{\frac{19}{70}}^{\frac{71}{260}}\frac{dt}{t}\int_{\frac{57}{100}-\frac{6}{5}t}^{\frac{23}{80}-\frac{t}{8}}\frac{du}{u(1-t-u)}\\
&\quad+\int_{\frac{71}{260}}^{\frac{3}{10}}\frac{dt}{t}\int_{\frac{57}{100}-\frac{6}{5}t}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du\\
&\quad+\int_{\frac{71}{260}}^{\frac{3}{10}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{23}{80}-\frac{t}{8}}\frac{du}{u(1-t-u)}+O(\varepsilon)\Bigg)\\
&\ge(0.000516+0.010796+0.011351)\eta x\log^{-1}x\\
&=0.022663\eta x\log^{-1}x.\end{aligned}$$

Using the above discussion and Lemma 20, we have

$$\begin{aligned}
&\Omega_{19}+\Omega_{22}+\Omega_{25}+\Omega_{28}+\Omega_{31}+\Omega_{34}+\Omega_{37}+\Omega_{40}+\Omega_{42}+\Omega_{45}\\
&\quad+\Omega_{48}+\Omega_{52}+\Omega_{56}+\Omega_{59}+\Omega_{62}+\Omega_{64}+\Omega_{65}+\Omega_{66}+\Omega_{68}+\Omega_{71}\\
&=\eta x\log^{-1}x\Bigg(\int_{\frac{3}{10}}^{\frac{19}{60}}\frac{dt}{t}\int_{\frac{57}{110}-\frac{12}{11}t}^{\frac{23}{80}-\frac{t}{8}}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du\\
&\quad+\int_{\frac{19}{60}}^{\frac{247}{770}}\frac{dt}{t}\int_{\frac{19}{110}}^{\frac{23}{80}-\frac{t}{8}}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du+\int_{\frac{247}{770}}^{\frac{359}{1100}}\frac{dt}{t}\int_{\frac{19}{110}}^{\frac{25}{44}-t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du\\
&\quad+\int_{\frac{359}{1100}}^{\frac{481}{1450}}\frac{dt}{t}\int_{\frac{19}{110}}^{\frac{11}{5}-6t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du\\
&\quad+\int_{\frac{481}{1450}}^{\frac{443}{1305}}\frac{dt}{t}\int_{\frac{19}{110}}^{\frac{157}{290}-t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du\\
&\quad+\int_{\frac{443}{1305}}^{\frac{19}{55}}\frac{dt}{t}\int_{\frac{19}{110}}^{\frac{29}{110}-\frac{2}{11}t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du\\
&\quad+\int_{\frac{19}{55}}^{\frac{2143}{5620}}\frac{dt}{t}\int_{\frac{57}{98}-\frac{58}{49}t}^{\frac{29}{110}-\frac{2}{11}t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du\\
&\quad+\int_{\frac{2143}{5620}}^{\frac{3963}{9860}}\frac{dt}{t}\int_{\frac{57}{98}-\frac{58}{49}t}^{\frac{59}{76}-\frac{29}{19}t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du\\
&\quad+\int_{\frac{3963}{9860}}^{\frac{4331}{9860}}\frac{dt}{t}\int_{\frac{9}{85}}^{\frac{59}{76}-\frac{29}{19}t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du+O(\varepsilon)\Bigg)\\
&\geq\eta x\log^{-1}x\Bigg(\int_{\frac{3}{10}}^{\frac{19}{60}}\frac{dt}{t}\int_{\frac{57}{110}-\frac{12}{11}t}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du\\
&\quad+\int_{\frac{3}{10}}^{\frac{19}{60}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{23}{80}-\frac{t}{8}}\frac{du}{u(1-t-u)}
\end{aligned}$$

$$+\int_{\frac{19}{60}}^{\frac{247}{770}}\frac{dt}{t}\int_{\frac{19}{110}}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{19}{60}}^{\frac{247}{770}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{23}{80}-\frac{t}{8}}\frac{du}{u(1-t-u)}$$

$$+\int_{\frac{247}{770}}^{\frac{359}{1100}}\frac{dt}{t}\int_{\frac{19}{110}}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{247}{770}}^{\frac{359}{1100}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{25}{44}-t}\frac{du}{u(1-t-u)}$$

$$+\int_{\frac{359}{1100}}^{\frac{28}{85}}\frac{dt}{t}\int_{\frac{19}{110}}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{359}{1100}}^{\frac{28}{85}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{11}{5}-6t}\frac{du}{u(1-t-u)}$$

$$+\int_{\frac{28}{85}}^{\frac{481}{1450}}\frac{dt}{t}\int_{\frac{19}{110}}^{\frac{11}{5}-6t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{481}{1450}}^{\frac{443}{1305}}\frac{dt}{t}\int_{\frac{19}{110}}^{\frac{157}{290}-t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{443}{1305}}^{\frac{19}{55}}\frac{dt}{t}\int_{\frac{19}{110}}^{\frac{29}{110}-\frac{2}{11}t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{19}{55}}^{\frac{65}{183}}\frac{dt}{t}\int_{\frac{57}{98}-\frac{58}{49}t}^{\frac{29}{110}-\frac{2}{11}t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+0.5607\int_{\frac{65}{183}}^{\frac{2143}{5620}}\frac{dt}{t}\int_{\frac{57}{98}-\frac{58}{49}t}^{\frac{1}{4}(1-t)}\frac{du}{u^2}$$

$$+\int_{\frac{65}{183}}^{\frac{2143}{5620}} \frac{dt}{t} \int_{\frac{1}{4}(1-t)}^{\frac{29}{110}-\frac{2}{11}t} \frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right) du$$

$$+ 0.5607 \int_{\frac{2143}{5620}}^{\frac{3963}{9860}} \frac{dt}{t} \int_{\frac{57}{98}-\frac{58}{49}t}^{\frac{1}{4}(1-t)} \frac{du}{u^2}$$

$$+\int_{\frac{2143}{5620}}^{\frac{3963}{9860}} \frac{dt}{t} \int_{\frac{1}{4}(1-t)}^{\frac{59}{76}-\frac{29}{19}t} \frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right) du$$

$$+ 0.5607 \int_{\frac{3963}{9860}}^{\frac{40}{97}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{1}{4}(1-t)} \frac{du}{u^2}$$

$$+\int_{\frac{3963}{9860}}^{\frac{40}{97}} \frac{dt}{t} \int_{\frac{1}{4}(1-t)}^{\frac{59}{76}-\frac{29}{19}t} \frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right) du$$

$$+ 0.5607 \int_{\frac{40}{97}}^{\frac{4331}{9860}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{59}{76}-\frac{29}{19}t} \frac{du}{u^2} + O(\varepsilon)\Bigg)$$

$$\geq (0.034609 + 0.009183 + 0.009870 + 0.002518$$
$$+ 0.012892 + 0.003183 + 0.006792 + 0.000789$$
$$+ 0.004547 + 0.012016 + 0.008083 + 0.015437$$
$$+ 0.021797 + 0.049656 + 0.056577 + 0.028631$$
$$+ 0.038798 + 0.004151 + 0.052251)\eta x \log^{-1} x$$
$$= 0.371780\eta x \log^{-1} x.$$

The proof of Lemma 24 is complete.

LEMMA 25.

$$\Phi = \Omega_{70} + \Omega_{73} + \Omega_{74} + \Omega_{76} + \Omega_{77} + \Omega_{79} + \Omega_{80} + \Omega_{81} + \Omega_{82} + \Omega_{83}$$
$$+ \Omega_{84} + \Omega_{85} + \Omega_{86} + \Omega_{87} + \Omega_{88} + \Omega_{89} + \Omega_{91} + \Omega_{93} + \Omega_{94}$$
$$\geq 0.321616\eta x \log^{-1} x.$$

P r o o f. On applying Lemmas 8, 19, 20 and 21, in the same way as in Lemma 24, we have

$$\Phi = \eta x \log^{-1} x \Bigg(\int_{\frac{19}{45}}^{\frac{19}{44}} \frac{dt}{t} \int_{\frac{57}{40}-\frac{29}{10}t}^{\frac{29}{100}-\frac{t}{5}} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du$$

$$+ \int_{\frac{19}{44}}^{\frac{897}{1972}} \frac{dt}{t} \int_{\frac{57}{40}-\frac{29}{10}t}^{\frac{29}{100}-\frac{t}{5}} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du + \int_{\frac{19}{44}}^{\frac{897}{1972}} \frac{dt}{t} \int_{\frac{19}{25}-\frac{6}{5}t}^{\frac{13}{70}+\frac{t}{7}} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du$$

$$+ \int_{\frac{897}{1972}}^{\frac{133}{290}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{29}{100}-\frac{t}{5}} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du + \int_{\frac{897}{1972}}^{\frac{133}{290}} \frac{dt}{t} \int_{\frac{19}{25}-\frac{6}{5}t}^{\frac{13}{70}+\frac{t}{7}} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du$$

$$+ \int_{\frac{133}{290}}^{\frac{19}{40}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{13}{70}+\frac{t}{7}} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du + \int_{\frac{19}{40}}^{\frac{53}{110}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{51}{70}-t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du$$

$$+ \int_{\frac{53}{110}}^{\frac{339}{700}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{59}{18}-\frac{58}{9}t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du + \int_{\frac{53}{110}}^{\frac{339}{700}} \frac{dt}{t} \int_{\frac{19}{110}}^{\frac{51}{70}-t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du$$

$$+ \int_{\frac{339}{700}}^{\frac{49}{100}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{59}{18}-\frac{58}{9}t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du + \int_{\frac{339}{700}}^{\frac{49}{100}} \frac{dt}{t} \int_{\frac{19}{110}}^{\frac{63}{20}-6t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du$$

$$+ \int_{\frac{49}{100}}^{\frac{1}{2}} \frac{dt}{t} \int_{\frac{19}{110}}^{\frac{7}{10}-t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du + O(\varepsilon) \Bigg)$$

$$\geq \eta x \log^{-1} x \Bigg(\int_{\frac{19}{45}}^{\frac{131}{308}} \frac{dt}{t} \int_{\frac{57}{40}-\frac{29}{10}t}^{\frac{29}{100}-\frac{t}{5}} \frac{du}{u(1-t-u)}$$

$$+ \int_{\frac{131}{308}}^{\frac{19}{44}} \frac{dt}{t} \int_{\frac{57}{40}-\frac{29}{10}t}^{\frac{1}{3}(1-t)} \frac{1}{u(1-t-u)} \left(1 + \log\left(\frac{1-t}{u} - 2\right)\right) du$$

$$+ \int_{\frac{131}{308}}^{\frac{19}{44}} \frac{dt}{t} \int_{\frac{1}{3}(1-t)}^{\frac{29}{100}-\frac{t}{5}} \frac{du}{u(1-t-u)}$$

$$+ \int_{\frac{19}{44}}^{\frac{47}{106}} \frac{dt}{t} \int_{\frac{57}{40}-\frac{29}{10}t}^{\frac{1}{3}(1-t)} \frac{1}{u(1-t-u)} \left(1 + \log\left(\frac{1-t}{u} - 2\right)\right) du$$

$$+ \int_{\frac{19}{44}}^{\frac{47}{106}} \frac{dt}{t} \int_{\frac{1}{3}(1-t)}^{\frac{29}{100}-\frac{t}{5}} \frac{du}{u(1-t-u)} + 0.5607 \int_{\frac{47}{106}}^{\frac{897}{1972}} \frac{dt}{t} \int_{\frac{57}{40}-\frac{29}{10}t}^{\frac{1}{4}(1-t)} \frac{du}{u^2}$$

$$+\int_{\frac{47}{106}}^{\frac{897}{1972}}\frac{dt}{t}\int_{\frac{1}{4}(1-t)}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{47}{106}}^{\frac{897}{1972}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{29}{100}-\frac{t}{5}}\frac{du}{u(1-t-u)}+\int_{\frac{19}{44}}^{\frac{897}{1972}}\frac{dt}{t}\int_{\frac{19}{25}-\frac{6}{5}t}^{\frac{13}{70}+\frac{t}{7}}\frac{du}{u(1-t-u)}$$

$$+0.5607\int_{\frac{897}{1972}}^{\frac{133}{290}}\frac{dt}{t}\int_{\frac{9}{85}}^{\frac{1}{4}(1-t)}\frac{du}{u^2}$$

$$+\int_{\frac{897}{1972}}^{\frac{133}{290}}\frac{dt}{t}\int_{\frac{1}{4}(1-t)}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{897}{1972}}^{\frac{133}{290}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{29}{100}-\frac{t}{5}}\frac{du}{u(1-t-u)}+\int_{\frac{897}{1972}}^{\frac{133}{290}}\frac{dt}{t}\int_{\frac{19}{25}-\frac{6}{5}t}^{\frac{13}{70}+\frac{t}{7}}\frac{du}{u(1-t-u)}$$

$$+0.5607\int_{\frac{133}{290}}^{\frac{19}{40}}\frac{dt}{t}\int_{\frac{9}{85}}^{\frac{1}{4}(1-t)}\frac{du}{u^2}$$

$$+\int_{\frac{133}{290}}^{\frac{19}{40}}\frac{dt}{t}\int_{\frac{1}{4}(1-t)}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{133}{290}}^{\frac{19}{40}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{13}{70}+\frac{t}{7}}\frac{du}{u(1-t-u)}+0.5607\int_{\frac{19}{40}}^{\frac{53}{110}}\frac{dt}{t}\int_{\frac{9}{85}}^{\frac{1}{4}(1-t)}\frac{du}{u^2}$$

$$+\int_{\frac{19}{40}}^{\frac{53}{110}}\frac{dt}{t}\int_{\frac{1}{4}(1-t)}^{\frac{1}{3}(1-t)}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{19}{40}}^{\frac{53}{110}}\frac{dt}{t}\int_{\frac{1}{3}(1-t)}^{\frac{51}{70}-t}\frac{du}{u(1-t-u)}+0.5607\int_{\frac{53}{110}}^{\frac{339}{700}}\frac{dt}{t}\int_{\frac{9}{85}}^{\frac{1}{4}(1-t)}\frac{du}{u^2}$$

$$+\int_{\frac{53}{110}}^{\frac{339}{700}}\frac{dt}{t}\int_{\frac{1}{4}(1-t)}^{\frac{59}{18}-\frac{58}{9}t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+\int_{\frac{53}{110}}^{\frac{339}{700}}\frac{dt}{t}\int_{\frac{19}{110}}^{\frac{51}{70}-t}\frac{du}{u(1-t-u)}+0.5607\int_{\frac{339}{700}}^{\frac{109}{223}}\frac{dt}{t}\int_{\frac{9}{85}}^{\frac{1}{4}(1-t)}\frac{du}{u^2}$$

$$+\int_{\frac{339}{700}}^{\frac{109}{223}}\frac{dt}{t}\int_{\frac14(1-t)}^{\frac{59}{18}-\frac{58}{9}t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+0.5607\int_{\frac{109}{223}}^{\frac{49}{100}}\frac{dt}{t}\int_{\frac{9}{85}}^{\frac{59}{18}-\frac{58}{9}t}\frac{du}{u^2}+\int_{\frac{339}{700}}^{\frac{49}{100}}\frac{dt}{t}\int_{\frac{19}{110}}^{\frac{63}{20}-6t}\frac{du}{u(1-t-u)}$$

$$+\int_{\frac{49}{100}}^{\frac12}\frac{dt}{t}\int_{\frac{19}{110}}^{\frac{7}{10}-t}\frac{du}{u(1-t-u)}+O(\varepsilon)\Bigg)$$

$$\begin{aligned}\geq (&0.000892+0.001922+0.002834+0.015864\\ &+0.005547+0.013560+0.025626+0.006139\\ &+0.014504+0.009587+0.008357+0.002151\\ &+0.004454+0.038219+0.036373+0.038613\\ &+0.014188+0.015100+0.016708+0.004887\\ &+0.004744+0.005821+0.008584+0.003871\\ &+0.001900+0.010238+0.010933)\eta x\log^{-1}x\\ =\ &0.321616\eta x\log^{-1}x.\end{aligned}$$

The proof of Lemma 25 is complete.

LEMMA 26.

$$\begin{aligned}\Omega_5+\Omega_7+\Omega_9+\Omega_{11}+\Omega_{13}+\Omega_{15}+\Omega_{18}+\Omega_{21}+\Omega_{24}+\Omega_{27}+\Omega_{30}\\ +\Omega_{33}+\Omega_{36}+\Omega_{39}+\Omega_{41}+\Omega_{43}+\Omega_{46}+\Omega_{49}+\Omega_{53}\\ \geq 0.277632\eta x\log^{-1}x.\end{aligned}$$

Proof. We have

$$\begin{aligned}(25)\quad \Omega_5&=\sum_{X^{\frac{133}{580}}<p\leq X^{\frac{19}{80}}}\ \sum_{p^{-1}X^{\frac{133}{290}}<q<p}S(\mathcal{A}_{pq},q)\\ &=\sum_{X^{\frac{133}{580}}<p\leq X^{\frac{19}{80}}}\ \sum_{p^{-1}X^{\frac{133}{290}}<q<p}S(\mathcal{A}_{pq},X^{\delta})\\ &\quad-\sum_{X^{\frac{133}{580}}<p\leq X^{\frac{19}{80}}}\ \sum_{p^{-1}X^{\frac{133}{290}}<q<p}\ \sum_{X^{\delta}<r<X^{\frac{9}{85}}}S(\mathcal{A}_{pqr},r)\\ &\quad-\sum_{X^{\frac{133}{580}}<p\leq X^{\frac{19}{80}}}\ \sum_{p^{-1}X^{\frac{133}{290}}<q<p}\ \sum_{X^{\frac{9}{85}}<r<\min(q,(\frac{2X}{pq})^{\frac12})}S(\mathcal{A}_{pqr},r)\\ &=\Phi_1-\Phi_2-\Phi_3.\end{aligned}$$

Let $z=X^{\delta}$ and $D=D(p,q)=X^{\frac{21}{40}}/(pq)$. Applying Iwaniec's sieve method, we have

$$\Phi_1 \le \frac{\eta x}{\log z} \sum_{X^{\frac{133}{580}}<p\le X^{\frac{19}{80}}} \sum_{p^{-1}X^{\frac{133}{290}}<q<p} \frac{1}{pq} F\left(\frac{\log D}{\log z}\right)$$
$$+ \sum_{X^{\frac{133}{580}}<p\le X^{\frac{19}{80}}} \sum_{p^{-1}X^{\frac{133}{290}}<q<p} \sum_{r<X^{\frac{21}{40}}/(pq)} a(r)\widetilde{r}(\mathcal{A}, pqr)$$
$$+ O(\varepsilon\eta x \log^{-1} x).$$

Let $m = pqr$. Then Lemma 16 yields

$$\sum_{X^{\frac{133}{580}}<p\le X^{\frac{19}{80}}} \sum_{p^{-1}X^{\frac{133}{290}}<q<p} \sum_{r<X^{\frac{21}{40}}/(pq)} a(r)\widetilde{r}(\mathcal{A}, pqr) = O(\eta x \log^{-5} x).$$

Hence,

$$\Phi_1 \le \frac{e^{-\gamma}}{\delta}\eta x \log^{-1} x \sum_{X^{\frac{133}{580}}<p\le X^{\frac{19}{80}}} \sum_{p^{-1}X^{\frac{133}{290}}<q<p} \frac{1}{pq} + O(\delta\eta x \log^{-1} x)$$
$$= \eta \sum_{X^{\frac{133}{580}}<p\le X^{\frac{19}{80}}} \sum_{p^{-1}X^{\frac{133}{290}}<q<p} S(\mathcal{B}_{pq}, X^{\delta}) + O(\delta\eta x \log^{-1} x).$$

In the same way, we can get the lower bound

$$\Phi_1 \ge \eta \sum_{X^{\frac{133}{580}}<p\le X^{\frac{19}{80}}} \sum_{p^{-1}X^{\frac{133}{290}}<q<p} S(\mathcal{B}_{pq}, X^{\delta}) + O(\delta\eta x \log^{-1} x).$$

Now we have the asymptotic formula

$$\Phi_1 = \eta \sum_{X^{\frac{133}{580}}<p\le X^{\frac{19}{80}}} \sum_{p^{-1}X^{\frac{133}{290}}<q<p} S(\mathcal{B}_{pq}, X^{\delta}) + O(\delta\eta x \log^{-1} x). \tag{26}$$

Using Lemma 18, we have

$$\Phi_2 = \eta \sum_{X^{\frac{133}{580}}<p\le X^{\frac{19}{80}}} \sum_{p^{-1}X^{\frac{133}{290}}<q<p} \sum_{X^{\delta}<r<X^{\frac{9}{85}}} S(\mathcal{B}_{pqr}, r) + O(\delta\eta x \log^{-1} x). \tag{27}$$

By Lemma 13 with the region (i), (7) holds. Then Lemma 19 with a small modification yields

$$\Phi_3 = \eta \sum_{X^{\frac{133}{580}}<p\le X^{\frac{19}{80}}} \sum_{p^{-1}X^{\frac{133}{290}}<q<p} \sum_{X^{\frac{9}{85}}<r<\min(q,(\frac{2X}{pq})^{\frac{1}{2}})} S(\mathcal{B}_{pqr}, r) + O(\delta\eta x \log^{-1} x). \tag{28}$$

Gathering together (25)–(28), we have

$$\begin{aligned}
\Omega_5 &= \eta \sum_{X^{\frac{133}{580}} < p \le X^{\frac{19}{80}}} \sum_{p^{-1}X^{\frac{133}{290}} < q < p} S(\mathcal{B}_{pq}, q) + O(\delta\eta x \log^{-1} x) \\
&= \eta x \log^{-1} x \int_{\frac{133}{580}}^{\frac{19}{80}} \frac{dt}{t} \int_{\frac{133}{290}-t}^{t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du + O(\delta\eta x \log^{-1} x) \\
&= \eta x \log^{-1} x \int_{\frac{133}{580}}^{\frac{19}{80}} \frac{dt}{t} \int_{\frac{133}{290}-t}^{t} \frac{1}{u(1-t-u)} \left(1 + \log\left(\frac{1-t}{u} - 2\right)\right) du \\
&\quad + O(\delta\eta x \log^{-1} x) \\
&\ge 0.003001 \eta x \log^{-1} x.
\end{aligned}$$

In the same way, it can be shown that

$$\begin{aligned}
&\Omega_7 + \Omega_9 + \Omega_{11} + \Omega_{13} + \Omega_{15} + \Omega_{18} + \Omega_{21} + \Omega_{24} + \Omega_{27} + \Omega_{30} \\
&\qquad + \Omega_{33} + \Omega_{36} + \Omega_{39} + \Omega_{41} + \Omega_{43} + \Omega_{46} + \Omega_{49} + \Omega_{53} \\
&= \eta x \log^{-1} x \Bigg(\int_{\frac{19}{80}}^{\frac{53}{220}} \frac{dt}{t} \int_{\frac{133}{290}-t}^{\frac{19}{40}-t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du \\
&\quad + \int_{\frac{53}{220}}^{\frac{151}{580}} \frac{dt}{t} \int_{\frac{133}{290}-t}^{\frac{53}{110}-t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du + \int_{\frac{151}{580}}^{\frac{17}{55}} \frac{dt}{t} \int_{\frac{897}{1972}-t}^{\frac{53}{110}-t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du \\
&\quad + \int_{\frac{17}{55}}^{\frac{3441}{9860}} \frac{dt}{t} \int_{\frac{897}{1972}-t}^{\frac{59}{134}-\frac{58}{67}t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du + \int_{\frac{3441}{9860}}^{\frac{37}{100}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{59}{134}-\frac{58}{67}t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du \\
&\quad + \int_{\frac{37}{100}}^{\frac{653}{1700}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{49}{100}-t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du + O(\varepsilon) \Bigg) \\
&\ge \eta x \log^{-1} x \Bigg(\int_{\frac{19}{80}}^{\frac{53}{220}} \frac{dt}{t} \int_{\frac{133}{290}-t}^{\frac{19}{40}-t} \frac{1}{u(1-t-u)} \left(1 + \log\left(\frac{1-t}{u} - 2\right)\right) du \\
&\quad + \int_{\frac{53}{220}}^{\frac{151}{580}} \frac{dt}{t} \int_{\frac{133}{290}-t}^{\frac{53}{110}-t} \frac{1}{u(1-t-u)} \left(1 + \log\left(\frac{1-t}{u} - 2\right)\right) du
\end{aligned}$$

$$+\int\limits_{\frac{151}{580}}^{\frac{404}{1479}}\frac{dt}{t}\int\limits_{\frac{897}{1972}-t}^{\frac{53}{110}-t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+0.5607\int\limits_{\frac{404}{1479}}^{\frac{17}{55}}\frac{dt}{t}\int\limits_{\frac{897}{1972}-t}^{\frac{1}{4}(1-t)}\frac{du}{u^2}$$

$$+\int\limits_{\frac{404}{1479}}^{\frac{17}{55}}\frac{dt}{t}\int\limits_{\frac{1}{4}(1-t)}^{\frac{53}{110}-t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+0.5607\int\limits_{\frac{17}{55}}^{\frac{3441}{9860}}\frac{dt}{t}\int\limits_{\frac{897}{1972}-t}^{\frac{59}{134}-\frac{58}{67}t}\frac{du}{u^2}+0.5607\int\limits_{\frac{3441}{9860}}^{\frac{37}{100}}\frac{dt}{t}\int\limits_{\frac{9}{85}}^{\frac{59}{134}-\frac{58}{67}t}\frac{du}{u^2}$$

$$+0.5607\int\limits_{\frac{37}{100}}^{\frac{653}{1700}}\frac{dt}{t}\int\limits_{\frac{9}{85}}^{\frac{49}{100}-t}\frac{du}{u^2}+O(\varepsilon)\Bigg)$$

$$\geq (0.002490+0.020848+0.018080+0.033046$$
$$+0.027398+0.105118+0.055429+0.012222)\eta x\log^{-1}x$$
$$=0.274631\eta x\log^{-1}x.$$

The proof of Lemma 26 is complete.

LEMMA 27.

$$\varPhi=\varOmega_{44}+\varOmega_{47}+\varOmega_{51}+\varOmega_{55}+\varOmega_{58}+\varOmega_{61}+\varOmega_{63}+\varOmega_{67}+\varOmega_{69}+\varOmega_{72}$$
$$\geq 0.033767\eta x\log^{-1}x.$$

Proof. On applying Lemmas 11, 19, 20 and 21, in the same way as in Lemma 24, we have

$$\varPhi=\eta x\log^{-1}x\Bigg(\int\limits_{\frac{1843}{5280}}^{\frac{37}{100}}\frac{dt}{t}\int\limits_{\frac{171}{295}-\frac{70}{59}t}^{\frac{57}{98}-\frac{58}{49}t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du$$

$$+\int\limits_{\frac{37}{100}}^{\frac{5541}{13940}}\frac{dt}{t}\int\limits_{\frac{133}{230}-\frac{82}{69}t}^{\frac{57}{98}-\frac{58}{49}t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du$$

$$+\int\limits_{\frac{5541}{13940}}^{\frac{3963}{9860}}\frac{dt}{t}\int\limits_{\frac{9}{85}}^{\frac{57}{98}-\frac{58}{49}t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du$$

$$+\int_{\frac{1063}{2640}}^{\frac{19}{45}}\frac{dt}{t}\int_{\frac{59}{76}-\frac{29}{19}t}^{\frac{179}{230}-\frac{35}{23}t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du$$

$$+\int_{\frac{19}{45}}^{\frac{4331}{9860}}\frac{dt}{t}\int_{\frac{59}{76}-\frac{29}{19}t}^{\frac{421}{540}-\frac{41}{27}t}\frac{1}{u^2}w\left(\frac{1-t-u}{u}\right)du+O(\varepsilon)\Bigg)$$

$$\geq \eta x\log^{-1}x\Bigg(\int_{\frac{1843}{5280}}^{\frac{389}{1105}}\frac{dt}{t}\int_{\frac{171}{295}-\frac{70}{59}t}^{\frac{57}{98}-\frac{58}{49}t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+0.5607\int_{\frac{389}{1105}}^{\frac{65}{183}}\frac{dt}{t}\int_{\frac{171}{295}-\frac{70}{59}t}^{\frac{1}{4}(1-t)}\frac{du}{u^2}$$

$$+\int_{\frac{389}{1105}}^{\frac{65}{183}}\frac{dt}{t}\int_{\frac{1}{4}(1-t)}^{\frac{57}{98}-\frac{58}{49}t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+0.5607\int_{\frac{65}{183}}^{\frac{37}{100}}\frac{dt}{t}\int_{\frac{171}{295}-\frac{70}{59}t}^{\frac{57}{98}-\frac{58}{49}t}\frac{du}{u^2}+0.5607\int_{\frac{37}{100}}^{\frac{5541}{13940}}\frac{dt}{t}\int_{\frac{133}{230}-\frac{82}{69}t}^{\frac{57}{98}-\frac{58}{49}t}\frac{du}{u^2}$$

$$+0.5607\int_{\frac{5541}{13940}}^{\frac{3963}{9860}}\frac{dt}{t}\int_{\frac{9}{85}}^{\frac{57}{98}-\frac{58}{49}t}\frac{du}{u^2}$$

$$+\int_{\frac{1063}{2640}}^{\frac{40}{97}}\frac{dt}{t}\int_{\frac{59}{76}-\frac{29}{19}t}^{\frac{179}{230}-\frac{35}{23}t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+0.5607\int_{\frac{40}{97}}^{\frac{27}{65}}\frac{dt}{t}\int_{\frac{59}{76}-\frac{29}{19}t}^{\frac{1}{4}(1-t)}\frac{du}{u^2}$$

$$+\int_{\frac{40}{97}}^{\frac{27}{65}}\frac{dt}{t}\int_{\frac{1}{4}(1-t)}^{\frac{179}{230}-\frac{35}{23}t}\frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right)du$$

$$+0.5607\int_{\frac{27}{65}}^{\frac{19}{45}}\frac{dt}{t}\int_{\frac{59}{76}-\frac{29}{19}t}^{\frac{179}{230}-\frac{35}{23}t}\frac{du}{u^2}$$

$$+0.5607\int_{\frac{19}{45}}^{\frac{4331}{9860}}\frac{dt}{t}\int_{\frac{59}{76}-\frac{29}{19}t}^{\frac{421}{540}-\frac{41}{27}t}\frac{du}{u^2}+O(\varepsilon)\Bigg)$$

$$\geq (0.000517 + 0.000285 + 0.000279 + 0.002997 + 0.013589 + 0.001413 + 0.002113 + 0.000371 + 0.000359 + 0.001830 + 0.010014)\eta x \log^{-1} x = 0.033767\eta x \log^{-1} x.$$

The proof of Lemma 27 is complete.

LEMMA 28.

$$\Phi = \Omega_{75} + \Omega_{78} + \Omega_{90} + \Omega_{92} \geq 0.019054\eta x \log^{-1} x.$$

Proof. On applying Lemmas 12 and 19–21, in the same way as in Lemma 24, we have

$$\Phi = \eta x \log^{-1} x \Bigg(\int_{\frac{4331}{9860}}^{\frac{2691}{5950}} \frac{dt}{t} \int_{\frac{57}{40}-\frac{35}{12}t}^{\frac{57}{40}-\frac{29}{10}t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du + \int_{\frac{2691}{5950}}^{\frac{897}{1972}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{57}{40}-\frac{29}{10}t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du + \int_{\frac{233}{480}}^{\frac{49}{100}} \frac{dt}{t} \int_{\frac{59}{18}-\frac{58}{9}t}^{\frac{179}{55}-\frac{70}{11}t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du + \int_{\frac{49}{100}}^{\frac{2944}{5950}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{179}{55}-\frac{70}{11}t} \frac{1}{u^2} w\left(\frac{1-t-u}{u}\right) du + O(\varepsilon)\Bigg)$$

$$\geq \eta x \log^{-1} x \Bigg(\int_{\frac{4331}{9860}}^{\frac{141}{320}} \frac{dt}{t} \int_{\frac{57}{40}-\frac{35}{12}t}^{\frac{57}{40}-\frac{29}{10}t} \frac{1}{u(1-t-u)} \left(1 + \log\left(\frac{1-t}{u} - 2\right)\right) du + 0.5607 \int_{\frac{141}{320}}^{\frac{47}{106}} \frac{dt}{t} \int_{\frac{57}{40}-\frac{35}{12}t}^{\frac{1}{4}(1-t)} \frac{du}{u^2} + \int_{\frac{141}{320}}^{\frac{47}{106}} \frac{dt}{t} \int_{\frac{1}{4}(1-t)}^{\frac{57}{40}-\frac{29}{10}t} \frac{1}{u(1-t-u)} \left(1 + \log\left(\frac{1-t}{u} - 2\right)\right) du + 0.5607 \int_{\frac{47}{106}}^{\frac{2691}{5950}} \frac{dt}{t} \int_{\frac{57}{40}-\frac{35}{12}t}^{\frac{57}{40}-\frac{29}{10}t} \frac{du}{u^2} + 0.5607 \int_{\frac{2691}{5950}}^{\frac{897}{1972}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{57}{40}-\frac{29}{10}t} \frac{du}{u^2}$$

$$+\int_{\frac{233}{480}}^{\frac{109}{223}} \frac{dt}{t} \int_{\frac{59}{18}-\frac{58}{9}t}^{\frac{179}{55}-\frac{70}{11}t} \frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right) du$$

$$+ 0.5607 \int_{\frac{109}{223}}^{\frac{49}{100}} \frac{dt}{t} \int_{\frac{59}{18}-\frac{58}{9}t}^{\frac{1}{4}(1-t)} \frac{du}{u^2}$$

$$+\int_{\frac{109}{223}}^{\frac{49}{100}} \frac{dt}{t} \int_{\frac{1}{4}(1-t)}^{\frac{179}{55}-\frac{70}{11}t} \frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right) du$$

$$+ 0.5607 \int_{\frac{49}{100}}^{\frac{661}{1345}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{1}{4}(1-t)} \frac{du}{u^2}$$

$$+\int_{\frac{49}{100}}^{\frac{661}{1345}} \frac{dt}{t} \int_{\frac{1}{4}(1-t)}^{\frac{179}{55}-\frac{70}{11}t} \frac{1}{u(1-t-u)}\left(1+\log\left(\frac{1-t}{u}-2\right)\right) du$$

$$+ 0.5607 \int_{\frac{661}{1345}}^{\frac{2944}{5950}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{179}{55}-\frac{70}{11}t} \frac{du}{u^2} + O(\varepsilon)\Bigg)$$

$$\geq (0.000612 + 0.000691 + 0.000645 + 0.005586 + 0.001031 + 0.002927 + 0.000331 + 0.000980 + 0.002633 + 0.000435 + 0.003183)\eta x \log^{-1} x$$

$$= 0.019054 \eta x \log^{-1} x.$$

The proof of Lemma 28 is complete.

LEMMA 29.

$$\Phi = \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 + \Omega_6 + \Omega_8 + \Omega_{10} + \Omega_{12} + \Omega_{14} + \Omega_{17} + \Omega_{20} + \Omega_{23} + \Omega_{26} + \Omega_{29} + \Omega_{32} + \Omega_{35} + \Omega_{38} + \Omega_{50} + \Omega_{54} + \Omega_{57} + \Omega_{60} \geq 1.902585 \eta x \log^{-1} x.$$

Proof. The main idea in this lemma is adopted from [20]. We have

$$\Omega_1 \geq \sum_{X^{\frac{9}{85}} < p \leq X^{\frac{21}{100}-10^{-8}}} \sum_{X^{\frac{9}{85}} < q < p} S(\mathcal{A}_{pq}, q) \tag{29}$$

$$= \sum_{X^{\frac{9}{85}} < p \leq X^{\frac{21}{100}-10^{-8}}} \sum_{X^{\frac{9}{85}} < q < p} S(\mathcal{A}_{pq}, X^{\delta}) - \sum_{X^{\frac{9}{85}} < p \leq X^{\frac{21}{100}-10^{-8}}} \sum_{X^{\frac{9}{85}} < q < p} \sum_{X^{\delta} < r < X^{\frac{9}{85}}} S(\mathcal{A}_{pqr}, r)$$

$$- \sum_{X^{\frac{9}{85}}<p\le X^{\frac{21}{100}-10^{-8}}} \sum_{X^{\frac{9}{85}}<q<p} \sum_{X^{\frac{9}{85}}<r<\min(q,(\frac{2X}{pq})^{\frac{1}{2}})} S(\mathcal{A}_{pqr}, r)$$
$$= \Phi_1 - \Phi_2 - \Phi_3.$$

Let $z = X^{\delta}$ and $D(p,q) = X^{\frac{21}{40}}/(pq)$. Using Iwaniec's sieve method, in the same way as in Lemma 26, we have

$$\Phi_1 = \eta \sum_{X^{\frac{9}{85}}<p\le X^{\frac{21}{100}-10^{-8}}} \sum_{X^{\frac{9}{85}}<q<p} S(\mathcal{B}_{pq}, X^{\delta}) + O(\delta\eta x \log^{-1} x). \tag{30}$$

Lemma 18 yields

$$\Phi_2 = \eta \sum_{X^{\frac{9}{85}}<p\le X^{\frac{21}{100}-10^{-8}}} \sum_{X^{\frac{9}{85}}<q<p} \sum_{X^{\delta}<r<X^{\frac{9}{85}}} S(\mathcal{B}_{pqr}, r) + O(\delta\eta x \log^{-1} x). \tag{31}$$

Hence,

$$\begin{aligned} \Phi_1 - \Phi_2 &= \eta \sum_{X^{\frac{9}{85}}<p\le X^{\frac{21}{100}-10^{-8}}} \sum_{X^{\frac{9}{85}}<q<p} S(\mathcal{B}_{pq}, X^{\frac{9}{85}}) + O(\delta\eta x \log^{-1} x) \\ &= \eta x \log^{-1} x \cdot \frac{85}{9} \int_{\frac{9}{85}}^{\frac{21}{100}-10^{-8}} \frac{dt}{t} \int_{\frac{9}{85}}^{t} \frac{1}{u} w\Big(\frac{85}{9}(1-t-u)\Big)\, du \\ &\quad + O(\delta\eta x \log^{-1} x) \\ &\ge \eta x \log^{-1} x \bigg(\frac{85}{9} \int_{\frac{9}{85}}^{\frac{21}{100}} \frac{dt}{t} \int_{\frac{9}{85}}^{t} \frac{1}{u} w\Big(\frac{85}{9}(1-t-u)\Big)\, du - 10^{-6}\bigg) \\ &\ge \eta x \log^{-1} x \bigg(0.5612 \cdot \frac{85}{9} \int_{\frac{9}{85}}^{\frac{21}{100}} \frac{dt}{t} \int_{\frac{9}{85}}^{t} \frac{du}{u} - 10^{-6}\bigg) \\ &\ge 1.242693 \eta x \log^{-1} x. \end{aligned} \tag{32}$$

Note that if $q < (2X/p)^{\frac{1}{3}}$, then $q < (2X/(pq))^{\frac{1}{2}}$. We have

$$\begin{aligned} \Phi_3 &\le \sum_{X^{\frac{9}{85}}<p\le X^{\frac{21}{100}-10^{-8}}} \sum_{X^{\frac{9}{85}}<q<p} \sum_{X^{\frac{9}{85}}<r<q} S(\mathcal{A}_{pqr}, X^{\frac{9}{85}}) \\ &= \sum_{X^{\frac{9}{85}}<p\le X^{\frac{21}{100}-10^{-8}}} \sum_{X^{\frac{9}{85}}<q<p} \sum_{X^{\frac{9}{85}}<r<q} S(\mathcal{A}_{pqr}, X^{\delta}) \\ &\quad - \sum_{X^{\frac{9}{85}}<p\le X^{\frac{21}{100}-10^{-8}}} \sum_{X^{\frac{9}{85}}<q<p} \sum_{X^{\frac{9}{85}}<r<q} \sum_{X^{\delta}<s<X^{\frac{9}{85}}} S(\mathcal{A}_{pqrs}, s) \\ &= \Phi_4 - \Phi_5. \end{aligned} \tag{33}$$

Let $z = X^{\delta}$ and $D(p,q,r) = X^{10^{-8}}$. Using Lemma 17 with region (i), in the same way as in Lemma 26, we have

$$\Phi_4 = \eta \sum_{X^{\frac{9}{85}} < p \le X^{\frac{21}{100} - 10^{-8}}} \sum_{X^{\frac{9}{85}} < q < p} \sum_{X^{\frac{9}{85}} < r < q} S(\mathcal{B}_{pqr}, X^{\delta}) + O(\delta \eta x \log^{-1} x). \tag{34}$$

Lemma 18 yields

$$\Phi_5 = \eta \sum_{X^{\frac{9}{85}} < p \le X^{\frac{21}{100} - 10^{-8}}} \sum_{X^{\frac{9}{85}} < q < p} \sum_{X^{\frac{9}{85}} < r < q} \sum_{X^{\delta} < s < X^{\frac{9}{85}}} S(\mathcal{B}_{pqrs}, s) + O(\delta \eta x \log^{-1} x). \tag{35}$$

We therefore have

$$\begin{aligned}
\Phi_3 &\le \eta \sum_{X^{\frac{9}{85}} < p \le X^{\frac{21}{100} - 10^{-8}}} \sum_{X^{\frac{9}{85}} < q < p} \sum_{X^{\frac{9}{85}} < r < q} S(\mathcal{B}_{pqr}, X^{\frac{9}{85}}) \\
&\quad + O(\delta \eta x \log^{-1} x) \\
&\le \eta x \log^{-1} x \cdot \frac{85}{9} \int_{\frac{9}{85}}^{\frac{21}{100}} \frac{dt}{t} \int_{\frac{9}{85}}^{t} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{1}{v} w\left(\frac{85}{9}(1-t-u-v)\right) dv \\
&\le \eta x \log^{-1} x \Bigg(0.5617 \cdot \frac{85}{9} \int_{\frac{9}{85}}^{\frac{49}{255}} \frac{dt}{t} \int_{\frac{9}{85}}^{t} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{dv}{v} \\
&\quad + 0.5617 \cdot \frac{85}{9} \int_{\frac{49}{255}}^{\frac{21}{100}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{1}{2}(\frac{49}{85}-t)} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{dv}{v} \\
&\quad + 0.5644 \cdot \frac{85}{9} \int_{\frac{49}{255}}^{\frac{21}{100}} \frac{dt}{t} \int_{\frac{1}{2}(\frac{49}{85}-t)}^{t} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{dv}{v} \Bigg) \\
&\le (0.187169 + 0.077309 + 0.019528) \eta x \log^{-1} x \\
&= 0.284006 \eta x \log^{-1} x.
\end{aligned}$$

Hence,

$$\Omega_1 \ge 0.958687 \eta x \log^{-1} x. \tag{36}$$

In the same way, it can be shown that

$$\Omega_2 + \Omega_3 + \Omega_4 + \Omega_6 + \Omega_8 + \Omega_{10} + \Omega_{12} + \Omega_{14} + \Omega_{17} + \Omega_{20} + \Omega_{23}$$

$$+ \Omega_{26} + \Omega_{29} + \Omega_{32} + \Omega_{35} + \Omega_{38} + \Omega_{50} + \Omega_{54} + \Omega_{57} + \Omega_{60}$$

$$\geq \eta x \log^{-1} x \Bigg(\frac{85}{9} \int_{\frac{21}{100}}^{\frac{13}{60}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{7}{20}-\frac{2}{3}t} \frac{1}{u} w\bigg(\frac{85}{9}(1-t-u)\bigg)\, du$$

$$- \frac{85}{9} \int_{\frac{21}{100}}^{\frac{13}{60}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{7}{20}-\frac{2}{3}t} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{1}{v} w\bigg(\frac{85}{9}(1-t-u-v)\bigg)\, dv$$

$$+ \frac{85}{9} \int_{\frac{13}{60}}^{\frac{12613}{49300}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{29}{120}-\frac{t}{6}} \frac{1}{u} w\bigg(\frac{85}{9}(1-t-u)\bigg)\, du$$

$$- \frac{85}{9} \int_{\frac{13}{60}}^{\frac{12613}{49300}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{29}{120}-\frac{t}{6}} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{1}{v} w\bigg(\frac{85}{9}(1-t-u-v)\bigg)\, dv$$

$$+ \frac{85}{9} \int_{\frac{12613}{49300}}^{\frac{3441}{9860}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{897}{1972}-t} \frac{1}{u} w\bigg(\frac{85}{9}(1-t-u)\bigg)\, du$$

$$- \frac{85}{9} \int_{\frac{12613}{49300}}^{\frac{3441}{9860}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{897}{1972}-t} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{1}{v} w\bigg(\frac{85}{9}(1-t-u-v)\bigg)\, dv$$

$$+ \frac{85}{9} \int_{\frac{297}{800}}^{\frac{653}{1700}} \frac{dt}{t} \int_{\frac{49}{100}-t}^{\frac{19}{160}} \frac{1}{u} w\bigg(\frac{85}{9}(1-t-u)\bigg)\, du$$

$$- \frac{85}{9} \int_{\frac{297}{800}}^{\frac{653}{1700}} \frac{dt}{t} \int_{\frac{49}{100}-t}^{\frac{19}{160}} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{1}{v} w\bigg(\frac{85}{9}(1-t-u-v)\bigg)\, dv$$

$$+ \frac{85}{9} \int_{\frac{653}{1700}}^{\frac{5073}{13120}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{19}{160}} \frac{1}{u} w\bigg(\frac{85}{9}(1-t-u)\bigg)\, du$$

$$- \frac{85}{9} \int_{\frac{653}{1700}}^{\frac{5073}{13120}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{19}{160}} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{1}{v} w\bigg(\frac{85}{9}(1-t-u-v)\bigg)\, dv$$

$$+ \frac{85}{9} \int_{\frac{5073}{13120}}^{\frac{5541}{13940}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{133}{230}-\frac{82}{69}t} \frac{1}{u} w\bigg(\frac{85}{9}(1-t-u)\bigg)\, du$$

$$- \frac{85}{9} \int_{\frac{5073}{13120}}^{\frac{5541}{13940}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{133}{230} - \frac{82}{69}t} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{1}{v} w\left(\frac{85}{9}(1-t-u-v)\right) dv$$

$$- 0.000100\Bigg)$$

$$\geq \eta x \log^{-1} x \Bigg(0.5612 \cdot \frac{85}{9} \int_{\frac{21}{100}}^{\frac{13}{60}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{7}{20} - \frac{2}{3}t} \frac{du}{u}$$

$$- 0.5644 \cdot \frac{85}{9} \int_{\frac{21}{100}}^{\frac{13}{60}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{7}{20} - \frac{2}{3}t} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{dv}{v}$$

$$+ 0.5612 \cdot \frac{85}{9} \int_{\frac{13}{60}}^{\frac{12613}{49300}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{29}{120} - \frac{t}{6}} \frac{du}{u}$$

$$- 0.5617 \cdot \frac{85}{9} \int_{\frac{13}{60}}^{\frac{12613}{49300}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{1}{2}(\frac{49}{85}-t)} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{dv}{v}$$

$$- 0.5644 \cdot \frac{85}{9} \int_{\frac{13}{60}}^{\frac{12613}{49300}} \frac{dt}{t} \int_{\frac{1}{2}(\frac{49}{85}-t)}^{\frac{29}{120} - \frac{t}{6}} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{dv}{v}$$

$$+ 0.5612 \cdot \frac{85}{9} \int_{\frac{12613}{49300}}^{\frac{3441}{9860}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{897}{1972} - t} \frac{du}{u}$$

$$- 0.5617 \cdot \frac{85}{9} \int_{\frac{12613}{49300}}^{\frac{1643}{4930}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{1}{2}(\frac{49}{85}-t)} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{dv}{v}$$

$$- 0.5644 \cdot \frac{85}{9} \int_{\frac{12613}{49300}}^{\frac{1643}{4930}} \frac{dt}{t} \int_{\frac{1}{2}(\frac{49}{85}-t)}^{\frac{897}{1972} - t} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{dv}{v}$$

$$- 0.5644 \cdot \frac{85}{9} \int_{\frac{1643}{4930}}^{\frac{3441}{9860}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{897}{1972} - t} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{dv}{v}$$

$$+ 0.5612 \cdot \frac{85}{9} \int_{\frac{297}{800}}^{\frac{653}{1700}} \frac{dt}{t} \int_{\frac{49}{100} - t}^{\frac{19}{160}} \frac{du}{u}$$

$$
\begin{aligned}
&- 0.5644 \cdot \frac{85}{9} \int_{\frac{297}{800}}^{\frac{653}{1700}} \frac{dt}{t} \int_{\frac{49}{100}-t}^{\frac{19}{160}} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{dv}{v} \\
&+ 0.5612 \cdot \frac{85}{9} \int_{\frac{653}{1700}}^{\frac{5073}{13120}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{19}{160}} \frac{du}{u} \\
&- 0.5644 \cdot \frac{85}{9} \int_{\frac{653}{1700}}^{\frac{5073}{13120}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{19}{160}} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{dv}{v} \\
&+ 0.5612 \cdot \frac{85}{9} \int_{\frac{5073}{13120}}^{\frac{5541}{13940}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{133}{230}-\frac{82}{69}t} \frac{du}{u} \\
&- 0.5644 \cdot \frac{85}{9} \int_{\frac{5073}{13120}}^{\frac{5541}{13940}} \frac{dt}{t} \int_{\frac{9}{85}}^{\frac{133}{230}-\frac{82}{69}t} \frac{du}{u} \int_{\frac{9}{85}}^{u} \frac{dv}{v} \\
&- 0.000100\Big) \\
\geq\ &(0.111673 - 0.037863 + 0.570625 - 0.100014 \\
&- 0.085420 + 0.599832 - 0.063002 - 0.072362 \\
&- 0.000822 + 0.010098 - 0.000782 + 0.004011 \\
&- 0.000232 + 0.008592 - 0.000335 - 0.000100)\eta x \log^{-1} x \\
=\ &0.943899\eta x \log^{-1} x.
\end{aligned}
$$

Hence,

$$\Phi \geq 1.902585\eta x \log^{-1} x.$$

The proof of Lemma 29 is complete.

10. The proof of the Theorem. From Lemmas 24–29, it follows that

$$\Omega \geq 2.949096\eta x \log^{-1} x.$$

Then using Lemmas 22 and 23, we obtain

$$\pi(x + \eta x) - \pi(x) = S(\mathcal{A}, (2X)^{\frac{1}{2}}) \geq 0.011\eta x \log^{-1} x.$$

Now (15) holds and the proof of the Theorem is complete.

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