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Asymptotic behaviour of some infinite products involving prime numbers

by

Hsien-Kuei Hwang (Taipei)

1. Introduction. Given an integer $n \ge 2$, let $\omega(n)$ denote the number of distinct prime factors of n in the decomposition of n into prime factors. Let Q denote the set of square-free positive integers, namely, integers having no repeated prime factors in their factorizations. The distribution of the values of the arithmetic function $\omega(n)$ has received much attention in the literature (cf. [15, 16, 7, 6, 17, 13]). In particular, the Sathe–Selberg formulae state (cf. [16], [17, p. 231]):

$$\frac{1}{x}\#\{n : 1 \le n \le x \text{ and } \omega(n) = m\} = \frac{(\log\log x)^{m-1}}{(m-1)!\log x}\left(G(r) + O\left(\frac{B_G\, m}{(\log\log x)^2}\right)\right), \tag{1}$$

$$\frac{1}{x}\#\{n : 1 \le n \le x,\ n \in Q \text{ and } \omega(n) = m\} = \frac{(\log\log x)^{m-1}}{(m-1)!\log x}\left(F(r) + O\left(\frac{B_F\, m}{(\log\log x)^2}\right)\right), \tag{2}$$

uniformly for $1 \le m \le M\log\log x$, for any fixed $M > 0$, where $r = (m-1)/(\log\log x)$, G and F are entire functions defined by

$$G(z) = \frac{1}{\Gamma(z+1)}\prod_{p\,\text{prime}}\left(1 + \frac{z}{p-1}\right)\left(1 - \frac{1}{p}\right)^{z}, \tag{3}$$

$$F(z) = \frac{1}{\Gamma(z+1)}\prod_{p\,\text{prime}}\left(1 + \frac{z}{p}\right)\left(1 - \frac{1}{p}\right)^{z}, \tag{4}$$

and $B_G := \sup_{|z|\le M}|G''(z)|$, B_F being similarly defined. These two functions are related by

$$zG(z) = F(z-1)$$

as can easily be seen.

The asymptotic behaviour of these two entire functions is of some independent interest. Intuitively, such information sheds new light on the asymptotic natures of (1) and (2). Indeed, using Stirling's formula for $1/\Gamma(z+1)$ and formally differentiating both sides of (8) below with respect to z leads to

$$\frac{F''(z)}{F(z)} = (\log z)^2 + 2(\log z)(\log\log z) + O(\log z) \qquad (|z| \to \infty,\ |\arg z| \le \pi - \varepsilon),$$

a result that can be rigorously justified by the methods used in this paper. Let $N(x,m)$ denote the quantity on the left-hand side of (2). Then the right-hand side of (2) can be further made explicit by (cf. [17, p. 240], [8, p. 210])

$$N(x,m) = \frac{(\log\log x)^{m-1}}{(m-1)!\log x} F(r)\left(1 - \frac{F''(r)(m-1)}{2F(r)(\log\log x)^2} + \text{smaller order terms}\right).$$

Thus, as $r = (m-1)/(\log\log x) \to \infty$,

$$\frac{F''(r)(m-1)}{2F(r)(\log\log x)^2} = \frac{F''(r)r}{F(r)\log\log x} \sim \frac{r(\log r)^2}{\log\log x},$$

the rightmost term is $o(1)$ if and only if $r = o((\log\log x)/(\log\log\log x)^2)$; or, equivalently, $m = o((\log\log x)^2/(\log\log\log x)^2)$. This suggests that (2) might still hold for m in this range. The justification of such a formal process requires, of course, a further argument.

Likewise, we can "guess" the same type of result for the left-hand side of (1), which is indeed true, as was shown by Hensley [6] (cf. also [7, 13]).

On the other hand, if we let $F(z) = \sum_{n\ge 0} \alpha_n z^n$, then the asymptotic behaviour of α_n as $n \to \infty$ is closely related to that of $F(z)$ as $z \to \infty$ by the formula

$$\alpha_n = \frac{1}{2i\pi} \oint_{|z|=r} z^{-n-1} F(z)\, dz \qquad (r > 0),$$

especially when we apply the saddle-point method (cf. [3]).

The aim of this paper is to derive asymptotic expansions of $F(z)$ as $|z| \to \infty$ and $|\arg z| < \pi$. Since the asymptotic behaviour of the entire function $1/\Gamma(z)$ is well known, it suffices to consider the series

$$(5) \qquad f(z) := \sum_{p\ \text{prime}} \left(\log\left(1 + \frac{z}{p}\right) + z\log\left(1 - \frac{1}{p}\right)\right) \qquad (z \ne -p,\ p \text{ prime}).$$

Let $\mathcal{H}_0$ denote a certain truncated Hankel contour around the origin (counter-clockwise) in the s-plane (see the next section for precise definition). Throughout this paper, the symbol ε always denotes a small positive quantity whose value may vary from one occurrence to another. Our main result is the following.

THEOREM 1. *The function f satisfies*

$$-\frac{f(z)}{z+1} = \frac{1}{2i\pi} \int_{\mathcal{H}_0} \frac{\pi z^s}{(1+s)\sin \pi s} \log(1/s)\, ds + R(z), \tag{6}$$

where the error term R satisfies

$$R(z) \ll \exp\biggl(-\frac{c \log|z|}{(\log\log|z|)^{2/3}(\log\log\log|z|)^{1/3}} \biggr), \tag{7}$$

uniformly as $|z| \to \infty$ and $|\arg z| \le \pi - \varepsilon$, $\varepsilon > 0$, $c > 0$ being some absolute constant. Moreover, $R(z) \ll |z|^{-1/2+\varepsilon}$ under the Riemann hypothesis.

From (6), we deduce the following expansion. The symbol $[z^n]h(z)$ represents the coefficient of z^n in the Taylor expansion of h.

THEOREM 2. *If $|z| \to \infty$ in the region $|\arg z| \le \pi - \varepsilon$, $\varepsilon > 0$, then f satisfies*

$$-\frac{f(z)}{z+1} = \log\log z - \gamma + \sum_{1 \le j < \nu} \frac{c_j (j-1)!}{(\log z)^j} + O((\log|z|)^{-\nu}), \tag{8}$$

uniformly with respect to z, where $\nu = 1, 2, \ldots,$ γ is Euler's constant and the coefficients c_j are defined by

$$\begin{aligned} c_j &:= [z^j] \frac{\pi z}{(1-z)\sin \pi z} \\ &= 2 \sum_{0 \le l \le [j/2]} \frac{(-1)^l}{(2l)!}(1 - 2^{2l-1})B_{2l}\pi^{2l} \qquad (j = 0, 1, 2, \ldots), \end{aligned} \tag{9}$$

the B_l's being Bernoulli numbers.

An alternative expression for c_j is

$$c_j = j + \frac{(-1)^j}{2} + \sum_{l \ge 2} \frac{(-1)^l}{l^j} \biggl(\frac{1}{l-1} - \frac{(-1)^j}{l+1} \biggr) \qquad (j = 0, 1, 2, \ldots), \tag{10}$$

which is easily obtained by standard expansion of meromorphic functions and is both exact and asymptotic (as $j \to \infty$). In particular, $c_0 = c_1 = 1$, $c_2 = c_3 = 1 + \frac{1}{6}\pi^2$, $c_4 = c_5 = 1 + \frac{1}{6}\pi^2 + \frac{7}{360}\pi^4$. Obviously, c_j is a polynomial in π^2 of degree $[j/2]$ with positive coefficients.

Two closely related infinite products arise in the distribution of the number of distinct irreducible factors of a monic polynomial over a finite field F_q (cf. [1, 19] and [8, Ch. 5]):

$$\widetilde{G}(z) = \prod_{k\geq 1} \left(\left(1 + \frac{z}{q^k - 1}\right) e^{-z/q^k} \right)^{I_k},$$

$$\widetilde{F}(z) = \prod_{k\geq 1} \left(\left(1 + \frac{z}{q^k}\right) e^{-z/q^k} \right)^{I_k},$$

where

$$I(z) := \sum_{j\geq 1} I_j z^j = \sum_{j\geq 1} \frac{\mu(j)}{j} \log \frac{1}{1 - qz^j} \qquad (|z| < 1/q), \tag{11}$$

the $\mu(j)$ being the Möbius function.

The asymptotic behaviours of $\log \widetilde{F}(z)$ and $\log \widetilde{G}(z)$ can be treated by the same approach.

Theorem 3. *Let $\widetilde{f}(z) = \log \widetilde{F}(z)$, $z \in \mathbb{C} \setminus (-\infty, -q]$. Then $\widetilde{f}$ satisfies*

$$\begin{aligned}
-\frac{\widetilde{f}(z)}{z} &= \frac{1}{2i\pi} \int_{\mathcal{H}_0} \frac{\pi z^s}{(1+s)\sin \pi s} \log(1/s)\, ds + K_q - \log\log q \qquad (12)\\
&\quad + O(|z|^{-1/2+\varepsilon})\\
&= \log\log z + K_q - \gamma - \log\log q + \sum_{1\leq j<\nu} \frac{c_j (j-1)!}{(\log z)^j} \qquad (13)\\
&\quad + O((\log |z|)^{-\nu}),
\end{aligned}$$

uniformly as $|z| \to \infty$ in the sector $|\arg z| \leq \pi - \varepsilon$, where $\nu = 1, 2, \ldots,$ c_j is as in Theorem 2 and

$$K_q := \sum_{j\geq 2} \frac{\mu(j)}{j} \log \frac{1}{1 - q^{1-j}}.$$

As to the function $\widetilde{G}$, since

$$\log \widetilde{G}(z) = \log \widetilde{F}(z) + \sum_{j\geq 1} I_j \left(\log\left(1 + \frac{z}{q^j - 1}\right) - \log\left(1 + \frac{z}{q^j}\right) \right),$$

and the Mellin transform of the last series exists in the strip $-1 < \Re s < 0$, we conclude that the asymptotic behaviour of $\log \widetilde{G}$ is also characterized by (12) and (13).

While the leading term on the right-hand side of (8) may be derived by Mertens's formula (cf. [13, p. 239]), the methods developed here have wide applications to other entire functions defined via infinite products issuing from arithmetical functions or combinatorial structures: integers subject to arithmetical constraints (cf. [14]), arithmetical semigroups under Axiom $A^{\#}$ (cf. [11, 12, 5]), the combinatorial schemes of Flajolet and Soria (cf. [5], [8, Ch. 5]), "factorisatio numerorum" in arithmetical semigroups (cf. [10]), and the combinatorial scheme developed by the author (having an exponential singularity) (cf. [8, Chs. 6, 9]), etc. Of these, an interesting example is the random mapping patterns (cf. [5]) in which the integrand in question (when applying the Mellin inversion formula) has both logarithmic and algebraic singularities, thus successive terms in the asymptotic expansion are of order $(\log z)^{-j/2}$ in lieu of $(\log z)^{-j}$.

To avoid technical complications, we content ourselves with the proof of Theorems 1–3. The infinite products of $\widetilde{G}$ and of $\widetilde{F}$ (when taking logarithm) are special classes of the so-called harmonic sums (see [4] for a general introduction and survey).

2. The proof of the theorems

Proof of Theorem 1. Let $\pi(x) = \sum_{p\le x} 1$ denote the number of primes $\le x$ for $x \ge 2$ and $\pi(x) = 0$ for $x < 2$. By writing f (defined in (5)) as a Stieltjes integral and by an integration by parts, we have

$$f(z) = -z(z+1)\int_{2^-}^{\infty} \frac{\pi(x)}{x(x+1)(x+z)}\,dx =: -z(z+1)h(z), \tag{14}$$

say. Thus h is the Stieltjes transform of the function $x \mapsto \pi(x)/(x(x-1))$. Observe that h can be written in the form

$$h(z) = \int_0^{\infty} u(x)v\left(\frac{z}{x}\right)dx,$$

with $v(x) = 1/(1+x)$ and

$$u(x) = \begin{cases} \dfrac{\pi(x)}{x^2(x-1)} & \text{for } x \ge 2,\\ 0 & \text{for } 0 \le x < 2.\end{cases}$$

Thus the Mellin transform of h satisfies

$$M[h;s] = M[v;s]M[u;1+s],$$

and h is expressible by the Parseval formula (cf. [20, Ch. 3]):

$$h(z) = \frac{1}{2i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} z^{-s}M[v;s]M[u;1+s]\,ds \tag{15}$$

for $z \notin (-\infty, 0]$, where $z^{-s} = \exp(-s\log z)$, log having its principal value, and σ is any real number lying on the common strip of $M[v;s]$ and $M[u;1+s]$. It remains to find explicit representations for the Mellin transforms of u and of v. Now, from the table of Mellin transforms in [20, p. 193], we find

$$M[v;s] = \frac{\pi}{\sin \pi s} \qquad (0 < \Re s < 1),$$

and the Mellin transform $M[u; 1+s]$ is easily seen to exist in the half plane $\Re s < 1$, since $u(x) \ll x^{-2}(\log x)^{-1}$ as $x \to \infty$. Substituting these into (15) and carrying out the change of variables $s \to -s$ yields

$$h(z) = \frac{1}{2i\pi} \int_{-1/2-i\infty}^{-1/2+i\infty} \frac{-\pi z^s}{\sin \pi s} M[u; 1-s]\, ds. \tag{16}$$

Using the formula (cf. [18, §3.7])

$$\int_2^\infty \frac{\pi(x)}{x(x^s-1)}\, dx = \frac{\log \zeta(s)}{s} \qquad (\Re s > 1),$$

ζ being Riemann's zeta function, we have

$$M[u; 1-s] = \int_2^\infty \frac{\pi(x)}{x^{s+2}(x-1)}\, dx = \frac{\log \zeta(s+2)}{s+2} + \varpi(s+2),$$

for $\Re s > -1$, where

$$\varpi(s) = \int_2^\infty \frac{(x^s - x)\pi(x)}{x^{s+1}(x^s-1)(x-1)}\, dx, \tag{17}$$

is regular and bounded for $\Re s \geq 1/2 + \varepsilon$ and satisfies $\varpi(1) = 0$. Thus

$$h(z) = I_1 + I_2, \tag{18}$$

where

$$I_1 = \frac{1}{2i\pi} \int_{-1/2-i\infty}^{-1/2+i\infty} \frac{-\pi z^s}{(s+2)\sin \pi s} \log \zeta(s+2)\, ds,$$

$$I_2 = \frac{1}{2i\pi} \int_{-1/2-i\infty}^{-1/2+i\infty} \frac{-\pi z^s}{(s+2)\sin \pi s} \varpi(s+2)\, ds.$$

The integrand of I_2 having a removable singularity at $s = -1$, it follows, by (17) and the absolute convergence of the integral, that

$$|I_2| \ll |z|^{-3/2+\varepsilon} \qquad (\varepsilon > 0). \tag{19}$$

To evaluate the integral I_1, we use the following zero-free region for $\zeta(s+2)$ (cf. [9]):

$$\sigma \geq -1 - \frac{c}{(\log |t|)^{2/3}(\log\log |t|)^{1/3}} \qquad (s = \sigma + it,\ |t| \geq t_0 > e^e,\ c > 0),$$

in which $\log \zeta(s+2)$ satisfies the estimate

$$\log \zeta(\sigma + 2 + it) \ll (\log |t|)^{2/3}(\log\log |t|)^{1/3}. \tag{20}$$

We now take a large positive number $T > t_0$ and a small quantity $0<\delta<1/2$ and set

$$a = -1 + (\log |z|)^{-1} \quad \text{and} \quad b = -1 - c(\log T)^{-2/3}(\log\log T)^{-1/3}.$$

Move the line of integration of I_2 to the contour $\mathcal{C}$ shown in Figure 1. The integration contour $\mathcal{C}$ consists of 7 parts described as follows:

$$\mathcal{C}_1 = (a-i\infty, a-iT], \quad \mathcal{C}_2 = [a-iT, b-iT], \quad \mathcal{C}_3 = \{s : s = b+it; -T \le t \le -\delta\},$$

$$\begin{aligned}\mathcal{H}_{-1} &= \mathcal{H}_{-1}(T,\delta)\\ &= [b-i\delta, -1-i\delta] \cup \{s : s = \delta e^{i\theta}, -\pi/2 \le \theta \le \pi/2\} \cup [-1+i\delta, b+i\delta],\end{aligned}$$

$$\mathcal{C}_4 = \{s : s = b+it; \delta \le t \le T\}, \quad \mathcal{C}_5 = [b+iT, a+iT], \quad \mathcal{C}_6 = [a+iT, a+i\infty).$$

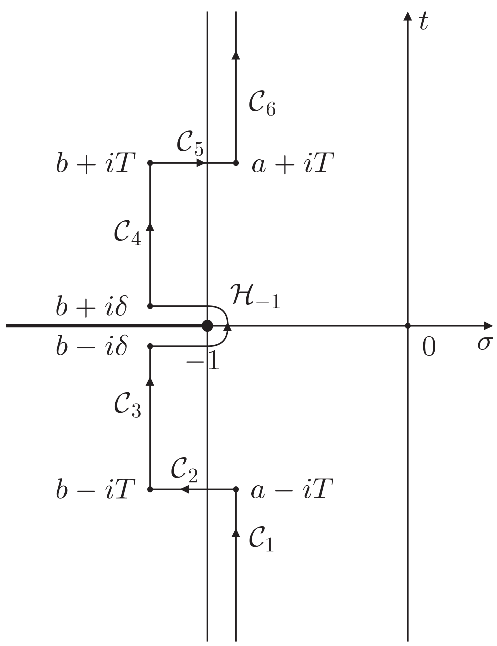

Fig 1. The contour $\mathcal{C}$

For convenience, let J_k denote the integral

$$J_k := \frac{1}{2i\pi} \int_{\mathcal{C}_k} \frac{-\pi z^s}{(s+2)\sin \pi s} \log \zeta(s+2)\, ds$$

for $k = 1, 2, \ldots, 6$.

From (20) and the bound

$$\frac{-\pi z^{\sigma+it}}{\sin \pi(\sigma+it)} \ll |z|^\sigma e^{-(\pi+\theta_z)|t|} \qquad (-\pi+\varepsilon \le \theta_z = \arg z \le \pi - \varepsilon),$$

we deduce the following estimates:

$$\begin{aligned}
J_1 + J_6 &= \frac{1}{2\pi} \int\limits_{|t| \ge T} \frac{-\pi z^{a+it}}{(a+2+it)\sin \pi(a+it)} \log \zeta(a+2+it)\, dt \\
&\ll |z|^a e^{-(\pi+\theta_z)T}, \\
J_2 + J_5 &\ll \frac{|z|^a}{\log |z|}\, e^{-(\pi+\theta_z)T}, \\
J_3 + J_4 &= \frac{1}{2\pi} \int\limits_{\delta \le |t| \le T} \frac{-\pi z^{b+it}}{(b+2+it)\sin \pi(b+it)} \log \zeta(b+2+it)\, dt \ll |z|^b.
\end{aligned}$$

Thus, letting

$$J_{\mathcal{H}} = \frac{1}{2i\pi} \int\limits_{\mathcal{H}_{-1}} \frac{-\pi z^s}{(s+2)\sin \pi s} \log \zeta(s+2)\, ds,$$

we have

$$I_1 = J_{\mathcal{H}} + R_0(z), \tag{21}$$

where

$$R_0(z) \ll |z|^a e^{-(\pi+\theta_z)T} + |z|^b \ll |z|^{-1} e^{-(\pi+\theta_z)T} + |z|^b.$$

Taking

$$T = \frac{c \log |z|}{(\pi + \theta_z)(\log\log |z|)^{2/3} (\log\log\log |z|)^{1/3}}\,,$$

so as to balance the two error terms of R_0, we obtain the estimate (compare (7))

$$R_0(z) \ll |z|^{-1} \exp\left(- \frac{c \log |z|}{(\log\log |z|)^{2/3} (\log\log\log |z|)^{1/3}} \right). \tag{22}$$

It remains to evaluate the integral $J_{\mathcal{H}}$ which can be decomposed into two parts:

$$\begin{aligned}
J_{\mathcal{H}} &= \frac{1}{2i\pi} \int\limits_{\mathcal{H}_{-1}} \frac{-\pi z^s}{(s+2)\sin \pi s} \log(1/(s+1))\, ds \\
&\quad + \frac{1}{2i\pi} \int\limits_{\mathcal{H}_{-1}} \frac{-\pi z^s}{(s+2)\sin \pi s} \log((s+1)\zeta(s+2))\, ds,
\end{aligned}$$

where the integrand of the second integral is regular on the path of the integration and single-valued along the cut; it is therefore regular inside the contour $\mathcal{H}_{-1}$. By Cauchy's theorem,

$$\frac{1}{2i\pi} \int\limits_{\mathcal{H}_{-1}} \frac{-\pi z^s}{(s+2)\sin \pi s} \log((s+1)\zeta(s+2))\, ds \ll |z|^b, \tag{23}$$

which is of the same order as $R_0(z)$.

Collecting the above results we obtain (6) by (14), (16), (18), (19), (21)–(23) and

$$\frac{1}{2i\pi}\int\limits_{\mathcal{H}_{-1}}\frac{-\pi z^s}{(s+2)\sin\pi s}\log(1/(s+1))\,ds=\frac{1}{2i\pi}\int\limits_{\mathcal{H}_0}\frac{\pi z^{s-1}}{(s+1)\sin\pi s}\log(1/s)\,ds,$$

$\mathcal{H}_0$ being the translated contour of $\mathcal{H}_{-1}$ from -1 to the origin.

Finally, from the above proof, it is obvious that $R_0(z)\ll|z|^{-3/2+\varepsilon}$ under the Riemann hypothesis (cf. [18, Ch. XIV]); or, equivalently, the error term $R(z)$ in (6) satisfies $R(z)\ll|z|^{-1/2+\varepsilon}$.

This completes the proof of Theorem 1. ■

Proof of Theorem 2. To evaluate the asymptotic behaviour of the integral

$$J:=\frac{1}{2i\pi}\int\limits_{\mathcal{H}_0}\frac{\pi z^s}{(s+1)\sin\pi s}\log(1/s)\,ds$$

as $|z|\to\infty$ in the sector $|\arg z|\le\pi-\varepsilon$, we start from the Laurent expansion (cf. (9))

$$\frac{\pi}{(1+s)\sin\pi s}=\frac{1}{s}+\sum_{1\le j<\nu}(-1)^j c_j s^{j-1}+\varrho_\nu(s)\qquad(s\ne0,|s|<1),$$

for any $\nu=1,2,\ldots$, where ϱ_ν is analytic in the unit circle and satisfies $\varrho_\nu(s)\ll|s|^{\nu-1}$ there. Substituting this expansion into J yields

$$J=\frac{1}{2i\pi}\int\limits_{\mathcal{H}_0}\frac{z^s}{s}\log(1/s)\,ds+\sum_{1\le j<\nu}(-1)^j c_j\frac{1}{2i\pi}\int\limits_{\mathcal{H}_0}s^{j-1}z^s\log(1/s)\,ds+Y_\nu(z),$$

where

$$Y_\nu(z)=\frac{1}{2i\pi}\int\limits_{\mathcal{H}_0}\varrho_\nu(s)z^s\log(1/s)\,ds\qquad(\nu=1,2,\ldots).$$

By Hankel's representation of the entire function $1/\Gamma(s)$ (cf. [17, p. 205]) and by extending the integration contour $\mathcal{H}_0$ to $-\infty\pm i\delta$, we deduce, for any fixed $\alpha\in\mathbb{R}$,

$$\frac{1}{2i\pi}\int\limits_{\mathcal{H}_0}\frac{z^s}{s^\alpha}\log(1/s)\,ds=(\log z)^{\alpha-1}\left(\frac{\log\log z}{\Gamma(\alpha)}+\frac{\Gamma'(\alpha)}{\Gamma(\alpha)^2}\right)+O(|z|^{b+1}).$$

Using the relations (cf. [2, p. 15]) $\Gamma'(1)=-\gamma$ and (cf. [2, p. 46])

$$\frac{\Gamma'(-k)}{\Gamma^2(-k)}=(-1)^{k-1}k!\qquad(k=0,1,2,\ldots),$$

we obtain

$$J=\log\log z-\gamma+\sum_{1\le j<\nu}\frac{c_j(j-1)!}{(\log z)^j}+O\Big(|z|^{b+1}\sum_{1\le j<\nu}c_j(j-1)!\Big)+Y_\nu(z).$$

Now, by (10), $c_j \ll j$, it follows that

$$|z|^{b+1} \sum_{1 \le j < \nu} c_j (j-1)! \ll \nu! \, |z|^{b+1} \ll |z|^{b+1} \asymp |R(z)|,$$

and it remains to estimate Y_ν for which the change of variables $w = s \log z$ gives

$$Y_\nu(z) = \frac{1}{2i\pi} \int_{\mathcal{H}_0'} \varrho_\nu\left(\frac{w}{\log z}\right) e^w (\log\log z - \log w) \frac{dw}{\log z},$$

where $\mathcal{H}_0'$ is the transformed contour of $\mathcal{H}_0$. The function ϱ_ν being regular in the unit circle, we deduce

$$\frac{1}{2i\pi} \int_{\mathcal{H}_0'} \varrho_\nu\left(\frac{w}{\log z}\right) e^w dw \ll |z|^{b+1}.$$

Now take $\delta = (\log |z|)^{-1}$. By the definition of $\mathcal{H}_0$ and the estimate $\varrho_\nu(s) \ll |s|^{\nu-1}$, we obtain

$$\begin{aligned} &\frac{-1}{2i\pi} \int_{\mathcal{H}_0'} \varrho_\nu\left(\frac{w}{\log z}\right) e^w \frac{\log w}{\log z} \, dw \\ &\qquad \ll \int_0^{|1+b|\log|z|} \frac{|-\sigma + i\delta|^{\nu-1}}{(\log |z|)^\nu} e^{-\sigma} |\log(\sigma^2 + \delta^2)| d\sigma + \delta^\nu \\ &\qquad \ll (\log |z|)^{-\nu}. \end{aligned}$$

Finally, (9) follows from the expansion (cf. [2, Eq. (5), p. 51])

$$\frac{z}{\sin z} = 2 \sum_{j \ge 0} (-1)^j \frac{B_{2j}}{(2j)!} (1 - 2^{2j-1}) z^{2j} \qquad (|z| < \pi).$$

This completes the proof. ∎

Proof of Theorem 3 (sketch). By definition,

$$\widetilde{f}(z) = \sum_{k \ge 1} I_k \left(\log\left(1 + \frac{z}{q^k}\right) - \frac{z}{q^k} \right),$$

the right-hand side is a harmonic sum (cf. [4]) and its Mellin transform (cf. [20, p. 193]) is given by

$$M[\widetilde{f}; s] = \frac{\pi}{s \sin \pi s} \, I(q^s) \qquad (-2 < \Re s < -1).$$

Thus

$$\widetilde{f}(z) = \frac{1}{2i\pi} \int_{-3/2-i\infty}^{-3/2+i\infty} \frac{\pi z^{-s}}{s \sin \pi s} I(q^s)\, ds.$$

Using the expansion (cf. (11))

$$I(q^s) = \log \frac{1}{1-q^{1+s}} + K_q + O(|1+s|) \qquad (s \sim -1),$$

and proceeding along the same lines as above (with much simpler analysis), we deduce the estimates (12) and (13). ∎

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Institute of Statistical Science
Academia Sinica
Taipei 11529, Taiwan
E-mail: hkhwang@stat.sinica.edu.tw

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