ACTA ARITHMETICA
LXXV.3 (1996)

A note on the equation $ax^n - by^n = c$

by

MAURICE MIGNOTTE (Strasbourg)

1. Introduction. We consider a form of degree $n \geq 3$ with positive rational integer coefficients

$$F(x, y) = ax^n - by^n, \qquad a \neq b,$$

and the equation

$$ax^n - by^n = c,$$

where c is a non-zero integer. Such forms have been studied by many authors.

The first result on such an equation is due to Thue [Th] and is a particular case of his general theorem for the inequality

$$|G(x, y)| \leq c,$$

where $G(x, y)$ is an irreducible binary form of degree $n \geq 3$ with rational integer coefficients. This result was improved by Siegel [S] who proved the following theorem.

THEOREM A. *The inequality*

$$|F(x, y)| \leq c,$$

where a, b, c are positive integers and $n \geq 3$, has at most one solution in positive co-prime integers x, y if

$$(ab)^{n/2-1} \geq 4c^{2n-2}\Big(n \prod_p p^{1/(n-1)}\Big)^n,$$

where p runs through all the different prime factors of n.

Many authors followed this way; the results up to 1968 have been quoted in Mordell's book [M], Chap. 28.

The first effective result on Thue equations is due to Baker [B]. This result has been sharpened several times. Concerning the special case studied here, in [ST], Chap. 2, the following lower bound for $F(x, y)$ is proved.

Theorem B. *There exist computable numbers C_1 and C_2 such that*

$$|ax^n - by^n| \ge (\max\{|x|,|y|\})^{n-C_2\log n}$$

for all rational integers n, x, y with $n \ge C_1$ and $|x| \ne |y|$.

This theorem implies a result due to Tijdeman [Ti]:

Corollary C. *If $abc \ne 0$, $n \ge 0$, $x > 1$ and $y \ge 0$ are rational integers satisfying*

$$ax^n - by^n = c,$$

then n is bounded by a computable number depending only on a, on b and on c.

Here we only consider effective results which are variants of Theorem B or Corollary C.

Theorem 1. *Let*

$$F(x,y) = ax^n - by^n, \qquad a \ne b,$$

be a binary form of degree $n \ge 3$, with positive integer coefficients a and b. Put $A = \max\{a,b,3\}$. Then, for $y > |x|$ and $F(x,y) \ne 0$, we have

$$\begin{aligned}|F(x,y)| \ge \frac{|b|}{1.1} y^n \cdot \exp\left\{-\left(\frac{2+\eta}{3}\cdot\frac{U^2}{\lambda}\log A + \frac{2(2+\eta)}{3}U + 1\right)\log y\right\}\\ \times \exp\{-\theta(1+h/\lambda)^{3/2}(\log A\cdot\log y)^{1/2}\}\\ \times \exp\{-3.04h - 2U\log A - 2.16\,\log A\},\end{aligned}$$

where

$$\lambda = \log\left(1 + \frac{\log A}{|\log(a/b)|}\right),$$

$$h = \max\left\{5\lambda, \log\lambda + 0.47 + \log\left(\frac{n}{\log A} + \frac{1.5}{\log(\max\{y,3\})}\right)\right\},$$

and

$$U = \frac{4h}{\lambda} + 4 + \frac{\lambda}{h}, \quad \eta = \frac{1}{223}, \quad \theta = \frac{16\sqrt{6(2+\eta)}}{3}.$$

Theorem 2. *Let n, F and A be defined as in Theorem 1. Suppose that*

$$F(x,y) = c$$

with $y > |x| > 0$. Then

$$n \le \max\left\{3\,\log(1.5|c/b|), 7400\,\frac{\log A}{\lambda}\right\}.$$

When λ is close to $\log A$ and c is not too large with respect to n, then the previous inequality gives an absolute upper bound on n. This is exactly the content of our main result:

Theorem 3. *Consider the special binary form* $F(x,y) = (b+1)x^n - by^n$, $b \ge 1$. *Suppose that*

$$0 < |F(x,y)| < \min\{(2^n - 2)b, \tfrac{2}{3}n^2b^3\}, \tag{1}$$

with

$$|x| \ne |y| \quad \text{and} \quad xy \ne 0.$$

Then x^n *and* y^n *are necessarily of the same sign*, *thus we may suppose* x *and* y *positive*, *and then*

$$y > x > 1, \qquad y \ge nb(y-x), \quad \text{and} \quad n < 600.$$

This may be the first time where an absolute upper bound is obtained for the exponent of such a family of exponential diophantine equations. Theorem 3 shows the power of Lemma 1 below, which contains all the known refinements on estimates of linear forms in two logs of algebraic numbers (except for the square for the term h). We use the fact that the logarithms of the algebraic numbers appearing in the linear form are small; such a fact was used for the first time in a paper by T. N. Shorey [Sh].

Theorem 3 is also a consequence of Waldschmidt's estimates [W], but Lemma 1 leads to smaller constants than [W].

2. Proof of Theorem 1. Consider a relation in rational integers x, y

$$ax^n - by^n = c, \quad n \ge 3,$$

where a, b are positive rational integers and c is non-zero. Let $A = \max\{|a|, |b|, 3\}$.

Without loss of generality, we may suppose that $|x| \le y$. Theorem 1 is trivially true if $y = 1$. For $y = 2$, considering the two cases $a \ge 2^{n-1}b$ and $a < 2^{n-1}b$, it is easy to verify that Theorem 1 is true. Thus, we assume that $y \ge 3$. From the relation

$$\left|\frac{a}{b}\left(\frac{x}{y}\right)^n - 1\right| = \frac{|c|}{|b|y^n},$$

if $|c| \ge |b|y^n/(4A)$ then Theorem 1 is true [the verification is easy], thus we assume $|c| < |b|y^n/(4A)$ and then the "linear form" $\Lambda := \log(a/b) - n\log|y/x|$ satisfies the inequality $|\Lambda| < 1.1|c/b|\,y^{-n}$.

On the other hand, estimates for linear forms in two logarithms produce lower bounds for $|\Lambda|$. We use the following result from [LMN] (Théorème 2):

Lemma 1. *Let* α_1, α_2 *be two positive real algebraic numbers. Consider*

$$\Lambda = b_2 \log\alpha_2 - b_1 \log\alpha_1,$$

where b_1 *and* b_2 *are positive rational integers. Put* $D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$. *Suppose that* $\log\alpha_1$ *and* $\log\alpha_2$ *are linearly independent over* $\mathbb{Q}$. *For any*

$\varrho > 1$, *take*

$$h \geq \max\left\{\frac{D}{2},\, 5\lambda,\, D\left(\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log\lambda + 1.56\right)\right\},$$
$$a_i \geq (\varrho - 1)|\log\alpha_i| + 2D\,h(\alpha_i) \qquad (i = 1, 2),$$

and

$$a_1 + a_2 \geq 4\max\{1, \lambda\}, \qquad \frac{1}{a_1} + \frac{1}{a_2} \leq \min\{1, \lambda^{-1}\},$$

where $\lambda = \log\varrho$. *Then*

$$\begin{aligned}\log|\Lambda| \geq\ & -\frac{\lambda a_1 a_2}{9}\left(\frac{4h}{\lambda^2} + \frac{4}{\lambda} + \frac{1}{h}\right)^2 - \frac{2\lambda}{3}(a_1 + a_2)\left(\frac{4h}{\lambda^2} + \frac{4}{\lambda} + \frac{1}{h}\right)\\ & - \frac{16\sqrt{2a_1a_2}}{3}\left(1 + \frac{h}{\lambda}\right)^{3/2} - 2(\lambda + h) - \log\left(a_1a_2\left(1 + \frac{h}{\lambda}\right)^2\right)\\ & + \lambda/2 + \log\lambda - 0.15.\end{aligned}$$

Remark. The result in [LMN] is proved under the stonger hypothesis $\min\{a_1, a_2\} \geq \max\{2, 2\lambda\}$, but the proof uses only the weaker conditions stated in Lemma 1.

Theorem 1 is trivially true when $n \leq U + (2/3)U^2(\log A)/\lambda$, moreover $h \geq 5\lambda$ implies $U \geq 24.2$, hence we may suppose

(2) $$n - 1 \geq \frac{N\log A}{\lambda} \quad \text{with } N = 300.$$

Now, we apply Lemma 1; here $D = 1$, $b_1 = n$, $b_2 = 1$, and $\alpha_1 = |y/x|$, $\alpha_2 = a/b$. We have to choose

$$a_1 \geq 2\log y + (\varrho - 1)\log|y/x|, \qquad a_2 \geq 2\log A + (\varrho - 1)|\log(a/b)|.$$

We choose

$$\varrho = 1 + \frac{\log A}{|\log(a/b)|}.$$

Then we can take

$$a_2 = 3\log A.$$

Clearly, $A/(A-1) \leq \max\{a/b, b/a\} \leq A$, hence $1/A < |\log(a/b)| \leq \log A$. Thus, $\lambda = \log\varrho$ satisfies

$$\log 2 \leq \lambda = \log\left(1 + \frac{\log A}{|\log(a/b)|}\right) < \log(1 + A\log A).$$

An elementary study shows that $\lambda < 1.39\log A$. Notice also that

(3) $$\log|y/x| \geq -\log\frac{y-1}{y} > \frac{1}{y} \quad \text{and} \quad \log|y/x| \leq \frac{1}{n}(|\Lambda| + \log(a/b)),$$

therefore

$$\text{(4)} \qquad \frac{1}{y} < \log|y/x| \le \frac{1}{n}\left(|\log(a/b)| + \frac{1}{3A}\right) \le \varepsilon_n |\log(a/b)|,$$

$$\text{where } \varepsilon_n = \varepsilon = \frac{4}{3n} \le \frac{4}{9}.$$

Thus

$$\frac{1}{y}(\varrho - 1) < (\varrho - 1)\log|y/x| \le \varepsilon \log A.$$

Hence, it is legitimate to take

$$a_1 = 2\log y + \varepsilon \log A.$$

Now,

$$\frac{b_1}{a_2} + \frac{b_2}{a_1} < \frac{n}{3\log A} + \frac{1}{2\log y} \le \frac{1}{3}\left(\frac{n}{\log A} + \frac{1.5}{\log y}\right),$$

and we can take

$$h = \max\left\{5\,\lambda, \log\lambda + 0.47 + \log\left(\frac{n}{\log A} + \frac{1.5}{\log y}\right)\right\}.$$

The inequalities

$$\varrho \le 1 + \varepsilon_n y \log A \le 2\varepsilon_n y \log A = \frac{8}{3n}\cdot y\log A \le \frac{8\lambda}{3N}\cdot y \qquad \text{(by (2))}$$

lead to $\lambda \le \log\lambda + \log y - \log(3N/8)$. Hence

$$\lambda \le \left(1 - \frac{\log\lambda}{\lambda} + \frac{\log(3N/8)}{\lambda}\right)^{-1}\log y \le \left(1 - \frac{8}{3eN}\right)^{-1}\log y < 1.004\,\log y,$$

and $a_1 \le (2+\eta)\log y$, where

$$\eta = \left(1 - \frac{8}{3eN}\right)^{-1}\frac{4}{3\,N} = \frac{4e}{3eN - 8} \le \frac{1.004\cdot 4}{3\,N} < 0.0045.$$

Now it is clear that a_1 and a_2 satisfy the conditions of Lemma 1.

Then Lemma 1 leads to

$$\begin{aligned}\log|\Lambda| \ge\; & -\frac{2+\eta}{3}\cdot\frac{U^2}{\lambda}\log A\,\log y - \frac{2(2+\eta)}{3}U\,\log y - 2U\log A \\ & - \theta\,(1 + h/\lambda)^{3/2}(\log y\cdot\log A)^{1/2} \\ & - 2h - 1.5\lambda + \log\lambda - 0.15 - \log(a_1 a_2 (1 + h/\lambda)^2),\end{aligned}$$

where

$$U = \frac{4h}{\lambda} + 4 + \frac{\lambda}{h} \quad \text{and} \quad \theta = \frac{16\sqrt{6(2+\eta)}}{3}.$$

We have

$$\tfrac{3}{2}\lambda - \log\lambda + 0.15 \le 1.79\,\log A.$$

The estimate $a_1a_2 \le 3(1+\eta)\log A \cdot \log y$ implies

$$\log(a_1a_2) \le 0.37\,\log A + \log y,$$

and (using the fact that $x \mapsto x^{-1}\log(1+x)$ is decreasing for $x > 1$), since $h \ge 5\lambda$, we have

$$\frac{\lambda}{h}\,\log(1+h/\lambda) \le \frac{\log 6}{5},$$

hence

$$2\log(1+h/\lambda) \le \frac{2\log 6}{5}\cdot\frac{h}{\lambda} \le 1.04\,h.$$

Collecting these estimates gives

$$\begin{aligned}\log|\Lambda| \ge {}& -\frac{2+\eta}{3}\cdot\frac{U^2}{\lambda}\log A\log y - \frac{2(2+\eta)}{3}U\log y - 2U\log A\\ & - \theta(1+h/\lambda)^{3/2}(\log y\log A)^{1/2} - 3.04\,h - 2.16\log A - \log y.\end{aligned} \tag{5}$$

Since $|\Lambda| \le 1.1|c/b|y^{-n}$, we get

$$\begin{aligned}|c| \ge {}& \frac{|b|}{1.1}\,y^n\exp\left\{-\left(\frac{2+\eta}{3}\cdot\frac{U^2}{\lambda}\log A + \frac{2(2+\eta)}{3}\,U + 1\right)\log y\right\}\\ & \times\exp\{-\theta(1+h/\lambda)^{3/2}(\log y\cdot\log A)^{1/2}\}\\ & \times\exp\{-3.04h - 2U\log A - 2.16\log A\},\end{aligned}$$

which ends the proof of Theorem 1.

3. Proof of Theorem 2. We keep the notations of the proof of Theorem 1. We may suppose that (2) holds with $N = 7300$ and that $n \ge 3\log(1.5|c/b|)$. Then $|c| < |b|y^{n/2}/1.5$, $|\Lambda| < 1.5|c/b|y^{-n}$ and

$$\log|\Lambda| \le -n\log y + \log(1.5|c/b|) \le -(n/2)\log y.$$

Since $\Lambda = \log(a/b) - n\log(y/|x|)$, we have

$$n\log(y/|x|) - \frac{1}{y^{n/2}} \le \log(a/b),$$

where

$$\log(y/|x|) \ge -\log\frac{y-1}{y} \ge \frac{1}{y},$$

thus $(n-1)/y \le \log(a/b)$. This implies $y \ge (n-1)/\log A$ and, as above, $\lambda \le 1.004\log y$.

Recall that

$$h = \max\left\{5\lambda, \log\lambda + 0.47 + \log\left(\frac{n}{\log A} + \frac{1.5}{\log y}\right)\right\}.$$

Notice that

$$\log\lambda + 0.47 + \log\left(\frac{n}{\log A} + \frac{1.5}{\log y}\right) \le \frac{\lambda}{e} + 0.47 + \log(3/2) + \log\left(y + \frac{1.5}{\log y}\right) < 5.02\log y,$$

thus, in any case, $h \le 5.02\log y$.

Using (5) and $\log|\Lambda| \le -(n/2)\log y$ and (3), we get

$$\frac{n}{2} \le \frac{2+\eta}{3}U^2\,\frac{\log A}{\lambda} + \frac{2(2+\eta)}{3}U + 2U\,\frac{\log A}{\log y} + \theta\,(1+h/\lambda)^{3/2}\Big(\frac{\log A}{\log y}\Big)^{1/2} + 3.04\cdot 5.02 + 2.16\,\frac{\log A}{\log y} + 1,$$

which implies

$$\text{(6)}\qquad \frac{n}{2} \le \frac{2+\eta}{3}\,U^2\,\frac{\log A}{\lambda} + \frac{2(2+\eta)}{3}U + 2.01U\,\frac{\log A}{\lambda} + \theta\sqrt{1.01}(1+h/\lambda)^{3/2}\Big(\frac{\log A}{\lambda}\Big)^{1/2} + 16.27 + 2.17\frac{\log A}{\lambda}.$$

Now, we distinguish two cases:

(i) $h \le 12.5\,\lambda$,

(ii) $h = \log\lambda + 0.47 + \log\left(\frac{n}{\log A} + \frac{1.5}{\log y}\right) > 12.5\,\lambda$.

In case (i), $U = 44.1$ and, applying (6) we get $n \le 7000(\log A)/\lambda$.

In case (ii),

$$h \le 0.47 + \log\left(\frac{n\lambda}{\log A} + 1.52\right) < 1.053\mathcal{L},$$

where $\mathcal{L} = \log(n\lambda/\log A)$, and (6) implies

$$\text{(7)}\qquad \frac{n}{2} \le 16.27 + \frac{2+\eta}{3}\Big(\frac{4.212\mathcal{L}}{\lambda} + 4.1\Big)^2\frac{\log A}{\lambda} + \frac{4.01}{3}\left(\frac{4.212\mathcal{L}}{\lambda} + 4.08\right) + 2.01\Big(\frac{4.212\mathcal{L}}{\lambda} + 4.08\Big)\frac{\log A}{\lambda} + \theta\,\sqrt{1.004}\Big(1 + \frac{1.053\mathcal{L}}{\lambda}\Big)^{3/2}.$$

Since $\lambda \ge \log 2$,

$$\text{(8)}\qquad \frac{1}{2}\frac{n\lambda}{\log A} \le 22.7 + \frac{2+\eta}{3}(6.077\mathcal{L} + 4.08)^2 + \frac{4.01}{3}(3.84\mathcal{L} + 5.7) + 2.01(6.077\mathcal{L} + 4.08) + \theta\sqrt{1.41}(1 + 1.52\mathcal{L})^{3/2}.$$

Finally, (8) implies $n\lambda/\log A < 7400$, which concludes the proof of Theorem 2.

4. Proof of Theorem 3. Now we consider the special case $a = b+1$, and we put $F(x,y) = c$. Then

$$(*)\quad (b+1)x^n - by^n = c, \quad b \ge 1, \quad \text{with} \quad 0 < |c| < \min\{(2^n-2)b, \tfrac{2}{3}n^2b^3\}.$$

If x^n and y^n are of opposite signs (with $|x| \neq |y|$ and $xy \neq 0$) then $(*)$ is impossible. Thus we may suppose that x and y are positive. Then the upper bound on $|c|$ implies $y > x > 1$.

Put $y = x + t$ (thus $t \ge 1$). Then

$$(b+1)x^n - b(x+t)^n = x^n - bt\left(nx^{n-1} + \binom{n}{2}x^{n-2}t + \ldots + t^{n-1}\right) = c,$$

and the condition on c leads to $x^n - btnx^{n-1} > 0$. Thus,

$$(9)\qquad y \ge nbt + 2.$$

We suppose $n \ge 500$, then $y > 500\,(a-1)$. We have $|c/b|y^{-n} < \frac{2}{3}(nb)^2y^{-n} \le \frac{2}{3}y^{-(n-2)}$, hence $|\Lambda| < y^{-(n-2)}$ and now inequality (5) implies

$$n - 2 \le \frac{2+\eta}{3} \cdot \frac{U^2}{\lambda}\log A + \frac{2(2+\eta)}{3}\,U + 2\,U\frac{\log A}{\log y} + \theta(1+h/\lambda)^{3/2}\left(\frac{\log A}{\log y}\right)^{1/2} + 3.04\,\frac{h}{\log y} + 4.$$

In the present case,

$$\lambda = \log\left(1 + \frac{A}{\log(a/(a-1))}\right).$$

Then the proof is almost the same as that of Theorem 2. Considering separately the cases $a = 2, 3, \ldots, 10$, and $a > 10$, we get $n < 600$.

Acknowledgements. The author is very grateful to the referee who suggested several improvements and corrected computational mistakes in the proof of Theorem 1.

References

[B] A. Baker, *On the representation of integers by binary forms*, Philos. Trans. Roy. Soc. London Ser. A 263 (1968), 173–191.

[LMN] M. Laurent, M. Mignotte et Y. Nesterenko, *Formes linéaires en deux logarithmes et déterminants d'interpolation*, J. Number Theory 55 (1995), 285–321.

[M] L. J. Mordell, *Diophantine Equations*, Academic Press, London, 1969.

[S] C. L. Siegel, *Die Gleichung* $ax^r - by^r = c$, Math. Ann. 144 (1937), 57–68. Also Gesammelte Abhandlungen, II (1966).

[Sh] T. N. Shorey, *Linear forms in the logarithms of algebraic numbers with small coefficients I*, J. Indian Math. Soc. 38 (1974), 271–284.

[ST] T. N. Shorey and R. Tijdeman, *Exponential Diophantine Equations*, Cambridge University Press, 1986.

[Th] A. Thue, *Berechnung aller Lösungen gewisser Gleichungen der Form $ax^r - by^r = f$*, Vid. Selskap Skrifter Kristiana Mat. Natur. Kl. (1918), No 4.

[Ti] R. Tijdeman, *Some applications of Baker's sharpened bounds to diophantine equations*, Séminaire Delange–Pisot–Poitou, 1974/75, Exp. 27, Paris, 7 pp.

[W] M. Waldschmidt, *Minorations de combinaisons linéaires de logarithmes de nombres algébriques*, Canad. J. Math. 45 (1993), 176–224.

Institut de Mathématiques
Université Louis Pasteur
7, rue René Descartes
67084 Strasbourg, France
E-mail: mignotte@math.u-strasbg.fr

Received on 16.7.1995
and in revised form on 17.11.1995 (2837)