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The two parameter hyperbola problem

by

G. Kuba (Wien)

1. Introduction. Consider a hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

in the Euclidian x, y-plane with $a, b > 0$ and let $R(a, b)$ be the number of lattice points (of the standard lattice $\mathbb{Z}^2$) "between" the hyperbola and its asymptotes, i.e.,

$$R(a, b) = \#\left\{(x, y) \in \mathbb{Z}^2 \;\middle|\; 0 < \frac{x^2}{a^2} - \frac{y^2}{b^2} \le 1\right\}.$$

The aim of this paper is to develop an asymptotic expansion of $R(a, b)$ in terms of a and b (the major and minor axis of the hyperbola). In [7] we considered the same problem for an ellipse, which turned out to be a pure geometric lattice point problem. In contrast, for hyperbolas the lattice point problem is to a large extent of arithmetic nature since the number $R(a, b)$ is finite if and only if the slope a/b of the asymptote is a rational number. As a consequence, we must assume that $a/b \in \mathbb{Q}$ for the lattice point problem to be well defined. Moreover, the magnitude of $R(a, b)$ will not only depend on the size of a and b, but also on the size of the numerator p and the denominator q of the reduced fraction which is equal to a/b. Another important difference to the ellipse problem is the fact that the implication $a_1 \ge a \wedge b_1 \ge b \Rightarrow R(a_1, b_1) \ge R(a, b)$ does not hold for the hyperbola.

In order to avoid expressions including all four quantities a, b, p, q which cannot be interpreted in a meaningful way we will assume a, b to be integers. Thus, we will investigate the behavior of the function $R(a, b)$ for $(a, b) \in \mathbb{N}^2$. The choice of $\mathbb{N}^2$ as parameter domain can in addition be justified by the fact that $\{(a, b) \in \mathbb{R}^2 \mid a, b > 0 \wedge a/b \in \mathbb{Q}\}$ is both a null set and a set of first category.

Thus, the objective of the present paper is a proof of the following result.

Theorem. *For arbitrary positive integers* a, b *let* $d = d(a,b)$ *be the greatest common divisor of* a *and* b, $a \sqcap b = \min\{a,b\}$, $a \sqcup b = \max\{a,b\}$, *and*

$$R(a,b) = \#\left\{(x,y) \in \mathbb{Z}^2 \;\middle|\; 0 < \frac{x^2}{a^2} - \frac{y^2}{b^2} \le 1\right\}.$$

Then as $ab \to \infty$,

$$R(a,b) \sim 2ab \log \frac{ab}{d}.$$

More precisely, *the following asymptotic expansion holds*:

$$R(a,b) = 2ab \log \frac{ab}{d} + (2\gamma - 1)ab + \Delta(a,b),$$

where $\gamma = 0.577215\ldots$ *is Euler's constant and*

(i) $$\Delta(a,b) \ll \frac{(ab)^{23/73}(a \sqcap b)^{50/73}}{d^{50/73}} (\log d(a \sqcup b))^{461/146}$$

for $d \ge (a \sqcup b)^{19/25}$,

(ii) $$\Delta(a,b) \ll \frac{(ab)^{23/73}(a \sqcup b)^{50/73}}{d^{50/73}} (\log d(a \sqcap b))^{461/146}$$

for $(a \sqcap b)^{19/25} \le d < (a \sqcup b)^{19/25}$,

(iii) $$\Delta(a,b) \ll \frac{ab}{d^{3/2}}$$ *for* $d < (a \sqcap b)^{19/25}$.

The $\ll$*-constants are absolute.*

2. Applications. In order to illustrate our Theorem, we give some examples of applications. First of all we consider the one parameter case where the hyperbola problem is simply a generalized form of the divisor problem:

Corollary 1. *For fixed* $p, q \in \mathbb{N}$ *and arbitrary* $k \in \mathbb{N}$, *let* $(p;q)$ *be the greatest common divisor of* p *and* q *and*

$$R(k) = \#\{(x,y) \in \mathbb{Z}^2 \mid 0 < (px+qy)(px-qy) \le p^2q^2k^2\}.$$

Then

$$R(k) = 2pqk^2 \log\left(\frac{pq}{(p;q)}k\right) + (2\gamma-1)pqk^2 + O(k^{46/73}(\log k)^{461/146}) \quad (k \to \infty).$$

Proof. With $(a,b) = (kq, kp)$ we have $d = k(p;q)$ and $R(k) = R(a,b)$.

Another example where $d \asymp a$ or $d \asymp b$ is given in the following corollary.

Corollary 2. *Let* n, m, k *be positive integers. Then as* $k^{n+m} \to \infty$,

$$R(k^n, k^m) = 2(\max\{n,m\})k^{n+m} \log k + (2\gamma - 1)k^{n+m} + \Delta_{k,n,m},$$

where

$$\Delta_{k,n,m} \ll k^{\frac{23}{73}(n+m)}((n+m)(\log k))^{461/146}$$
$$\text{if } 25(\min\{n,m\}) \ge 19(\max\{n,m\}),$$

and

$$\Delta_{k,n,m} \ll k^{\frac{23}{73}(n+m)+\frac{50}{73}|n-m|}((n+m)(\log k))^{461/146}$$
$$\text{if } 25(\min\{n,m\}) < 19(\max\{n,m\}).$$

The examples in Corollaries 1 and 2 belong to the most likely case where the greatest common divisor d of a and b is large for large a and b. In the following corollary we consider an extreme instance of the case where d is bounded.

COROLLARY 3. *For relatively prime* $a, b \in \mathbb{N}$,

$$R(a,b) = 2ab\log ab + O(ab).$$

In this case we cannot determine an estimate of the error term $\Delta(a,b)$ which is better than the trivial $O(ab)$.

3. Proof of the Theorem. Evaluation of the main term. In order to calculate $R(a,b)$ it is sufficient to count all lattice points in the domain

$$D(a,b) := \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; x,y>0 \wedge 0 < \frac{x^2}{a^2} - \frac{y^2}{b^2} \le 1\right\}.$$

Then we have

$$R(a,b) = 4\,\#(D(a,b)\cap\mathbb{Z}^2) + 2a.$$

The main idea now is to count the lattice points in $D(a,b)$ along lines parallel to the asymptote $x/a - y/b = 0$. For abbreviation, we put

$$\hat{a} = \frac{a}{d} \quad \text{and} \quad \hat{b} = \frac{b}{d},$$

where d is the greatest common divisor of a and b. Then the "counting" lines are all lines g_n,

$$g_n : \hat{b}x - \hat{a}y = n \qquad \text{with } n = 1, 2, \ldots, ab/d.$$

(For $n = ab/d$ the vertex $(a,0)$ of the hyperbola is the only lattice point on g_n in $\overline{D(a,b)}$.)

Let S_n be the intersection point of g_n with the x-axis and T_n be the intersection point of g_n with the hyperbola $(bx-ay)(bx+ay) = a^2b^2$. Then $S_n = (n/\hat{b}, 0)$ and the x-coordinate x_n of T_n is given by

$$x_n = \frac{1}{2b}\left(dn + \frac{a^2b^2}{dn}\right).$$

Now we can write

$$R_1(a,b) := \#(D(a,b)\cap\mathbb{Z}^2) = \sum_{1\le n\le ab/d} r_n,$$

where r_n is the number of all lattice points on the line g_n between S_n (excluded) and T_n (included).

In order to calculate r_n, let $x = x(n)$ be the unique solution of the congruence

$$\hat{b}x \equiv n \pmod{\hat{a}}$$

in the interval $1\le x\le \hat{a}$. Then for every $x\in x(n)+\hat{a}\mathbb{Z}$ there is exactly one integer y with $(x,y)\in g_n$. Of course, $x(n)$ only depends on the residue class $n+\hat{a}\mathbb{Z}$, so we will write $x(n+\hat{a}\mathbb{Z})$ instead of $x(n)$. Then we have

$$\begin{aligned} r_n &= \#\{k\in\mathbb{Z}\mid n/\hat{b} < x(n+\hat{a}\mathbb{Z})+k\hat{a}\le x_n\}\\ &= \left[\frac{x_n - x(n+\hat{a}\mathbb{Z})}{\hat{a}}\right] - \left[\frac{n}{\hat{a}\hat{b}} - \frac{x(n+\hat{a}\mathbb{Z})}{\hat{a}}\right]. \end{aligned}$$

([ ] are the Gauss brackets.)

Now let $\psi(\cdot)$ be defined by

$$\psi(z) = z - [z] - 1/2 \qquad (z\in\mathbb{R}).$$

Then we compute

$$R_1(a,b) = \sum_{1\le n\le ab/d}\left(\frac{1}{2\hat{a}b}\left(dn + \frac{a^2b^2}{dn}\right) - \frac{n}{\hat{a}\hat{b}}\right) + \Psi_1(a,b) - \Psi_2(a,b),$$

where

$$\Psi_1(a,b) = \sum_{1\le n\le ab/d}\psi\left(\frac{n}{\hat{a}\hat{b}} - \frac{x(n+\hat{a}\mathbb{Z})}{\hat{a}}\right),$$

$$\Psi_2(a,b) = \sum_{1\le n\le ab/d}\psi\left(\frac{x_n - x(n+\hat{a}\mathbb{Z})}{\hat{a}}\right).$$

The calculation of the main term of $R_1(a,b)$ is straightforward. We make use of the well-known formulas (for the second see Fricker [2])

$$\sum_{1\le n\le N} n = N(N+1)/2$$

and

$$\sum_{1\le n\le N}\frac{1}{n} = \log N + \gamma + \frac{1}{2N} + O(N^{-2}),$$

and obtain

$$R_1(a,b) = \frac{ab}{2}\log\frac{ab}{d} + \gamma\frac{ab}{2} - \frac{ab}{4} + \Psi_1(a,b) - \Psi_2(a,b) + O(1).$$

In order to calculate $\Psi_1(a,b)$, we introduce two new parameters A and B. Let A, B be integers satisfying $A\hat{a}+B\hat{b}=1$. Then for every integer n,

$$x(n+\hat{a}\mathbb{Z}) \equiv Bn \pmod{\hat{a}}.$$

Now, let $y = y(n+\hat{b}\mathbb{Z})$ be the unique solution of the congruence $\hat{a}y \equiv n \pmod{\hat{b}}$ in the interval $1 \le y \le \hat{b}$. Then for every n,

$$y(n+\hat{b}\mathbb{Z}) \equiv An \pmod{\hat{b}},$$

and we obtain

$$\frac{n}{\hat{a}\hat{b}} - \frac{x(n+\hat{a}\mathbb{Z})}{\hat{a}} \equiv \frac{y(n+\hat{b}\mathbb{Z})}{\hat{b}} \pmod{1}.$$

Since $\psi(\cdot)$ is a periodic function with period 1, the sum $\Psi_1(a,b)$ becomes a sum over a complete set of residues mod $\hat{b}$, repeated a times. We obtain

$$\Psi_1(a,b) = \sum_{1\le n\le a\hat{b}} \psi\left(\frac{y(n+\hat{b}\mathbb{Z})}{\hat{b}}\right) = a\sum_{1\le n\le\hat{b}} \left(\psi\left(\frac{n}{\hat{b}}\right)\right) = -\frac{a}{2}.$$

Thus we derive

$$R(a,b) = 2ab\log\frac{ab}{d} + (2\gamma-1)ab - 4\Psi_2(a,b) + O(1).$$

The estimation of $\Psi_2(a,b)$ now concludes the proof of the Theorem.

4. Estimation of Ψ_2. First, we want to get rid of the deranging term $x(n+\hat{a}\mathbb{Z})/\hat{a}$ in $\Psi_2(a,b)$. We substitute $x(n+\hat{a}\mathbb{Z})/\hat{a}$ by $Bn/\hat{a}$. Then we have

$$\Psi_2(a,b) = \sum_{1\le n\le ab/d} \psi\left(\frac{1}{2\hat{a}b}\left(dn + \frac{a^2b^2}{dn}\right) - \frac{Bn}{\hat{a}}\right).$$

In order to compute this sum, we divide n into residue classes mod $\hat{a}$ as well as into residue classes mod $\hat{b}$ and make use of the equation

$$\sum_{1\le n\le ab/d} F(n) = \sum_{1\le m\le\hat{a}}\ \sum_{0\le l<b} F(l\hat{a}+m) = \sum_{1\le m\le\hat{b}}\ \sum_{0\le l<a} F(l\hat{b}+m),$$

which holds for every function F defined on $\{1,2,\ldots,ab/d\}$.

Furthermore, we note that

$$\frac{1}{2\hat{a}b}\left(dn + \frac{a^2b^2}{dn}\right) - \frac{Bn}{\hat{a}} = \frac{1}{2a\hat{b}}\left(\frac{a^2b^2}{dn} - dn\right) + \frac{An}{\hat{b}}.$$

Therefore

$$\Psi_2(a,b) = \sum_{1\le m\le\hat{a}} \Psi_m^{*}(a,b) = \sum_{1\le m\le\hat{b}} \Psi_m^{**}(a,b),$$

where

$$\Psi_m^*(a,b)=\sum_{0\le l<b}\psi(f_m(l))\quad\text{and}\quad \Psi_m^{**}(a,b)=\sum_{0\le l<a}\psi(g_m(l)),$$

with

$$f_m(t)=\left(\frac{1}{2\hat{a}b}\left(\frac{a^2b^2}{d\hat{a}t+dm}+d\hat{a}t+dm\right)-\frac{Bm}{\hat{a}}\right),$$
$$g_m(t)=\left(\frac{1}{2a\hat{b}}\left(\frac{a^2b^2}{d\hat{b}t+dm}-d\hat{b}t-dm\right)+\frac{Am}{\hat{b}}\right).$$

In order to establish clauses (i) and (ii), we make use of an essential tool from Huxley's "Discrete Hardy–Littlewood Method" in the shape presented in Huxley [3] and [4]. The following lemma is a combination of Huxley [4], Theorem 3 and Theorem 4:

LEMMA 1. *Let M, M' and T be positive real parameters satisfying $M\le M'<2M$ and $M\le C_1T^{83/146}(\log T)^{-63/292}$ with a constant C_1. Furthermore, let $F(t)$ be a four times continuously differentiable function on $1\le t\le 2$ satisfying*

$$F'(t),F''(t),F^{(3)}(t),F'(t)F^{(3)}(t)-3(F''(t))^2,F''(t)F^{(4)}(t)-3(F^{(3)}(t))^2\ne 0$$

for all $1\le t\le 2$. Then

$$\sum_{M\le k\le M'}\psi\left(\frac{T}{M}F\left(\frac{k}{M}\right)\right)\ll T^{23/73}(\log T)^{315/146}.$$

The $\ll$-constant depends on C_1 and on the range of values taken by the derivatives of F.

To verify (iii), we use van der Corput's classical estimate of ψ-sums:

LEMMA 2 (see van der Corput [1]). *Let f be a real-valued function, twice continuously differentiable on $[a,b]\subset\mathbb{R}$. Furthermore, let f'' be monotonic and nonzero on $[a,b]$. Then*

$$\sum_{a\le k\le b}\psi(f(k))\ll\int_a^b|f''(t)|^{1/3}\,dt+|f''(a)|^{-1/2}+|f''(b)|^{-1/2},$$

where the $\ll$-constant is absolute.

In order to estimate $\Psi_m^*(a,b)$ for every $m=1,\dots,\hat{a}$, we put $\beta=b-2$ and

$$f(t):=f_m(t)=\frac{d}{2b}t+\frac{1}{2}\cdot\frac{ab}{\hat{a}t+m}+\frac{d^2m}{2ab}-\frac{Bm}{\hat{a}}\qquad(0<t\le\beta).$$

Then

$$\Psi_m^*(a,b) = \sum_{l=1}^{\beta} \psi(f(l)) + O(1).$$

We note that

$$f'(t) = \frac{d}{2b} - \frac{\hat{a}ab}{2(\hat{a}t+m)^2}, \qquad f''(t) = \frac{\hat{a}^2ab}{(\hat{a}t+m)^3},$$
$$f^{(3)}(t) = -\frac{3\hat{a}^3ab}{(\hat{a}t+m)^4}, \qquad f^{(4)}(t) = \frac{12\hat{a}^4ab}{(\hat{a}t+m)^5},$$

and observe that $f'(t), f''(t), f^{(3)}(t) \neq 0$ for all $t \in]0,\beta]$. Furthermore, we see that for $0 < t \leq \beta$,

$$f'(t)f^{(3)}(t) - 3(f''(t))^2 = -\frac{3a\hat{a}^3(a\hat{a}b^2 + d(\hat{a}t+m)^2)}{2(\hat{a}t+m)^6} \neq 0,$$

and

$$f''(t)f^{(4)}(t) - 3(f^{(3)}(t))^2 = -\frac{15a^2\hat{a}^6b^2}{(\hat{a}t+m)^8} \neq 0.$$

The sum $\Psi_m^*(a,b)$ may now be written as

$$\Psi_m^*(a,b) = \sum_{1\leq j<J} S_j + O(1)$$

with

$$S_j = \sum_{M_j \leq k < M_{j+1}} \psi(f(k)) \qquad (1 \leq j < J),$$

where $(M_j)_{1\leq j\leq J}$ is a geometric series, $M_j = 2^j M_1$, $M_J = \beta + 1$, and $1/2 \leq M_1 < 1$.

Now we apply Lemma 1 to each S_j. Let

$$M = M_j, \quad M' = -[-2M] - 1, \quad T = db \quad \text{and} \quad F(u) = \frac{M}{T} f(Mu).$$

Then $F(u)$ satisfies all conditions of Lemma 1. In addition we have $|F^{(r)}(u)| \ll 1$ for $1 \leq u \leq 2$ and $r = 1,2,3,4$.

Furthermore, we note that for $b \leq d^{83/63-\varepsilon}$ (which is true if $d \geq b^{19/25}$) the condition $M \leq C_1 T^{83/146}(\log T)^{-63/292}$ is fulfilled for all $M \in \{M_1, \ldots, M_J\}$.

Thus, all conditions of Lemma 1 are satisfied, and we obtain

$$S_j \ll (db)^{23/73}(\log db)^{315/146} \qquad (j = 1, \ldots, J-1).$$

Observing that $J \ll \log b$ we derive

$$\Psi_m^*(a,b) \ll (db)^{23/73}(\log db)^{1+315/146},$$

and this leads to

$$\Psi_2(a,b) \ll \frac{a}{d}(db)^{23/73}(\log db)^{461/146} = \frac{ab^{23/73}}{d^{50/73}}(\log db)^{461/146},$$

provided that $d \geq b^{19/25}$.

The sums $\Psi_m^{**}(a,b)$ from the second representation of $\Psi_2(a,b)$ can be treated in an analogous way. We derive

$$\Psi_m^{**}(a,b) \ll (da)^{23/73}(\log da)^{461/146} \quad \text{for } d \geq a^{19/25}.$$

Consequently, we also have

$$\Psi_2(a,b) \ll \frac{a^{23/73}b}{d^{50/73}}(\log da)^{461/146} \quad \text{if } d \geq a^{19/25}.$$

Both estimates put together yield clauses (i) and (ii) of the Theorem.

The estimation of $\Psi_2(a,b)$, with the help of van der Corput's method (Lemma 2), is straightforward. We obtain

$$\Psi_2(a,b) = \sum_{1\leq m\leq \hat{a}} \Psi_m^{*}(a,b) \ll \frac{ab^{1/3}}{d^{2/3}}\log\frac{ab}{d} + \frac{ab}{d^{3/2}},$$

and

$$\Psi_2(a,b) = \sum_{1\leq m\leq \hat{b}} \Psi_m^{**}(a,b) \ll \frac{a^{1/3}b}{d^{2/3}}\log\frac{ab}{d} + \frac{ab}{d^{3/2}}.$$

We combine both estimates and substitute $\log\frac{ab}{d}$ by $(ab)^{\varepsilon}$. Then as $ab \to \infty$,

$$\Psi_2(a,b) \ll \frac{(ab)^{1/3+\varepsilon}(a \sqcap b)^{2/3}}{d^{2/3}} + \frac{ab}{d^{3/2}}.$$

Note that the second term dominates the first if and only if $d(a \sqcap b)^{6\varepsilon/5} < (a \sqcup b)^{4/5-6\varepsilon/5}$. And this condition is surely fulfilled for $d < (a \sqcap b)^{19/25}$ if we put $\varepsilon = \frac{5}{12}\left(\frac{4}{5} - \frac{19}{25}\right)$.

This proves clause (iii) of the Theorem.

Remark. A direct (and better) estimation of the sum $\Psi_2(a,b)$ without splitting up the interval of summation $1 \leq n \leq ab/d$ into residue classes (as in Huxley and Watt [6], p. 162) seems impracticable. Huxley had a similar difficulty in the *Corrigenda: "Exponential sums and lattice points II"* [5]. What makes the problem difficult is the deranging term $Bn/\hat{a}$. Even van der Corput's method (Lemma 1) where linear terms are negligible is of no use because the linear term $Bn/\hat{a}$ contains the parameter B from $A\hat{a} + B\hat{b} = 1$. Obviously, the parameter B depends uncontrollably on the basic parameters a, b, d. The only way to get rid of this problem is, as we have done it, to sum over subintervals where the term $Bn/\hat{a}$ is constant modulo 1.

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Institut für Mathematik u.a.St.
Universität für Bodenkultur
Gregor Mendel-Straße 33
A-1180 Wien, Austria
E-mail: kuba@edv1.boku.ac.at

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