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On character sums of rational functions over local fields

by

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1. Introduction. Characters of $(\mathbb{Z}/m\mathbb{Z})^*$ were introduced by Dirichlet while studying the distribution of prime numbers in an arithmetic progression. Hecke generalized the notion of Dirichlet characters by interpreting it as a collection of characters of local fields.

Let F be an extension of $\mathbb{Q}_p$ of degree n. Let O be the ring of integers and U be the group of units. Let P be the prime ideal and π be a prime element. Denote the normalized valuation by $|\cdot|_v$ and the ordinal by ord. Let $q = NP$ be the norm of P. Let $f(x), g(x) \in O[x]$ and $d = \deg f$, $e = \deg g$. This paper studies the character sum

$$\sum_{x \bmod P^\alpha} \chi(f(x))\overline{\chi}(g(x)),$$

where χ is a character of U of conductor P^α, which extends to O by extension by zero. Let $i, \theta \in \mathbb{Z}$ and

$$R(x) = f(x)/g(x),$$
$$O(i,\theta) = \{x \in O : P \nmid g(x), \operatorname{ord}(R^{(i)}(x)/j!) \ge \alpha/(j+1)\ (1 \le j \le i),\ \operatorname{ord}(R^{(i)}(x)/i!) = \theta\},$$
$$W = \{x \in O : P \nmid g(x), \operatorname{ord}(R^{(i)}(x)/j!) \ge \alpha/(j+1)\ (j \in \mathbb{N})\}.$$

Let A be a fixed complete system of representatives modulo P^α, and $A(i,\theta) = A \cap O(i,\theta)$. By Lemma 1, $\operatorname{ord}(R^{(j)}(x)/j!) \ge 0$ if $P \nmid g(x)$. So

$$A \cap \{x \in O : P \nmid g(x)\} = (A \cap W) \cup \bigcup_{i \ge 1} \bigcup_{0 \le \theta < \alpha/(i+1)} A(i,\theta).$$

Hence we concentrate on the study of $\sum_{x \in A(i,\theta)} \chi(R(x))$ with $\alpha > 1$, $\theta < \alpha/(i+1)$.

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2. Main results

Theorem 1.

$$\Big|\sum_{x\in A(1,\theta)}\chi(R(x))\Big|\le 2|2|_v^{-1}(d+e-1)q^{\alpha/2}\quad \textit{if } \theta<\alpha/2.$$

Theorem 2.

$$\Big|\sum_{x\in A(2,\theta)}\chi(R(x))\Big|\le 2|2|_v^{-1}(d+e-1)q^{(\alpha+\theta)/2}\quad \textit{if } \theta<\alpha/3.$$

Theorem 3. *If* $i\ge 3$, $\theta=0$, *then*

$$\Big|\sum_{x\in A(i,\theta)}\chi(R(x))\Big|\le |i(i-1)|_v^{-1}(d+e-1)q^{[\frac{i-1}{i}\alpha]}.$$

Let

$$\beta=\left[\frac{\alpha-\theta+i-1}{i}\right],\qquad \gamma=\left[\frac{\alpha-\theta+i-2}{i}\right],$$

and

$$M_1(i,\theta)=\left\{x\in O(i,\theta):\operatorname{ord}\left(\frac{R^{(j)}(x)}{j!}\right)\ge\alpha-j\beta\ (1<j<i)\right\},$$
$$M_2(i,\theta)=O(i,\theta)\backslash M_1(i,\theta),\quad A_j(i,\theta)=A\cap M_j(i,\theta)\quad (j=1,2).$$

Theorem 4. *If* $i>2$ *and* $0<\theta<\alpha/(i+1)$, *then*

$$\Big|\sum_{x\in A_1(i,\theta)}\chi(R(x))\Big|\le |i|_v^{-i}(d+e-1)q^{\alpha-\gamma}.$$

Theorem 5. *Let* $i>2$ *and* $0<\theta<\alpha/(i+1)$. *Suppose that*

(1) $$x\in A(i,\theta)\Leftrightarrow x+P^{\beta}\in A(i,\theta).$$

Then

$$\Big|\sum_{x\in A_2(i,\theta)}\chi(R(x))\Big|\le \frac{\alpha}{2}|i|_v^{-i}(d+e-1)c(n,i-1)q^{\alpha-(\alpha-\beta-\theta)/(i-1)},$$

where $c(n,i-1)$ *is the constant in Lemma* 5.

3. Lemmas

Lemma 1. *Let* $u(x),w(x)\in O[x]$, $r(x)=u(x)/w(x)$. *Suppose that* $P\nmid w(x_0)$. *Then*

$$r(x_0+y)=\sum_{j=0}^{\infty}\frac{r^{(j)}(x_0)}{j!}y^j\in O[[y]].$$

Proof. $P \nmid w(x_0)$ implies that

$$r(x_0+y)=\sum_{j=0}^{\infty} a_j y^j \in O[[y]].$$

Differentiating term-by-term gives our result.

LEMMA 2. *Let $u(x)\in F[x]$. Let $\lambda,\mu,\kappa\in\mathbb{Z}$ be such that $\kappa>0$ and $\lambda>\mu$. Let $\tau=[(\lambda-\mu+\kappa-1)/\kappa]$ and B be a set of representatives modulo P^τ. Then*

$$\#\left\{x\in B : P^\lambda \mid u(x), P^\mu \,\Big\|\, \frac{u^{(\kappa)}(x)}{\kappa!}\right\}\le \deg u.$$

Proof. Let E be an extension of F containing all roots of u. Let $m=\deg u$ and $u(X)=a\prod_{i=1}^m (X-\alpha_i)$. Then

$$\frac{u^{(\kappa)}(X)}{\kappa!}=\sum_S a\prod_{j\notin S}(X-\alpha_j),$$

where $S\subset\{1,\ldots,m\}$ runs through all subsets of cardinality k. So, if $\operatorname{ord}(u(x))\ge\lambda$ and $\operatorname{ord}(u^{(\kappa)}(x)/\kappa!)=\mu$, then for some S,

$$\operatorname{ord}\Big(a\prod_{j\notin S}(X-\alpha_j)\Big)\le\mu$$

and hence

$$\operatorname{ord}\Big(\prod_{j\in S}(x-\alpha_j)\Big)\ge\lambda-\mu,$$

which implies that $\operatorname{ord}(x-\alpha_j)\ge(\lambda-\mu)/\kappa$ for some j. Therefore

$$\begin{aligned}&\#\{x\in B:\operatorname{ord}(u(x))\ge\lambda,\ \operatorname{ord}(u^{(\kappa)}/\kappa!)=\mu\}\\&\qquad\le\sum_{j=1}^m\#\{x\in B:\operatorname{ord}(x-\alpha_j)\ge(\lambda-\mu)/\kappa\le m\}.\end{aligned}$$

This completes the proof.

LEMMA 3. *Let $s(x)\in F[x]$, $w(x)\in O[x]$ and $r(x)=s(x)/w(x)$. Let $\kappa,\lambda,\mu,\sigma\in\mathbb{Z}$ be such that $\kappa>0$, $\sigma\ge0$ and $\lambda>\mu$. Let $\tau=[(\lambda-\mu+\kappa-1)/\kappa]$ and B be a set of representatives modulo P^τ. Then*

$$\begin{aligned}\#\Big\{x\in B: P^\lambda \,\Big|\, \frac{r^{(j+1)}(x)}{j!}\ (0\le j\le\sigma),\\ P^{\mu+1}\Big|\frac{r^{(j+1)}(x)}{j!(\kappa+\sigma+1)}\ (0\le j<\kappa+\sigma),\ P^\mu\,\Big\|\,\frac{r^{(\kappa+\sigma+1)}(x)}{(\kappa+\sigma+1)!},\ P\nmid w(x)\Big\}\\ \le(\deg s+\deg w-1)\left|\frac{(\kappa+\sigma+1)!}{\sigma!}\right|_v^{-1/\kappa}\left(N\left(\sqrt{\frac{(\kappa+\sigma+1)!}{\sigma!}}\right)\right)^{\operatorname{sgn}(\kappa-1)},\end{aligned}$$

where $\sqrt{a}$ is the radical of a and N denotes the norm.

Proof. Denote the set to be estimated by T. The proof splits into two cases, according to whether Lemma 2 can be employed or not.

Case 1: $\lambda \le \mu + \operatorname{ord}((\kappa+\sigma-1)!) - \operatorname{ord}(\sigma!)$. The trivial estimate yields

$$\#T \le q^{[(\lambda-\mu+\kappa-1)/\kappa]} \le q^{[(\operatorname{ord}((\kappa+\sigma+1)!)-\operatorname{ord}(\sigma!)+\kappa-1)/\kappa]}$$
$$\le \left|\frac{(\kappa+\sigma+1)!}{\sigma!}\right|_v^{-1/\kappa} \left(N\left(\sqrt{\frac{(\kappa+\sigma+1)!}{\sigma!}}\right)\right)^{\operatorname{sgn}(\kappa-1)}.$$

Case 2: $\lambda > \mu + \operatorname{ord}((\kappa+\sigma-1)!) - \operatorname{ord}(\sigma!)$. Since

$$\frac{(w^2r')^{(m)}}{m!} = \sum_{j=0}^{m} \frac{r^{(j+1)}(w^2)^{(m-j)}}{j!(m-j)!},$$

we get, for $x \in T$,

$$P^\lambda \,\Big|\, \frac{(w^2r')^{(\sigma)}(x)}{\sigma!} \quad \text{and} \quad P^\mu \,\Big\|\, \frac{(w^2r')^{(\kappa+\sigma)}(x)}{(\kappa+\sigma+1)!},$$

which is equivalent to

$$\operatorname{ord}\left(\frac{(w^2r')^{(\kappa+\sigma)}(x)}{\kappa!\sigma!}\right) = m + \operatorname{ord}((\kappa+\sigma-1)!) - \operatorname{ord}(\sigma!).$$

Hence, by Lemma 2 with $u(x) = (w^2r')^{(\sigma)}(x)/\sigma!$,

$$\#T \le (\deg s + \deg w - 1)q^\tau / q^{[(\operatorname{ord}((\kappa+\sigma+1)!)-\operatorname{ord}(\sigma!)+\kappa-1)/\kappa]}$$
$$\le (\deg s + \deg w - 1)\left|\frac{(\kappa+\sigma+1)!}{\sigma!}\right|_v^{-1/\kappa} \left(N\left(\sqrt{\frac{(\kappa+\sigma+1)!}{\sigma!}}\right)\right)^{\operatorname{sgn}(\kappa-1)}.$$

This completes the proof.

LEMMA 4. *If* $\operatorname{ord}(a) = h > \alpha/2$, *then* $\chi(1+ax)$ *is*, *with respect to* x, *an additive character of conductor* $P^{\alpha-h}$.

Proof. Obvious.

LEMMA 5. *If* ψ *is an additive character of conductor* P^t, $u(x) = \sum_{i=0}^m a_i x^i \in O[x]$ *and* $(a_i, \ldots, a_m) = 0$, *then*

$$\sum_{x \bmod P^t} \psi(u(x)) \le \begin{cases} c(n,m)q^{t(1-1/m)}, \\ (m-1)q^{1/2} \quad \text{if } t = 1, \end{cases}$$

where $c(n,m) \ge 1$ *depends at most on* m *and* n.

Proof. See [1], [2].

4. Proof of main results

4.1. Proof of Theorem 1. Grouping together elements of $A(1,\theta)$ whose images modulo $P^{[(\alpha+1)/2]}$ are the same, we get

$$\sum_{x\in A(1,\theta)} \chi(R(x)) = \sum_{y\in B\cap O(1,\theta)} \sum_{z\in C(y)} \chi(R(y)+R'(y)\pi^{[(\alpha+1)/2]}z),$$

where B is a fixed complete system of representatives modulo $P^{[(\alpha+1)/2]}$, and

$$C(y)=\{z\in O : y+\pi^{[(\alpha+1)/2]}z\in A(1,\theta)\},$$

which is a complete system of representatives modulo $P^{[\alpha/2]}$ if $y\in O(1,\theta)$.

(i) $\theta\le[\alpha/2]$. The inner sum is 0 by Lemma 4 and the estimate follows.

(ii) $(\alpha-1)/2=[\alpha/2]=\theta<\alpha/2$. We have

$$\sum_{x\in A(1,\theta)} \chi(R(x)) = q^{(\alpha-1)/2} \sum_{x\in B\cap O(1,\theta)} \chi(R(x)).$$

Grouping elements of $B\cap O(1,\theta)$ according to their images modulo $P^{(\alpha-1)/2}$, we get

$$\sum_{x\in B\cap O(1,\theta)} \chi(R(x)) = \sum_{y\in C\cap C_1} \sum_{z\in D(y)} \chi\biggl(R(y)+R'(y)\pi^{(\alpha-1)/2}z+\frac{R''(y)}{2}\pi^{\alpha-1}z^2\biggr),$$

where C is a fixed complete system of representatives modulo $P^{(\alpha-1)/2}$, and

$$C_1=\{y\in O : P^{(\alpha-1)/2}\mid R'(y)\}.$$

If $P\mid R''(y)/2$, then $D(y)$ is empty or a complete system of representatives modulo P. By Lemma 4, the inner sum is 0 and makes no contribution.

If $P\nmid R''(y)/2$, then $D(y)$ is empty or a complete system or a complete-but-one system of representatives modulo P. By Lemmas 4 and 5, the inner sum is $\le q^{1/2}+1$. It remains to bound $\#\{y\in C\cap C_1 : P\nmid R''(y)/2\}$.

If $(\alpha-1)/2>\operatorname{ord}(2)$ so that we can make use of Lemma 3 with $r(x)=R(x)$, we have

$$\#\{y\in C\cap C_1 : P\nmid R''(y)/2\}\le (d+e-1)|2|_v^{-1}.$$

If $(\alpha-1)/2\le\operatorname{ord}(2)$, the trivial estimate yields

$$\#\{y\in C\cap C_1 : P\nmid R''(y)/2\}\le q^{(\alpha-1)/2}\le |2|_v^{-1}.$$

Therefore

$$\sum_{x\in A(1,\theta)} \chi(R(x)) \le q^{(\alpha-1)/2}q^{1/2}\#\{y\in C\cap C_1 : P\nmid R''(y)/2\} \le 2(d+e-1)|2|_v^{-1}q^{\alpha/2}.$$

This completes the proof.

4.2. Proof of Theorem 2. Grouping elements of $A(2,\theta)$ according to their images modulo $P^{[(\alpha-\theta+1)/2]}$, we get

$$\sum_{x\in A(2,\theta)}\chi(R(x))=\sum_{y\in B\cap O(2,\theta)}\sum_{z\in C(y)}\chi(R(y)+R'(y)\pi^{[(\alpha-\theta+1)/2]}z),$$

where B is a fixed complete system of representatives modulo $P^{[(\alpha-\theta+1)/2]}$. By Lemma 4, the inner sum is 0 unless $\operatorname{ord}(R'(x))\ge\alpha-[(\alpha-\theta+1)/2]$. Hence

$$\sum_{x\in A(2,\theta)}\chi(R(x))=q^{\alpha-[(\alpha-\theta+1)/2]}\sum_{x\in B\cap B_1}\chi(R(x)),$$

where $B_1=\{y\in O(2,\theta):\operatorname{ord}(R'(x))\ge\alpha-[(\alpha-\theta+1)/2]\}$.

(i) $\theta=0$. In this case we can bound $\#B\cap B_1$ directly. If $[(\alpha+1)/2]>\operatorname{ord}(2)$ so that we can make use of Lemma 3 with $r(x)=R(x)$ and $\lambda=[(\alpha+1)/2]$, we have $\#B\cap B_1\le(d+e-1)|2|_v^{-1}$. If $[(\alpha+1)/2]\le\operatorname{ord}(2)$, the trivial estimate yields

$$\#B\cap B_1\le q^{[(\alpha+1)/2]}\le q^{\operatorname{ord}(2)}\le|2|_v^{-1}.$$

Hence

$$\sum_{x\in A(2,\theta)}\chi(R(x))\le(d+e-1)|2|_v^{-1}q^{[\alpha/2]}.$$

(ii) $\theta\ne0$. Grouping elements of $B\cap B_1$ according to their images modulo $P^{[(\alpha-\theta)/2]}$, we get

$$\sum_{x\in B\cap B_1}\chi(R(x))=\sum_{y\in C\cap B_1}\sum_{z\in D(y)}\chi\biggl(\sum_{j=0}^{3}\frac{R^{(j)}(y)}{j!}\pi^{j[(\alpha-\theta)/2]}z^j\biggr),$$

where C is a fixed complete system of representatives modulo $P^{[(\alpha-\theta)/2]}$, and $D(y)=\{z\in O:y+\pi^{[(\alpha-\theta)/2]}\in B\cap B_1\}$, which is a complete system of representatives modulo $P^{[(\alpha-\theta+1)/2]-[(\alpha-\theta)/2]}$.

By Lemmas 4 and 5, the inner sum is $\le 2q^{\frac12([(\alpha-\theta+1)/2]-[(\alpha-\theta)/2])}$. It remains to bound $\#C\cap B_1$.

If $\alpha-\theta-[(\alpha-\theta+1)/2]>\operatorname{ord}(2)$, so that we can make use of Lemma 3 with $r(x)=R(x)$, we have $\#C\cap B_1\le(d+e-1)|2|_v^{-1}$.

If $[(\alpha-\theta)/2]\le\operatorname{ord}(2)$, the trivial estimate yields

$$\#C\cap B_1\le q^{[(\alpha-\theta)/2]}\le|2|_v^{-1}.$$

Hence

$$\sum_{x\in A(2,\theta)}\chi(R(x))\le 2q^{\frac12([(\alpha-\theta+1)/2]-[(\alpha-\theta)/2])}(d+e-1)|2|_v^{-1}q^{\alpha-[(\alpha-\theta+1)/2]}$$
$$\le(d+e-1)2|2|_v^{-1}q^{(\alpha+\theta)/2}.$$

This completes the proof.

4.3. Proof of Theorem 3. We bound $\#A(i,\theta)$ directly.

If $[(\alpha+i-1)/i] > \operatorname{ord}(i)$, so that Lemma 3 can be employed with $r(x)=R(x)$, $\theta=i-2$ and $k=1$, we have

$$\begin{aligned}\#A(i,\theta) &\le (d+e-1)|i(i-1)|_v^{-1} q^{\alpha-[(\alpha+i-1)/i]}\\ &\le (d+e-1)|i(i-1)|_v^{-1} q^{[(i-1)\alpha/i]}.\end{aligned}$$

If $[(\alpha+i-1)/i] \le \operatorname{ord}(i)$, the trivial estimate yields

$$\#A(i,\theta) \le q^{\alpha} \le q^{\alpha-[(\alpha+i-1)/i]} q^{\operatorname{ord}(i)} \le |i|_v^{-1} q^{[(i-1)\alpha/i]}.$$

Theorem 3 now follows.

4.4. Proof of Theorem 4

(i) $\alpha/i \le \operatorname{ord}(i)+\theta$. Recall also $\theta < \alpha/(i+1)$. Calculation shows that $[(\alpha-\theta+i-2)/i] \le i\operatorname{ord}(i)$. Therefore the trivial estimate yields

$$\sum_{x\in A_1(i,\theta)} \chi(R(x)) \le q^{\alpha} \le q^{\alpha-[(\alpha-\theta+i-2)/i]} q^{i\operatorname{ord}(i)} \le |i|_v^{-i} q^{\alpha-\gamma}.$$

(ii) $\alpha/i > \operatorname{ord}(i)+\theta$. Grouping elements of $A_1(i,\theta)$ according to their images modulo P^{β}, we get

$$\sum_{x\in A_1(i,\theta)} \chi(R(x)) = \sum_{y\in B\cap M_1(i,\theta)} \sum_{z\in C(y)} \chi(R(y+\pi^{\beta}z)),$$

where B is a fixed complete system of representatives modulo P^{β}, and

$$C(y) = \{z\in O : y+\pi^{\beta}z \in A_1(i,\theta)\},$$

which is a complete system of representatives modulo $P^{\alpha-\beta}$ if $y\in M_1(i,\theta)$.

Our β is so chosen that $\alpha+i > i+\beta+\theta \ge \alpha$, and $\theta<\alpha/(i+1)$ justifies $j\beta\ge\alpha$ for $j>i$. Hence

$$\sum_{x\in A_1(i,\theta)} \chi(R(x)) = \sum_{y\in B\cap M_1(i,\theta)} \sum_{z\in C(y)} \chi(R_y(z)),$$

where

$$R_y(z) = \sum_{j=0}^{i-1} \frac{R^{(j)}(y)}{j!} \pi^{j\beta} z^j.$$

Since $y\in M_1(i,\theta)$ implies that

$$R_y(z) = R(y) + R'(y)\pi^{\beta+1} z \pmod{P^{\alpha}},$$

we have

$$\sum_{x\in A_1(i,\theta)} \chi(R(x)) = \sum_{y\in B\cap M_1(i,\theta)} \sum_{z\in C(y)} \chi(R(y)+R'(y)\pi^{\beta+1} z).$$

By Lemma 4, the inner sum is 0 unless $P^{\alpha-\beta} \mid R'(y)$. Hence

$$\sum_{x\in A_1(i,\theta)} \chi(R(x)) = q^{\alpha-\beta} \sum_{y\in D(i,\theta)} \chi(R(y)),$$

where

$$D(i,\theta) = \{y \in B \cap M_1(i,\theta) : P^{\alpha-\beta} \mid R'(y)\}.$$

Since $\alpha/i > \operatorname{ord}(i) + \theta$, we can apply Lemma 3 with $r(x) = R(x)$, $\theta = 0$, $\lambda = \alpha - \beta$, $k = i - 1$ and $\mu = \theta$. Thus

$$\begin{aligned}\#D(i,\theta) &\le q^{\beta-[(\alpha-\theta-\beta+i-2)/(i-1)]}|i|_v^{-1/(i-1)} N(\sqrt{i})(d+e-1)\\ &\le q^{\beta-\gamma}|i|_v^{-1/(i-1)} N(\sqrt{i})(d+e-1).\end{aligned}$$

Therefore

$$\sum_{x\in A_1(i,\theta)} \chi(R(x)) \le q^{\alpha-\gamma}|i|_v^{-1/(i-1)} N(\sqrt{i})(d+e-1).$$

This completes the proof.

4.5. Proof of Theorem 5

(i) $\alpha/i \le \operatorname{ord}(i) + \theta$. Recall also $\theta < \alpha/(i+1)$. Calculation shows that $(\alpha - \theta)/i \le i \operatorname{ord}(i)$. Therefore the trivial estimate yields

$$\sum_{x\in A_2(i,\theta)} \chi(R(x)) \le q^{\alpha} \le q^{\alpha-(\alpha-\beta-\theta)/(i-1)} q^{i\operatorname{ord}(i)} \le |i|_v^{-i} q^{\alpha-(\alpha-\beta-\theta)/(i-1)}.$$

(ii) $\alpha/i > \operatorname{ord}(i) + \theta$. Grouping elements of $A_2(i,\theta)$ according to their images modulo P^{β}, we get as in the proof of Theorem 4,

$$\sum_{x\in A_2(i,\theta)} \chi(R(x)) = \sum_{y\in B\cap M_2(i,\theta)} \sum_{z\in C(y)} \chi\Big(\sum_{j=0}^{i-1} \frac{R^{(j)}(y)}{j!}\pi^{j\beta} z^j\Big),$$

where B is a fixed complete system of representatives modulo P^{β}, and

$$C(y) = \{z \in O : y + \pi^{\beta} z \in A_2(i,\theta)\},$$

which by (1) is a complete system of representatives modulo $P^{\alpha-\beta}$.

Let $\delta \in \mathbb{Z}$ and

$$A_3(\delta) = \{y \in B \cap M_2(i,\theta) : \delta = \min_{1\le j\le i-1}\{j\beta + \operatorname{ord}(R^{(j)}(y)/j!)\}\}.$$

We can check that

$$B \cap M_2(i,\theta) = \bigcup_{\alpha/2\le\delta<\alpha} A_3(\delta).$$

If $y \in A_3(\delta)$, then by Lemmas 4 and 5, the inner sum is

$$\le c(n, i-1) q^{(\alpha-\delta)(1-1/(i-1))} q^{\alpha-\beta}/q^{\alpha-\delta}.$$

Since $\alpha/i > \mathrm{ord}(i) + \theta$, we can apply Lemma 3 with $r(x) = R(x)$, $\theta = 0$, $\lambda = \delta - \beta$, $k = i - 1$ and $\mu = \theta$. Thus

$$\#A_3(\delta) \le |i|_v^{-1/(i-1)} N(\sqrt{i})(d + e - 1)q^{\beta}/q^{(\delta-\beta-\theta)/(i-1)}.$$

Therefore

$$\sum_{x\in A_2(i,\theta)} \chi(R(x)) \le \frac{\alpha}{2}|i|_v^{-1/(i-1)} N(\sqrt{i})(d + e - 1)q^{\alpha-(\alpha-\beta-\theta)/(i-1)}.$$

This completes the proof of Theorem 5.

5. Corollaries

COROLLARY 1. *If* $0 < \theta < \alpha/(i + 1)$, $2 < i \le 5$, *then*

$$\Big|\sum_{x\in A_2(i,\theta)} \chi(R(x))\Big| \le \frac{\alpha}{2}|i|_v^{-i}(d + e - 1)c(n, i - 1)q^{\alpha-(\alpha-\beta-\theta)/(i-1)}.$$

P r o o f. It suffices to check the validity of (1) in Theorem 5.

COROLLARY 2. *If* $f(x) = \sum_{j=0}^{d} a_j x^j$, $g(x) = \sum_{j=0}^{d} b_j x^j$, *and* $P \nmid a_0b_1 - a_1b_0$, *suppose that* (1) *in Theorem* 5 *holds. Then*

$$\Big|\sum_{x \bmod P^{\alpha}} \chi(R(x))\Big| \le \alpha c_1(n, d + e)q^{\alpha-(\alpha-1)/(d+e)},$$

where $c_1(n, d + e)$ *is a constant depending at most on* n *and* $d + e$.

P r o o f. Since

$$\frac{(g^2R')^{(m)}}{m!} = \sum_{j=0}^{m} \frac{R^{(j+1)}(g^2)^{(m-j)}}{j!(m - j)!}$$

and

$$\mathrm{ord}\left(\frac{(g^2R')^{(d+e-1)}(x)}{(d + e - 1)!}\right) = 0,$$

we have $W = \emptyset$ and $A(i, \theta) = \emptyset$ if $i > d + e$, and $A(d + e, \theta) = \emptyset$ if $\theta \ne 0$. Corollary 2 now follows.

COROLLARY 3. *If* $f(x) = \sum_{j=0}^{d} a_j x^j$, $g(x) = \sum_{j=0}^{d} b_j x^j$, $P \nmid a_0b_1 - a_1b_0$, *and* $d + e \le 6$, *then*

$$\Big|\sum_{x \bmod P^{\alpha}} \chi(R(x))\Big| \le \alpha c(n)q^{\alpha-(\alpha-1)/(d+e)},$$

where $c(n)$ *is a constant depending at most on* n.

P r o o f. This follows from Corollary 2 and the validity of (1) in Theorem 5.

COROLLARY 4. *If* $(a,b)=0$, *then*

$$\Big|\sum_{x \bmod P^\alpha} \chi(ax+bx^{-1})\Big| \le \alpha c(n) q^{(2\alpha+1)/3}.$$

Proof. This is a special case of Corollary 3.

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