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On addition of two distinct sets of integers

by

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1. Introduction. For any nonempty finite set $K \subseteq \mathbb{Z}$ we denote by $d(K)$ the greatest common divisor of K and by $|K|$ the cardinality of K. By the length $\ell(K)$ of K we mean the difference between its maximal and minimal elements.

We write $[m, n] = \{x \in \mathbb{Z} \mid m \le x \le n\}$.

Let $A = \{0 = a_1 < \ldots < a_k\}$ and $B = \{0 = b_1 < \ldots < b_l\}$ be two sets of integers. As usual, their sum is defined by

$$A + B = \{x \in \mathbb{Z} \mid x = a + b,\ a \in A,\ b \in B\}$$

and we put $2A = A + A$.

Let $h_A = \ell(A) - |A| + 1$ denote the number of holes in A, that is,

$$h_A = |[1, \ell(A)] \setminus A|.$$

Here $\ell(A)$ is the largest element of A.

It is easily seen that

$$|A + B| \ge |A| + |B| - 1. \tag{1}$$

In [1] G. Freiman proved the following:

THEOREM 1. (a) *If* $a_k \le 2k - 3$, *then* $|2A| \ge a_k + k = 2|A| - 1 + h_A$.

(b) *If* $a_k \ge 2k - 2$ *and* $d(A) = 1$, *then* $|2A| \ge 3|A| - 3$.

Note that (a) improves the lower bound in (1) by exactly h_A. The first generalization of Theorem 1 in the case of two different summands was given by G. Freiman in [2]:

THEOREM 2. (a) *If* $\ell(B) \le \ell(A) \le |A| + |B| - 3$ *then*

$$|A + B| \ge \ell(A) + |B| = |A| + |B| - 1 + h_A.$$

(b) *If* $\max(\ell(A), \ell(B)) \ge |A| + |B| - 2$ *and* $d(A \cup B) = 1$ *then*

$$|A + B| \ge (|A| + |B| - 3) + \min(|A|, |B|).$$

Later, J. Steinig gave in [4] a somewhat simplified proof of Theorem 2(b), by using Mann's inequality.

A sharpening of Theorem 2(b) with a new beautiful proof based on Kneser's Theorem was recently obtained by V. F. Lev and P. Y. Smeliansky in [3]. Their main result is

Theorem 3. *Assume* $\ell(A) \geq \ell(B)$ *and define*

$$\delta = \begin{cases} 1 & \text{if } a_k = b_l, \\ 0 & \text{if } a_k > b_l. \end{cases}$$

(a) *If* $\ell(A) \leq |A| + |B| - 2 - \delta$, *then*

$$|A+B| \geq \ell(A) + |B| = |A| + |B| - 1 + h_A.$$

(b) *If* $\ell(A) \geq |A| + |B| - 1 - \delta$ *and* $d(A) = 1$, *then*

$$|A+B| \geq |A| + 2|B| - 2 - \delta.$$

Note that neither Theorem 2(a) nor Theorem 3(a) improve the trivial lower bound (1) in the case:

$$\ell(B) \leq \ell(A) \leq |A| + |B| - 3 \quad \text{and} \quad h_A = 0.$$

At the same time the lower bound $|A| + |B| - 1 + h_A$ gives only a modest improvement of (1) if h_A is very close to 0.

More precisely, the lower bound given by Theorems 2 and 3 in case (a) depends only on h_A and we would desire a symmetric one, which uses both sets A and B. What happens if h_A is much smaller than h_B? Is it still possible to improve the lower bound in this situation?

We prove

Theorem 4. *Define*

$$\delta = \begin{cases} 1 & \text{if } \ell(A) = \ell(B), \\ 0 & \text{if } \ell(A) \neq \ell(B). \end{cases}$$

If $\max(\ell(A), \ell(B)) \leq |A| + |B| - 2 - \delta$, *then*

$$|A+B| \geq (|A| + |B| - 1) + \max(h_A, h_B). \tag{2}$$

2. Proof of Theorem 4. There is no loss of generality in assuming $\ell(A) \geq \ell(B)$. If $h_A \geq h_B$, Theorem 3(a) gives the desired inequality (for an elementary proof see for example [4], Theorem x).

Suppose that $h_B > h_A$. If $\ell(A) = \ell(B)$, then Theorem 3(a) gives

$$\begin{aligned} |A+B| &\geq \ell(B) + |A| = |A| + |B| - 1 + h_B \\ &= |A| + |B| - 1 + \max(h_A, h_B). \end{aligned}$$

Hence, we assume below that

$$h_B > h_A, \tag{3}$$

$$\ell(A) = a_k > b_l = \ell(B) \tag{4}$$

and this also yields $\delta = 0$.

Define m by

$$a_m < b_l \le a_{m+1} \tag{5}$$

and let

$$A' = \{a_1, a_2, \ldots, a_m\}, \quad A'' = \{a_{m+1}, a_{m+2}, \ldots, a_k\}. \tag{6}$$

Note that $m \ge 2$ in view of (3) and (4). Obviously $(A' + B) \cap (A'' + b_l) = \emptyset$ and therefore

$$|A + B| \ge |A' + B| + |A''|. \tag{7}$$

If we show that B and A' satisfy the assumptions of Theorem 3(a) then using this theorem to estimate $|B + A'|$ one obtains

$$|B + A'| \ge |B| + |A'| - 1 + h_B,$$

and thus

$$\begin{aligned} |A + B| &\ge |A' + B| + |A''| \ge (|B| + |A'| - 1) + h_B + |A''| \\ &= (|A| + |B| - 1) + h_B, \end{aligned}$$

completing the proof.

Thus it remains to show that

$$\ell(B) > \ell(A'), \tag{8}$$

$$\ell(B) \le |B| + |A'| - 2. \tag{9}$$

By inequality (5), the number of holes in A between 1 and b_l is given by $b_l - (m - 1)$, and is at most h_A. We get

$$\begin{aligned} \ell(A') = a_m < b_l = \ell(B) &= [b_l - (m - 1)] + (m - 1) \\ &\le h_A + (m - 1) = a_k - (k - 1) + (m - 1) \\ &\le (k + l - 2) - (k - 1) + (m - 1) = |B| + |A'| - 2. \end{aligned}$$

The theorem is proved. ∎

3. Consequences. We usually utilize nontrivial lower bounds for $|A+B|$ in order to estimate the length of A and B for a given value of $|A+B|$. Our theorem sharpens the corresponding results of V. F. Lev and P. Y. Smeliansky.

In this section we do not assume that the minimal elements of A and B are 0 and for $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_l\}$ we define

$$\delta = \begin{cases} 1 & \text{if } \ell(A) = \ell(B), \\ 0 & \text{if } \ell(A) \ne \ell(B). \end{cases}$$

COROLLARY 1. *Let $B \subseteq A$ be two finite sets of integers. Denote by d the greatest common divisor of $a_2 - a_1, \ldots, a_k - a_1$, let $a = \ell(A)/d$ be the reduced length of A and put $b = \ell(B)/d$. If $T = |A + B| < |A| + 2|B| - 2 - \delta$ then $a \le T - l$ and $b \le T - k$.*

Proof. Define $A' = \{(a_i - a_1)/d : i = 1, \ldots, k\}$ and $B' = \{(b_j - b_1)/d : j = 1, \ldots, l\}$. Note that $\ell(A') = a \geq \ell(B')$ and $d(A') = 1$.

If $a \geq |A'| + |B'| - 1 - \delta$ then Theorem 3(b) gives $|A + B| = |A' + B'| \geq |A'| + 2|B'| - 2 - \delta = |A| + 2|B| - 2 - \delta$, a contradiction.

Therefore $a \leq |A'| + |B'| - 2 - \delta$. Theorem 4 gives $|A + B| = |A' + B'| \geq (|A'| + |B'| - 1) + \max(h_{A'}, h_{B'}) = \max(a + |B|, b + |A|)$. The corollary is proven. ∎

Corollary 2. *Let A and B be two finite sets of integers. Denote by d the greatest common divisor of $a_2 - a_1, \ldots, a_k - a_1, b_2 - b_1, \ldots, b_l - b_1$ and put $a = \ell(A)/d$, $b = \ell(B)/d$. If $T = |A+B| < |A|+|B|+\min(|A|, |B|)-2-\delta$ then $a \leq T - l$ and $b \leq T - k$.*

Proof. Define $A' = \{(a_i - a_1)/d : i = 1, \ldots, k\}$ and $B' = \{(b_j - b_1)/d : j = 1, \ldots, l\}$. Note that $d(A' \cup B') = 1$.

It is not difficult to prove (see [3], Lemma 2) that $|A' + B'| \geq 2|A'| + |B'| - 2$ if $d(A') > 1$ and $d(A' \cup B') = 1$.

If $c = \max(a, b) \geq |A'| + |B'| - 1 - \delta$ then by Theorem 3(b) and the previous remark we obtain $|A + B| = |A' + B'| \geq |A'| + |B'| - 2 - \delta + \min(|A'|, |B'|) = |A| + |B| - 2 - \delta + \min(|A|, |B|)$, a contradiction.

Therefore $c \leq |A'| + |B'| - 2 - \delta$. Theorem 4 gives $|A + B| = |A' + B'| \geq \max(a + |B'|, b + |A'|) = \max(a + |B|, b + |A|)$. The corollary is proven. ∎

References

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