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# The Dirichlet series of $\zeta(s)\zeta^{\alpha}(s+1)f(s+1)$: On an error term associated with its coefficients

by

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## 1. Introduction

**1.1.** Our main motivation for considering the class of Dirichlet series in the title (where $\alpha \in \mathbb{R}$ and $f(s+1)$ is assumed to have a Dirichlet series expansion absolutely convergent in the half plane $\sigma > -\lambda$, for some $\lambda > 0$) is that the sequences

$$\left\{\left(\frac{\sigma(n)}{n}\right)^{\alpha}\right\}_{n=1}^{\infty} \quad \text{and} \quad \left\{\left(\frac{\phi(n)}{n}\right)^{-\alpha}\right\}_{n=1}^{\infty} \tag{1}$$

(where as usual $\sigma$ and $\phi$ denote the sum-of-divisors and Euler's functions) both are sequences of coefficients $a(n)$ of such series.

Our goal is to establish explicit expressions for $P$ and $E$ in

$$\sum_{n\le x} a(n) = P(x) + E(x) = \text{principal term} + \text{error term} \tag{2}$$

(Theorem 1 in Section 2), and then to obtain $O$- and $\Omega$-estimates for $E$ (Theorems 2 and 3 in Sections 3 and 4). In Section 5 we derive expression (2) in the special cases where $\{a(n)\}$ is a sequence in (1), and infer the corresponding $O$- and $\Omega$-estimates (Theorems 4 and 5); our results cover all real values of $\alpha$ and, apart from the cases $\alpha = \pm 1$ and $\alpha = 0$, all supersede what is known today (and is described in Subsection 1.3). We also deduce similar results for the sequences $\{\sigma^{\alpha}(n)\}$ and $\{\phi^{\alpha}(n)\}$ (Corollaries 1–3).

**1.2.** *A selection of other applications.* We mention below without proof some other applications of our theorems. We can apply them to the study of $\sum_{n\le x}(\sigma(n)/\phi(n))^{\alpha/2}$. For instance, we have

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$$\sum_{n=1}^{\infty} \frac{\sigma(n)/\phi(n)}{n^s} = \zeta(s)\zeta^2(s+1)f(s+1), \tag{3}$$

where $f(s+1)$ is analytic and has an Euler product absolutely convergent in $\sigma > -1/2$, and the method we describe in this paper yields

$$\sum_{n\le x} \frac{\sigma(n)}{\phi(n)} = \zeta^2(2)f(2)x + \sum_{r=0}^{2} B_r(\log x)^r + e(x), \tag{4}$$

with

$$e(x) = \begin{cases} O((\log x)^{4/3}(\log\log x)^{8/3}), \\ \Omega_{\pm}((\log\log x)^2), \end{cases} \tag{5}$$

and where the $B_r$ are computable real constants.

Another interesting application is to the sum

$$\sum_{n\le x} \phi_a^{-\alpha/a}(n), \quad \text{where} \quad \phi_a(n) := n\prod_{p\mid n}\left(1-\frac{a}{p}\right) \tag{6}$$

(and where $a$ is not a prime number in case $\alpha > 0$). When $a$ is a positive integer and the prime factors of $n$ are all larger than $a$, then $\phi_a(n)$ counts the number of distinct groups of $a$ consecutive numbers all prime to and smaller than $n$. These functions were studied by V. Schemmel in 1869 [14]. For instance, we have

$$\sum_{n=1}^{\infty} \frac{\phi_a(n)/n}{n^s} = \zeta(s)\prod_{p}\left(1-\frac{a}{p^{s+1}}\right) := \zeta(s)C_a(s+1) \tag{7}$$

and our method yields

$$\sum_{n\le x} \frac{\phi_a(n)}{n} = C_a(2)x + \sum_{r=0}^{[-a]} B'_r(\log x)^r + e_a(x), \tag{8}$$

with

$$e_a(x) = \begin{cases} O((\log x)^{2|a|/3}(\log\log x)^{4|a|/3}), \\ \Omega_{\pm}((\log\log x)^{\varepsilon|a|}) \quad (\varepsilon = 1 \text{ if } a<0,\ \varepsilon = 1/2 \text{ if } a>0). \end{cases} \tag{9}$$

Similar results on the sum $\sum_{n\le x}\phi_a(n)$ can now be obtained with the method described in Section 5 (Corollaries 1–3).

**1.3.** *Former work on the cases* $a(n) = (\sigma(n)/n)^\alpha$ *and* $a(n) = (\phi(n)/n)^\alpha$

**1.3.1.** $\sigma(n)/n$ *and* $\phi(n)/n$. It was known to Dirichlet in 1849 [4] that $\sigma(n)/n$ and $\phi(n)/n$ are "on average" respectively $\pi^2/6$ and $6/\pi^2$, in the sense that

$$E'(x) := \sum_{n\le x}\frac{\sigma(n)}{n} - \frac{\pi^2}{6}x = o(x) \tag{10}$$

and

$$H(x) := \sum_{n\le x}\frac{\phi(n)}{n} - \frac{6}{\pi^2}x = o(x). \tag{11}$$

And the fact that $E(x) := E'(x) + \frac{1}{2}\log x + (\gamma/2+1)$ may be written in the form $\sum_{n\le x} v(n)\psi(x/n) + o(1)$ (as in Theorem 1 in Section 2 below) was known to Wigert [20] in 1914. He also obviously guessed that a truncation of the sum (producing some extra constant term) might help obtaining estimates of type $O$. Walfisz produced a number of papers since the 1920's on that problem and related ones; he published (posthumously) in 1963 [19] proofs of

$$E(x) = O((\log x)^{2/3}) \tag{12}$$

and

$$H(x) = O((\log x)^{2/3}(\log\log x)^{4/3}) \tag{13}$$

(see "Added in proof"), using Weyl's and Korobov's methods for the estimate of exponential sums (and the expansion of $\psi$ as a Fourier series). It is his argument producing (13) that inspired our Section 3.

On the other hand, the easy fact that $\sigma(n+1)/(n+1) - \sigma(n)/n$ can be as large as $C\log\log n$ for an infinity of values of $n$ (noticed in 1913–14 both by Gronwall and Wigert) shows that

$$E(x) = \Omega(\log\log x). \tag{14}$$

With a similar (but much more involved) argument Erdős and Shapiro proved in 1951 [5] that

$$H(x) = \Omega(\log\log\log x) \tag{15}$$

and then, by averaging $H$ on certain adequately chosen arithmetical progressions $An+B$ $(n\le x)$ of very large moduli $A = A(x)$, that (15) implies

$$H(x) = \Omega_{\pm}(\log\log\log\log x). \tag{16}$$

In 1987 Pétermann [10] proved in a similar manner that (14) implies

$$E(x) = \Omega_{\pm}(\log\log x). \tag{17}$$

Codecà [3], in 1984, had a hunch that truncation of the sum in the error term might also help obtaining estimates of type $\Omega$, in conjunction with the Erdős–Shapiro method; but his results on $H$ and $E$ are not as good as (16) and (17). His general argument was refined by Pétermann in 1988 [11] and then yielded (17) as well as the relation

$$H(x) = \Omega_{\pm}((\log\log x)^{1/2}), \tag{18}$$

first obtained by Montgomery [9] in 1987 by a similar *ad hoc* argument. Our Section 4 is based on this method.

**1.3.2.** $(\phi(n)/n)^\alpha$. In 1969 Il'yasov [6] extended Walfisz' argument for $H(x)$ to the error term $H_\alpha(x)$ associated with $(\phi(n)/n)^\alpha$ (which is the $e_{g_\alpha}(x)$ of Section 5) and proved that

$$H_\alpha(x) = O((\log x)^{2/3}(\log\log x)^{4/3}) \quad (0 < \alpha < 1). \tag{19}$$

In his thesis (1979) Sivaramasarma [16] established

$$H_\alpha(x) = O((\log x)^{\alpha-1/3}(\log\log x)^{4/3}) \quad (1 \le \alpha), \tag{20}$$

thus improving and completing the old result of Chowla ([2], 1930) $H_\alpha(x) = O((\log x)^\alpha)$, proved for positive integral values of $\alpha$. The case $\alpha = -1$ appears to have been considered first by Landau [8] in 1900. Sitaramachandrarao [15] proved in 1982 with the help of (12) that

$$H_{-1}(x) = O((\log x)^{2/3}), \tag{21}$$

and Pétermann [11] in 1988 that

$$H_{-1}(x) = \Omega_\pm(\log\log x). \tag{22}$$

**1.3.3.** $(\sigma(n)/n)^\alpha$. The error term $E_2'(x)$ associated with $\sigma^2(n)$ was considered by Ramanujan [13] in 1916.

[Note. In virtue of the corollaries in Section 5.2, in which $E_\alpha'(x)$ associated with $\sigma^\alpha(n)$ is expressed in terms of $E_\alpha(x)$ associated with $(\sigma(n)/n)^\alpha$ (which is the $e_{f_\alpha}(x)$ of Section 5.2), we choose to restrict our comments to $E_\alpha(x)$ (translating the results on $E_\alpha'(x)$ in the literature).]

R. A. Smith [17] proved in 1970 that

$$E_2(x) = O((\log x)^{5/3}). \tag{23}$$

(The error term he considers is in fact not exactly $E_2(x)$ but (in our notation) $F_2(x) := \sum_{n\le x} v(n)\psi(x/n) + O(\log x)$, where the $O(\log x)$ can be seen to be of the form $C\log x + o(\log x)$, and where the sum is *untruncated* and thus contains another term $C'\log x$. These two terms $(C+C')\log x$ are in our Theorem 1 part of the principal term. They have no influence on estimate (23); but it must be pointed out that Smith's Theorem 3, implying $F_2(x) = \Omega(\log x)$, only yields (at best, when adapted) $E_2(x) = \Omega(1)$.) Theorem 1 of our Section 2 was inspired by Smith's "fundamental lemma".

The meromorphic continuation of

$$\sum_{n=1}^{\infty} \frac{\prod_{r=1}^{\alpha} \sigma_{a_r}(n)}{n^s}$$

is rather easy to obtain for $\alpha = 1$, and is given by Ramanujan's identity (see [13]) for $\alpha = 2$. In the case where $\alpha$ is an integer exceeding 2, Balakrishnan [1] extended the half plane of convergence of the above series by obtaining a representation of it as a product of terms involving values of $\zeta$ at certain points associated with the $\alpha$ complex numbers $a_r$, thus generalizing Ramanujan's identity. Balakrishnan also generalized the estimates (12) of Walfisz and (23) of Smith by proving that

$$E_\alpha(x) = O((\log x)^{\alpha-1/3}) \tag{24}$$

holds for every positive integral value of $\alpha$. For the proof of Theorem 1 we use the idea in [1] in order to obtain the necessary extension of the series considered in this paper, and develop further the method of [1] by a finer analysis of the error term involved. This involves (in Lemmata 2.1–2.3) the use of a classical complex analysis method exploiting Perron's effective formula, Hankel's formula and the theorem of residues, originating from de la Vallée Poussin's proof of the prime number theorem, and sometimes referred to as the "Selberg–Delange method" (see [18, Chap. II.5]).

**1.4.** *Notation.* As usual $[u]$ and $\{u\}$ denote the integral and fractional parts of $u$, and we put $\psi(u) := \{u\} - 1/2$. We adopt the old convention that whenever a complex number is denoted by $s$, then $\sigma$ stands for its real part and $it$ for its imaginary part. There is no possible confusion with the arithmetical function $\sigma(n)$.

In Section 2 just below, $\mathcal{B}$ denotes a real number with $0 < \mathcal{B} \le 1/2$ and such that $\zeta(s)$ is zero-free in the region $D_0$ defined by $\sigma > 1 - 1/(\log t)^{1-\mathcal{B}}$ and $t > t_0$ for some $t_0$. In a region $D$ included in $D_0$, where $\zeta(s)$ is analytic and zero-free, we write $\zeta^\alpha(s)$ for $\exp(\alpha \log \zeta(s))$, where $\log z$ is that branch of the logarithm which is real on the positive real axis. Hence $\zeta^\alpha(s)$ is analytic in $D$ and satisfies $\zeta^\alpha(s) = \prod_p (1-p^{-s})^{-\alpha}$ in the half plane $\sigma > 1$. Moreover the relation $|\zeta^\alpha(s)| \ll (\log t)^{C|\alpha|}$ holds for some absolute constant $C$ everywhere in $D$: this is a direct consequence of the bound $|\log \zeta(s)| \ll \log\log t$ in $D$ (see [18, p. 181]).

Finally, it should be noted that all the constants in our results are computable.

**Acknowledgments.** We are indebted to the referee for a simplified proof of Lemma 2.5, which in addition permitted a simplified and shortened proof of Theorem 1. Also see Remark 2 below Lemma 2.3.

## 2. An asymptotic expression.

In this section we prove

THEOREM 1. *Let $\{a(n)\}$ be a sequence of complex numbers satisfying*

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \zeta(s)\zeta^\alpha(s+1)f(s+1)$$

*for a complex* $\alpha$ *and* $f(s+1)$ *having a Dirichlet series expansion*

$$f(s+1)=\sum_{n=1}^{\infty}\frac{b(n)}{n^{s+1}},$$

*which is absolutely convergent in the half plane* $\sigma > -\lambda$ *for a* $\lambda > 0$ (*and thus with* $|b(n)| \ll n^{\delta}$ *for some* $\delta < 1$). *Let*

$$\zeta^{\alpha}(s+1)f(s+1)=\sum_{n=1}^{\infty}\frac{v(n)}{n^{s}}.$$

*Then there is a number* $b$, $0<b<1/2$, *such that*

$$\sum_{n\le x}a(n)=\zeta^{\alpha}(2)f(2)x+\sum_{r=0}^{[\alpha_0]}B_r(\log x)^{\alpha-r}-\sum_{n\le y}v(n)\psi\left(\frac{x}{n}\right)+o(1),$$

*where* $\psi(z) := \{z\}-1/2$ ($\{z\}$ *denoting as usual the fractional part of the real number* $z$), *where* $y := x/\exp((\log x)^{b})$ *and* $\alpha_0$ *denotes the real part of* $\alpha$, *and where* $b$ *and the* $B_r$ *are computable constants.*

We need some lemmata for the proof of this theorem.

LEMMA 1. *Let* $L$ *be the contour*

$$L=\{re^{-i\pi}\mid \infty>r\ge\varepsilon\}\cup\{\varepsilon e^{i\theta}\mid -\pi<\theta<\pi\}\cup\{re^{i\pi}\mid \varepsilon\le r<\infty\},$$

*where* $\varepsilon$ *is an arbitrary positive real number*, *that is*, *the line on the lower edge of the negative real axis from* $-\infty$ *to* $-\varepsilon$, *followed by a circle of radius* $\varepsilon$ *with centre origin* (*traversed counterclockwise*), *combined with the upper edge of the negative real axis from* $-\varepsilon$ *to* $-\infty$. *Then*, *for any complex number* $\alpha$,

$$\frac{1}{2\pi i}\int_{L}\frac{x^{s}}{s^{\alpha}}\,ds=\frac{(\log x)^{\alpha-1}}{\Gamma(\alpha)}\qquad\left[=\frac{\sin(\pi\alpha)}{\pi}\Gamma(1-\alpha)(\log x)^{\alpha-1}\quad \text{if } \alpha\notin\mathbb{Z}\right].$$

Proof. This follows from Hankel's formula

$$\frac{1}{2\pi i}\int_{L}s^{-\alpha}e^{s}\,ds=\frac{1}{\Gamma(\alpha)}\tag{1}$$

(see for instance [18, Théorème II.5.2]) and the functional equation

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(z\pi)}\qquad\text{(if $z$ is not an integer).}\tag{2}$$

LEMMA 2. *Let* $L_1$ *be the contour* $L$ *of Lemma* 1 *with* $-\eta$ *to* $-\infty$ *cut off from both edges of the real axis. That is*,

$$L_1=\{re^{-i\pi}\mid \eta>r\ge\varepsilon\}\cup\{\varepsilon e^{i\theta}\mid -\pi<\theta<\pi\}\cup\{re^{i\pi}\mid \varepsilon\le r<\eta\},$$

*with* $\varepsilon < \eta$. *Then*, *uniformly for* $\alpha \in \mathbb{C}$,

$$\frac{1}{2\pi i}\int_{L_1} \frac{x^s}{s^\alpha}\,ds = \frac{(\log x)^{\alpha-1}}{\Gamma(\alpha)} + O((\log x)^{\alpha-1}47^{|\alpha|}\Gamma(1+|\alpha|)x^{-\eta/2}).$$

Proof. See [18, Corollaire II.5.2.1].

LEMMA 3. *Under the notation in Theorem* 1, *with* $\mathcal{B} \le 1/2$ *as in Subsection* 1.4, $B < \mathcal{B}$ *and* $R := [(\log x)^B]$,

$$\text{(3)}\qquad \sum_{n\le x} v(n) = \sum_{r=0}^{R} E_r(\log x)^{\alpha-r} + O(e^{-R})$$
$$= \sum_{r=0}^{[\alpha_0]} E_r(\log x)^{\alpha-r} + \begin{cases} O((\log x)^{\alpha-[\alpha_0]-1}), \\ O(e^{-R}), \quad \text{if } \alpha\in\mathbb{Z}, \end{cases}$$

*and*

$$\text{(4)}\qquad \sum_{n\le x} \frac{v(n)}{n} = \zeta^\alpha(2)f(2) + \frac{1}{x}\sum_{r=0}^{R} G_r(\log x)^{\alpha-r} + O(e^{-R}/x)$$
$$= \zeta^\alpha(2)f(2) + \frac{1}{x}\sum_{r=0}^{[\alpha_0]} G_r(\log x)^{\alpha-r} + \begin{cases} O((\log x)^{\alpha-[\alpha_0]-1}/x), \\ O(e^{-R}/x), \quad \text{if } \alpha\in\mathbb{Z}, \end{cases}$$

*where* $|E_r|$ *and* $|G_r|$ *are bounded by* $(Cr)^r$ *for some constant* $C$.

Remarks. 1. There is no constant term on the right of (3) (except of course when $\alpha = [\alpha_0] \ge 0$). It follows that if $\alpha_0 < 0$, then $\sum_{n=1}^\infty v(n)$ converges to 0.

2. The referee pointed to us that Lemma 3 and Remark 1 just above could also be established by invoking Theorem II.5.3 of [18], which provides an asymptotic expansion for $\sum_{n\le x} nv(n)$, the coefficients of which, say $H_r$, can be seen to satisfy $|H_r| \le K^r$. From this one can infer by partial integration expansions for $\sum_{n\le x} v(n)$ and $\sum_{n\le x} v(n)/n$, as well as satisfactory bounds on $|E_r|$ and $|G_r|$. Remark 1 is then obtained with a further integration by parts on $\zeta^\alpha(s+1)f(s+1) = \sum v(n)n^{-s}$, by letting $s\to 0^+$ and by using the fact that $s^\alpha\zeta^\alpha(s+1)f(s+1)$ is regular at the origin. In fact, a result more general than our Theorem 1 follows from Theorem II.5.3 of [18], and the proof of the latter is very similar to that of the former. Rather than referring to it we nevertheless preferred to stick to our original proof, in which only Hankel's and Perron's formulae and bounds on the zeta function in the critical strip, and direct consequences of these classical theorems, are used without proof, and from which Remark 1 above follows directly.

Proof of Lemma 3. Let $\delta < 1$ as in the assumptions of Theorem 1. We have $\sum_{n=1}^\infty nv(n)/n^s = \zeta^\alpha(s)f(s)$, where $f(s) = \sum b(n)/n^s$ with

$|b(n)| \ll n^{\delta}$. Hence we get

$$nv(n) = \sum_{r|n} d_{\alpha}(r) b(n/r) \ll n^{\delta+\varepsilon}$$

for every $\varepsilon > 0$ (where $d_{\alpha}(r)$ is the coefficient of $r^{-s}$ in the Dirichlet series expansion for $\zeta^{\alpha}(s)$) and, from a standard application of the "truncated" (or "effective") Perron formula (see for instance [18, Théorème II.2.2, p. 150]),

$$\sum_{n\le x} v(n) = \frac{1}{2\pi i} \int_{\vartheta-ix}^{\vartheta+ix} \zeta^{\alpha}(s+1) f(s+1) \frac{x^s}{s}\, ds + O(x^{\vartheta-1}),$$

where $\vartheta$ satisfies $\delta < \vartheta < 1$. The function $f(s+1)$ is absolutely convergent in the half plane $\sigma > -\lambda$. Also, $s\zeta(s+1)$ is analytic and non-zero in the disc $|s| < 1$. Hence the power series expansion for $s^{\alpha+1}\zeta^{\alpha}(s+1)f(s+1)/s$ at $s = 0$ has radius of convergence $\ge \min(\lambda, 1) = 1/\lambda_1$, say, with $\lambda_1 \ge 1$. Let

$$s^{\alpha+1}\zeta^{\alpha}(s+1)f(s+1)/s = \sum_{r=0}^{\infty} A_r s^r.$$

Then $|A_r| \ll (\lambda_1 + \mu)^r$ for every $\mu > 0$, and hence, if $|s| \le 1/(3\lambda_1)$,

$$\sum_{r=R+1}^{\infty} A_r s^r \ll e^{-R}. \tag{5}$$

We complete the segment from $\vartheta - ix$ to $\vartheta + ix$ into a closed contour, penetrating to the left up to the border of the available zero-free region of $\zeta(s+1)$. In fact, we take the contour $L_1$ of Lemma 2 with $\varepsilon < \vartheta$ and $\eta = 1/(3\lambda_1)$. Then we connect $-\eta$ on the upper edge of the real axis to the curve $\sigma = -1/(\log t)^{1-\mathcal{B}}$ $(t > 0)$, avoiding zeros of $\zeta(s+1)$, follow it up to $-1/(\log x)^{1-\mathcal{B}} + ix$, and join $\vartheta + ix$ by a horizontal line. Finally, we also take the symmetrical image of this with respect to the real axis, and obtain a closed contour $L_2$. Now we have

$$\begin{aligned}
\frac{1}{2\pi i} \int_{L_1} \zeta^{\alpha}(s+1) f(s+1) \frac{x^s}{s}\, ds
&= \frac{1}{2\pi i} \int_{L_1} \Big( \sum_{r=0}^{R} A_r s^{r-\alpha-1} x^s + \sum_{r=R+1}^{\infty} A_r s^{r-\alpha-1} x^s \Big)\, ds \\
&= \sum_{r=0}^{R} A_r \frac{(\log x)^{\alpha-r}}{\Gamma(\alpha-r+1)} + O(x^{-\eta/3}) + O(e^{-R}),
\end{aligned}$$

by using (5) and Lemma 2. After a straightforward estimate of $\int_{L_2\setminus L_1} \zeta^{\alpha}(s+1) \times f(s+1) x^s s^{-1}\, ds$, where we use the bound $|\log \zeta(s)| \ll \log\log t$ in the zero-free region bordered by $L_2$ (see for instance [18, p. 181] and [7, Theorem 6.6.1]), the first line in (3) of the lemma follows. The bound given for $|E_r|$ is obtained from the bound given above for $A_r$, and from the trivial $\Gamma(1+r-\alpha) \ll r^r$, and is then used (with the fact that $E_r = 0$ if $\alpha \in \mathbb{Z}$ and $r > \alpha_0$) to complete the proof of (3). Finally, a similar treatment yields (4).

LEMMA 4. *For $a(n)$ as in Theorem 1, we have*

$$\sum_{n\le x} a(n) = \zeta^{\alpha}(2)f(2)x + \sum_{r=0}^{[\alpha_0]} C_r(\log x)^{\alpha-r} - \sum_{n\le x} v(n)\psi(x/n) + o(1),$$

*where the $C_r$ are some constants.*

Proof. We see from the statement of Theorem 1 that

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \zeta(s)\sum_{n=1}^{\infty}\frac{v(n)}{n^s},$$

which in turn implies $a(n) = \sum_{r|n} v(r)$, and we get

$$\sum_{n\le x} a(n) = \sum_{n\le x} v(n)\left[\frac{x}{n}\right] = x\sum_{n\le x}\frac{v(n)}{n} - \frac{1}{2}\sum_{n\le x} v(n) - \sum_{n\le x} v(n)\psi\left(\frac{x}{n}\right),$$

where we recall that $\psi(u) := u - [u] - 1/2$. Now the lemma follows with the help of (3) and (4) of Lemma 3.

LEMMA 5. *If $y := x\exp(-(\log x)^b)$ for some $b < B$, then for some constants $c(m)$,*

$$\sum_{y\le n\le x} v(n)\psi(x/n) = \sum_{0\le m\le \alpha_0-1} c(m)(\log x)^{\alpha-1-m} + o(1).$$

Proof. If we put $V(x) := \sum_{n\le x} v(n)$, a straightforward calculation using (3) of Lemma 3, $|E_r| \le (Cr)^r$, $y = x\exp(-(\log x)^b)$ and $b < B < 1/2$ yields

$$\begin{aligned}(6)\qquad &\sum_{y<n\le x} v(n)\psi\left(\frac{x}{n}\right)\\ &= \int_y^x \psi\left(\frac{x}{u}\right)dV(u) = \int_y^x \psi\left(\frac{x}{u}\right)d\Big(\sum_{r=0}^{R(u)} E_r(\log u)^{\alpha-r} + e_1(u)\Big)\\ &= \int_y^x \psi\left(\frac{x}{u}\right)d\Big(\sum_{r=0}^{R(x)=R} E_r(\log u)^{\alpha-r}\Big) + \int_y^x \psi\left(\frac{x}{u}\right)d(e(u)) =: I + II,\end{aligned}$$

where $e_1(u)$ and $e(u)$ are $O(e^{-R(u)})$ and $R(u) = [(\log u)^B]$. Now

$$II = o(1), \tag{7}$$

since

$$II = -\psi\left(\frac{x}{y}\right)e(y) + \psi(1)e(x) + \int_1^{x/y} e\left(\frac{x}{u}\right) d\psi(u)$$
$$= o(1) + O\left(\exp(-(\log y)^B)\frac{x}{y}\right),$$

and

$$I = \sum_{r=0}^{R} \int_y^x E_r(\alpha - r)\frac{(\log u)^{\alpha-r-1}}{u}\psi\left(\frac{x}{u}\right) du \tag{8}$$
$$= \sum_{r=0}^{R} D_r \int_y^x \frac{(\log u)^{\alpha-r-1}}{u}\psi\left(\frac{x}{u}\right) du$$

(where $D_r := (\alpha - r)E_r$).

Now, if the real part $a_0$ of $a$ satisfies $a_0 \le 0$, then

$$\int_y^x \frac{1}{u}(\log u)^a \psi\left(\frac{x}{u}\right) du \ll (\log y)^{a_0} \int_y^x \frac{du}{u} = (\log y)^{a_0} \log(x/y).$$

Hence

$$\sum_{r=[\alpha_0]+1}^{R} D_r \int_y^x \frac{1}{u}(\log u)^{\alpha-r-1}\psi\left(\frac{x}{u}\right) du$$
$$\ll \log(x/y)(\log y)^{\alpha_0} \sum_{r=[\alpha_0]+1}^{R} (Cr)^{r+1}/(\log y)^{r+1}.$$

Since $r \le R = [(\log x)^B]$, $B < 1$, and $\log y \asymp \log x$, we have $Cr/\log y < 1/2$, say, and we get

$$\sum_{r=[\alpha_0]+1}^{R} D_r\left(\int_y^x \frac{1}{u}(\log u)^{\alpha-r-1}\psi\left(\frac{x}{u}\right) du\right) \tag{9}$$
$$\ll \log(x/y)(\log y)^{\alpha_0}(C([\alpha_0]+2)/\log y)^{[\alpha_0]+2}\sum_{r=0}^{\infty} 2^{-r} = o(1).$$

For $0 \le r \le [\alpha_0]$ we have (observe that we have this case only when $\alpha_0 \ge 0$)

$$\int_y^x \frac{1}{u}(\log u)^{\alpha-r-1}\psi\left(\frac{x}{u}\right)du$$

$$= \int_1^{x/y} \frac{\psi(t)}{t}(\log x)^{\alpha-r-1}\left(1-\frac{\log t}{\log x}\right)^{\alpha-r-1} dt$$

$$= \int_1^{x/y} \frac{\psi(t)}{t}(\log x)^{\alpha-r-1}\left(\sum_{m=0}^{M}\binom{\alpha-r-1}{m}(-1)^m\left(\frac{\log t}{\log x}\right)^m + O\left(\sum_{m=M+1}^{\infty}\left(\frac{\log t}{\log x}\right)^m\right)\right)dt,$$

since then

$$\left|\binom{\alpha-r-1}{m}\right| =: \left|\binom{\beta}{m}\right| = \frac{m^{-\beta-1}}{|\Gamma(-\beta)|}(1+o(1)) \ll 1.$$

(In order to see this—and even more—one may appeal for instance to

$$\Gamma(-\beta) = \lim_{m\to\infty}\frac{m!m^{-\beta}}{-\beta(-\beta+1)\ldots(-\beta+m)}.)$$

We choose $M := \lceil \alpha_0/(1-b)\rceil$ (where $\lceil a\rceil$ denotes the smallest integer not less than $a$) and note that

$$\frac{\log t}{\log x} \ll \frac{\log(x/y)}{\log x} = (\log x)^{b-1}$$

and

$$\sum_{m=M+1}^{\infty}\left(\frac{\log t}{\log x}\right)^m \ll (\log x)^{b-1-\alpha_0}.$$

Hence

$$\int_1^{x/y} \frac{\psi(t)}{t}(\log x)^{\alpha-r-1} O\left(\sum_{m=M+1}^{\infty}\left(\frac{\log t}{\log x}\right)^m\right)dt$$

$$\ll (\log x)^{b-1-\alpha_0}\int_1^{x/y}\frac{1}{t}(\log x)^{\alpha-1}\,dt \ll (\log x)^{b-1}.$$

Thus we see that

$$\int_y^x \frac{1}{u}(\log u)^{\alpha-r-1}\psi\left(\frac{x}{u}\right)du$$

$$= \sum_{m=0}^{M}(-1)^m\binom{\alpha-r-1}{m}(\log x)^{\alpha-r-1-m}\int_1^{x/y}\frac{\psi(t)}{t}(\log t)^m\,dt + O((\log t)^{b-1}).$$

Also we have the estimate

$$\int_{x/y}^{\infty} \frac{\psi(t)}{t}(\log t)^m\,dt \ll \frac{y}{x}(\log x)^m \ll (\log x)^m \exp(-(\log x)^b).$$

Thus we get, for $0 \le r \le [\alpha_0]$,

$$\int_y^x \frac{1}{u}(\log u)^{\alpha-r-1}\psi\left(\frac{x}{u}\right)du = \sum_{0\le m\le \alpha_0-1} c(m,r)(\log x)^{\alpha-1-r-m} + o(1),$$

where

$$c(m,r) = (-1)^m \binom{\alpha-r-1}{m} \int_1^{\infty} \frac{\psi(t)}{t}(\log t)^m\,dt.$$

Hence

$$\sum_{r=0}^{[\alpha_0]+1} D_r \int_y^x \frac{1}{u}(\log u)^{\alpha-r-1}\psi\left(\frac{x}{u}\right)du \tag{10}$$
$$= \sum_{0\le m\le \alpha_0-1} c(m)(\log x)^{\alpha-1-m} + o(1).$$

Combining (8)–(10) we get

$$I = \sum_{0\le m\le \alpha_0-1} c(m)(\log x)^{\alpha-1-m} + o(1)$$

and this with (6) and (7) implies the lemma.

Proof of Theorem 1. Theorem 1 is evident from Lemmata 4 and 5.

**3. A $O$-estimate.** As usual, the letter $p$ denotes exclusively prime numbers. In this section we prove

THEOREM 2. *Let $v_n = v(n)$ be a real multiplicative arithmetical function satisfying, for some real numbers $\alpha > 0$ and $\beta \ge 0$,*

$$\sum_{n\le x} |v_n| = O((\log x)^{\alpha}); \tag{h1}$$

$$\sum_{n\le x} (nv_n)^2 = O(x(\log x)^{\beta}); \tag{h2}$$

(h3) *$p^k v(p^k)$ is an ultimately monotonic function of $p$ for $k = 1$ and $k = 2$, and is bounded for every $k \ge 1$.*

*Set $y := x\exp(-(\log x)^b)$ for some positive number $b$, $t := \log x$, and $u := \log t = \log\log x$. Then*

$$\sum_{n\le y} v_n\psi(x/n) = O(t^{2\alpha/3}u^{4\alpha/3}) \tag{1}$$

(see "Added in proof").

N o t e. Our assumption (h3) can be replaced by various weaker conditions. We adopt it here for the sake of simplicity.

R e m a r k. Our proof heavily relies on Walfisz' proof in [19] of the estimate

$$H(x) := \sum_{n\le x} \frac{\phi(n)}{n} - \frac{6}{\pi^2}x = O(t^{2/3}u^{4/3}), \tag{2}$$

which is contained in our Theorem 2 ($nv_n = \mu(n)$, $\alpha = 1$ and $\beta = 0$). Walfisz' complete argument is 30 pages long, but the largest and most difficult part of it needs very little change, and instead of reproducing it almost in extenso, we choose to refer the reader to it. Thus our Lemmata 1, 2 and 6 are almost exactly his Hilfssätze 4.4.7, 4.4.8 and 4.5.5. This of course means that the thorough reader will need a copy of [19]. We did our best however to allow a reading without it.

We first need some notation.

*Notation.*

$$c_3 := \max\left(18+2\alpha, 4\log 2 + \frac{\beta\log 2}{2}\right), \quad c_2 := \frac{4c_3}{\log 2},$$
$$c_4 := \left(\frac{96}{266\,000 c_3 \max(144, c_2)}\right)^{1/3}, \quad c_1 := \frac{c_4}{4c_2},$$
$$X := [c_1 t^{1/3} u^{-4/3}], \quad s := \log N, \quad \text{where } x^{2/X} < N \le y,$$
$$N_0 := \exp\left(\frac{s}{c_2\log s}\right), \quad D := s^{c_3},$$

and

$$t(U) = t(U,V,V') := U \sum_{\substack{q_1<q\le NU^{-1}\\ N^{-1}UD^{-1}\le z\le N^{-1}UD}} |qq_1v(q)v(q_1)| \left|\sum_{n=V}^{V'} e\left(\frac{xz}{n}\right)\right|,$$

where $z := q_1^{-1} - q^{-1}$ and

$$U \le V = V(q,q_1) \le V' = V'(q,q_1) \le 2U. \tag{3}$$

Now we state the essential auxiliary results.

LEMMA 1. *With the notation introduced above we have*

$$t(U) = O(N^2D^{-1}s^{2\alpha-1}). \tag{4}$$

LEMMA 2. *With $e(x) := \exp(2i\pi x)$ we have*

$$\sum_{p\le N} e(x/p) = O(ND^{-1/2}s^5). \tag{5}$$

As mentioned in the remark above, Lemmata 1 and 2 are almost Hilfssätze 4.4.7 and 4.4.8 of [19], where Walfisz has 20 in place of our constant $c_3$, and where $T(M,U)$ in his notation should be replaced by our $s^3t(U)$ ($M$ being set equal to 1).

For the proof we refer the reader to Section 4.4 of [19], pp. 125–137. Only minor modifications are needed in the proofs of Hilfssätze 3–6. In the definition of $n$, numbered (45) on p. 134, in the proof of Hilfssatz 7, the coefficient 1/200 should be replaced by $c_4$.

Now we prove

Lemma 3. *Let*

$$S_2(k,U) := \sum_{U\le p^k\le U'} p^k v(p^k) \sum_{\substack{N_0<q\le Np^{-k}\\ g(q)<p}} q v_q e\left(\frac{x}{p^k q}\right).$$

*Then, under the assumption*

$$N_0 < U \le U' \le 2U \le 2NN_0^{-1}, \tag{6}$$

*there are some* $V$, $V'$ *satisfying* (3) *such that for* $k = 1,2,\ldots,$

$$S_2^2(k,U) = O(t(U)) + O(N^2D^{-1}s^{2\alpha-2}), \tag{7}$$

*whence, by Lemma* 1,

$$S_2^2(k,U) = O(N^2s^{-12}). \tag{8}$$

Proof. We have, by (h3) and the Cauchy–Schwarz inequality,

$$S_2^2(k,U) \le \sum_{U\le p^k\le U'} (p^k v(p^k))^2 \sum_{U\le p^k\le U'} \Big| \sum_{\substack{N_0<q\le Np^{-k}\\ g(q)<p}} q v_q e\left(\frac{x}{p^k q}\right)\Big|^2$$
$$= O\Big(U \sum_{U\le n\le U'} \Big| \sum_{\substack{N_0<q\le Nn^{-1}\\ g(q)<n}} q v_q e\left(\frac{x}{nq}\right)\Big|^2\Big).$$

If we set $V := \max(U, g(q)+1, g(q_1)+1)$ and $V' := \min(U', [Nq^{-1}], [Nq_1^{-1}])$, we have

$$S_2^2(k,U) = O\Big(U \sum_{q,q_1\le NU^{-1}} |qv(q)q_1v(q_1)| \Big|\sum_{n=V}^{V'} e\left(\frac{xz}{n}\right)\Big|\Big). \tag{9}$$

With the help of (h2) the contribution from the terms with $q = q_1$ on the right of (9) is easily seen to be $O(N^2D^{-1})$. Now the conditions

$$NU^{-1}D^{-1} \le q_1 < q \le NU^{-1}, \quad q - q_1 > NU^{-1}D^{-1}, \tag{10}$$

imply

$$q_1 < q \le NU^{-1}, \quad N^{-1}UD^{-1} \le z \le N^{-1}UD, \tag{11}$$

and the terms moved aside to replace the condition $q, q_1 \le NU^{-1}$ by (10) in (9) can be seen, with the help of (h1), to contribute a $O(N^2D^{-1}s^{2\alpha-2})$. On comparing (11) with the definition of $t(U)$, we conclude the proof of (7).

Then we obtain

LEMMA 4. *If* $x^{2/X} \le N \le x\exp(-t^b)$ *then*

$$S := \sum_{n\le N} nv_n e(x/n) = O(Ns^{-4}). \tag{12}$$

Proof. We write

$$S = \sum_{\substack{n\le N\\ g(n)>N_0}} nv_n e(x/n) + \sum_{\substack{n\le N\\ g(n)\le N_0}} nv_n e(x/n) =: S_3 + S', \tag{13}$$

where $g(n)$ denotes the largest prime divisor of $n$. We first estimate $S'$: it is $O$ of

$$\sum_{n\le N^{1/2}} n|v_n| + \sum_{\substack{N^{1/2}<n\le N\\ g(n)\le N_0}} n|v_n| =: I + II. \tag{14}$$

From (h1) we have

$$I = O(N^{1/2}(\log N)^{\alpha-1}). \tag{15}$$

As for $II$ we have

$$II^2 \le \Psi(N, N_0)\sum_{n\le N}(nv_n)^2, \tag{16}$$

where $\Psi(N, N_0)$ denotes the number of integers not exceeding $N$ and free of prime factors larger than $N_0$. Since

$$\Psi(N, N_0) \ll N^{1-1/(2\log N_0)} = Ns^{-2c_3/\log 2}$$

(see for instance Theorem III.5.1 in [18]), from (h2) we have

$$II = O(s^{-c_3/\log 2}\sqrt{N}(\sqrt{N}(\log N)^{\beta/2})) = O(Ns^{-4}). \tag{17}$$

Thus from (14), (15) and (17) we have

$$S' = O(Ns^{-4}). \tag{18}$$

Now we write

$$S_3 = \sum_{k=1}^{[s]} S_3(k) \quad\text{and}\quad S_3(k) = S_1(k) + S_2(k), \tag{19}$$

where

$$S_1(k) := \sum_{q\le N_0} qv_q \sum_{N_0<p\le(Nq^{-1})^{1/k}} p^k v(p^k) e\left(\frac{x}{p^k q}\right)$$

and

$$S_2(k) := \sum_{N_0 \le p \le N} p^k v(p^k) \sum_{\substack{N_0 < q \le N p^{-k} \\ g(q) < p}} q v_q e\left(\frac{x}{p^k q}\right),$$

and we evaluate $S_1(k)$. Its inner sum is

$$\sum_{N_0 < p_n \le (Nq^{-1})^{1/k}} W(p_n)(z(p_n) - z(p_{n+1})) + W(p_\lambda) z(p_{\lambda+1}), \tag{20}$$

where

$$W(p_n) := \sum_{N_0 < p \le p_n} e\left(\frac{x}{p^k q}\right), \qquad \lambda := \pi((Nq^{-1})^{1/k}),$$

and

$$z(p_n) = z(p_n, k) := p_n^k v(p_n^k).$$

If $k \ge 2$ then $W(p_n) = O(N^{1/k})$. And if $k = 1$ we have $W(p_n) = O(N^{3/4})$ when $p_n \le N^{3/4}$, and

$$W(p_n) = -\sum_{p < N_0} e\left(\frac{x}{pq}\right) + \sum_{p < p_n} e\left(\frac{x}{pq}\right) = O(Nq^{-1} D^{-1/2} s^5)$$

when $p_n > N^{3/4}$ by Lemma 2 above. Hence the expression in (20) above is $O$ of

$$Nq^{-1} D^{-1/2} s^5 \Big( \sum_{N_0 < p_n \le Nq^{-1}} |z(p_n) - z(p_{n+1})| + |z(p_{\lambda+1})| \Big) \quad \text{if } k = 1, \tag{21}$$

$$N^{1/2} \Big( \sum_{N_0 < p_n \le (Nq^{-1})^{1/2}} |z(p_n) - z(p_{n+1})| + |z(p_{\lambda+1})| \Big) \quad \text{if } k = 2, \tag{22}$$

$$N^{1/k} \sum_{p \le N^{1/k}} p^k |v(p^k)| \quad \text{if } k > 2. \tag{23}$$

With (h1) and (h3) it follows from (23) and (22) that

$$S_1(k) = O(N^{2/3}) \quad \text{if } k \ge 2. \tag{24}$$

As for $k = 1$ we have, from (h1), (h3) and (21),

$$S_1(1) = O\Big( \sum_{q \le N_0} q v_q q^{-1} N D^{-1/2} s^5 \Big) = O(N s^{5+\alpha-c_3/2}) = O(N s^{-4}). \tag{25}$$

Finally, by (8) we see that

$$S_2(k) = O(N s^{-5}) \tag{26}$$

for every $k$, and the lemma now follows from (26), (25), (24), (19), (18) and (13).

Two applications of Lemma 4 ($x = z, N = Q$ and $x = z, N = Q'$) then easily yield

LEMMA 5. *Let* $z > e^e$, $v := \log z$, $Z := [c_1 v^{1/3}(\log v)^{-4/3}]$, $Q \le Q' \le 2Q$, $z^{2/Z} \le Q \le Q' \le z\exp(-v^b)$. *Then*

$$\sum_{q=Q}^{Q'} qv_q e(z/q) = O(Qv^{-2}). \tag{27}$$

From Lemma 5 it follows that

LEMMA 6. *If* $x$ *is an integer and* $w := x^{6/X}$ *then*

$$\sum_{w\le n\le y} v_n \psi(x/n) = O(1). \tag{28}$$

Proof. The proof is exactly similar to that of Walfisz' Hilfssatz 4.5.5 in [19] (pp. 142–144), in the special case where $nv_n = \mu(n)$. Thus we refer the reader once more to Walfisz' book. Briefly, the argument is as follows. We first obtain

$$\sum_{q=Q}^{Q'} qv_q \psi(x/q) = O(Q/t), \tag{29}$$

where $Q \le Q' \le 2Q$ and $w \le Q \le Q' \le y$, by approximating the sum with

$$x\sum_{q=Q}^{Q'} qv_q \int_0^{1/x} \psi\left(\frac{x}{q} + \vartheta\right) d\vartheta, \tag{30}$$

replacing $\psi(y)$ by its Fourier series expansion, and making use of Lemma 5 above. Partial summation then yields

$$\sum_{q=Q}^{Q'} v_q \psi(x/q) = O(1/t), \tag{31}$$

whence the lemma.

Theorem 2 now follows, when $x$ is an integer, from Lemma 6 and the estimate

$$\sum_{n\le w} v_n \psi(x/n) = O(t^{2\alpha/3}u^{4\alpha/3}), \tag{32}$$

which is an immediate consequence of (h1). If $x$ is not an integer, then

$$\sum_{n\le y} v_n\left(\psi\left(\frac{x}{n}\right) - \psi\left(\frac{[x]}{n}\right)\right) = O\left(\sum_{n\le y}\frac{|v_n|}{n}\right) = O(1), \tag{33}$$

which is also a consequence of (h1).

**4. An $\Omega$-estimate.** In this section we prove

THEOREM 3. *Let $v_n = v(n)$ be a real multiplicative arithmetical function satisfying, for some real positive number $\alpha$,*

$$\sum_{n\le x} |v_n| = O((\log x)^{\alpha}); \tag{h1}$$

$$v(p^j) \text{ is of the same sign } * \text{ for all } p \text{ and all } j \ge 1; \tag{h4}$$

$$\sum_{i\ge 0} \frac{v(p^i)}{p^i} \ne 0 \quad \text{for every } p. \tag{h5}$$

*Let $P$ be the set of prime numbers if $* = +$ in* (h4), *and the set of primes $p \equiv 2 \pmod 3$ if $* = -$. Let $m$ be a real positive unbounded variable, $0 < a < 1$, and define $A = A(m)$ and $x = x(m)$ as follows*:

$$A := \prod_{\substack{p\le m\\ p\in P}} p =: \exp((\log x)^a). \tag{1}$$

*Finally, let $y(X) := X\exp(-(\log X)^b)$ for some $b > a$, $b < 1$. Then there is a positive constant $C$ such that for all sufficiently large $m$ there are numbers $X = X(m) \le (A+1)x$ and $X' = X'(m) \le (A+1)x$ satisfying*

$$\sum_{n\le y(X)} v_n \psi\left(\frac{X}{n}\right) \ge C\Big(\prod_{\substack{p\le m\\ p\in P}} (1+|v_p|)\Big) + O(1) \tag{2}$$

*and*

$$\sum_{n\le y(X')} v_n \psi\left(\frac{X'}{n}\right) \le -C\Big(\prod_{\substack{p\le m\\ p\in P}} (1+|v_p|)\Big) + O(1). \tag{3}$$

Note. The conditions (h4) and (h5) can be weakened or adapted in various manners in order to treat other examples; they are sufficient for the applications we have in mind.

We begin by stating

LEMMA. *Let $A = A(x)$ be a positive integer and $B = B(x)$ a non-negative real number with $B < A$. Let $z = z(x)$ be a positive, strictly increasing, continuous and unbounded function. Suppose that $z$ is regularly 0-varying, i.e.*

$$\limsup_{x\to\infty} \frac{z(2x)}{z(x)} < \infty, \tag{4}$$

*and that*

$$u(x) := z(Ax+B) = o(x(\log x)^{1-\alpha}). \tag{5}$$

*Suppose further that*

$$A = o(z(A\eta x + B)(\log x)^{-\alpha}) \tag{6}$$

*for some* $\eta = o((\log x)^{1-\alpha})$, $\eta \le 1$. *Set*

$$g(x) := \sum_{n\le z} v(n)\psi(x/n), \tag{7}$$

*where* $v$ *is a real multiplicative arithmetical function satisfying* (h1). *Then*

$$\frac{1}{x}\sum_{n\le x} g(An+B) = \sum_{k\le u(x)} \frac{v(k)}{k^*}\sum_{n\le k^*}\psi\left(\frac{n}{k^*}+\frac{B}{k}\right) + O(1), \tag{8}$$

*where* $k^*$ *denotes* $k/(A,k)$.

Remark. In the special case where $\sum_{n\le x} n|v(n)| = O(x)$ (which implies (h1) for some $\alpha \le 1$), this lemma is Theorem 1 of [11], the proof of which only needs minor modifications in order to take care of the general case, and to which we refer the reader. There $\alpha(k)/k$ plays the role of $v(k)$, and $f$ that of $\psi$. In fact, (8) remains true if (as in [11]) we replace $\psi$ by any periodic function $f$ of period 1, of bounded variation, and with $\int f(u)\,du = 0$ on the period. The fact that $B$ does not need to be an integer is pointed out in the Addendum of [11].

Proof of Theorem 3. For $A, v, a, b$ as in the theorem, $z := y$ and (for instance) $\eta := \exp(-(\log x)^b)$, the hypotheses of the lemma are satisfied. Thus, by (8) and the fact that the function $\psi(t)$ (which is the first Bernoulli polynomial of argument $\{t\}$) satisfies a Kubert identity of order 1 (see [12]), we have

$$\frac{1}{x}\sum_{n\le x} g(An+B) = \sum_{k\le u(x)} \frac{v(k)}{k}(A,k)\psi\left(\frac{B}{(A,k)}\right) + O(1) =: G + O(1). \tag{9}$$

Suppose first that $* = +$ in (h4), which means that $v(n)$ is positive for all $n$. If $B := 0$ then

$$G = -\frac{1}{2}\sum_{k\le u(x)}\frac{v(k)(A,k)}{k} = -\frac{1}{2}\sum_{n|A} v(n) \sum_{\substack{k\le u(x)/n\\ \heartsuit}} \frac{v(nk)/v(n)}{k},$$

where $\heartsuit = \heartsuit(n)$ under the last sum means that the summation is over the $k$ with prime factors satisfying $p\nmid A$ or $p\,|\,n$. The number 1 is such a $k$ and thus

$$G \le -\frac{1}{2}\sum_{n|A} v(n) = -\frac{1}{2}\prod_{p\le m}(1+v(p)), \tag{10}$$

and we have proved (3) in this case. The choice $B := A-1$ similarly yields (2).

Now suppose that $* = -$ and put $B := A/3$. If $n \mid A$ and $k$ has only prime divisors $p$ with either $p \mid n$ or $p \nmid A$, then $\psi(B/(A,nk)) = 6^{-1}(-1)^{\omega(A)-\omega(n)+1}$. And (h4) ensures that $v(m)(-1)^{\omega(m)} = |v(m)|$ for all $m$. Hence $G$ of (9) is in this case

$$\frac{(-1)^{\omega(A)+1}}{6} \sum_{\substack{n|A\\ v(n)\neq 0}} |v(n)| \sum_{\substack{k\le u(x)/n\\ \heartsuit}} \frac{|v(nk)/v(n)|}{k}(-1)^{\omega(nk)-\omega(n)},$$

and thus

$$(11)\quad (-1)^{\omega(A)+1}6G = \sum_{\substack{n|A\\ v(n)\neq 0}} |v(n)|\biggl(\sum_{\substack{k=1\\ \heartsuit}}^{\infty} \frac{|v(nk)/v(n)|}{k}(-1)^{\omega(nk)-\omega(n)} + O\biggl(\frac{n(\log u)^{\alpha}}{u(x)|v(n)|}\biggr)\biggr)$$
$$= \sum_{n|A} |v(n)| \prod_{\substack{p|n\\ v(p)\neq 0}} \biggl(1+\frac{1}{|v(p)|}\biggl(\frac{|v(p^2)|}{p}+\frac{|v(p^3)|}{p^2}+\dots\biggr)\biggr) \times \prod_{p\nmid A}\biggl(1+\frac{v(p)}{p}+\frac{v(p^2)}{p^2}+\dots\biggr) + O\biggl(\frac{A\log\log A\,(\log u)^{\alpha}}{u(x)}\biggr).$$

It follows that

$$(12)\qquad 6|G| \ge \prod_{\substack{p\le m\\ p\in P}} (1+|v(p)|)\Bigl|\prod_{p\nmid A}(1+a_p)\Bigr| + o(1),$$

where

$$a_p := \frac{v(p)}{p}+\frac{v(p^2)}{p^2}+\dots$$

Now the second product in (12) converges to a non-zero value $C_A$, since by (h5) no factor is zero, and since by (h1), $\sum |v(n)|/n$ converges. Moreover, we have $|C_A| > C_0$, where $C_0$ is some strictly positive constant independent of $A$: indeed, the product $\prod_{p\equiv 1(3) \text{ or } p=3}(1+a_p)$ would otherwise diverge to 0, and this again is excluded by (h1) and (h5).

Hence (2) or (3) is proved in this case. Finally, (3) or (2) is obtained similarly with the choice $B := 2A/3$.

## 5. Applications to the functions $\sigma$ and $\phi$

**5.1.** *Auxiliary results.* In the next two lemmata we establish the fact that Theorem 1 is applicable to the sequences $a(n) = (\sigma(n)/n)^{\alpha}$ and $a(n) = (\phi(n)/n)^{\alpha}$.

LEMMA 1. *Put* $\sigma_{-1}(n) := \sigma(n)/n$. *Then*

$$\sum_{n=1}^{\infty} \frac{\sigma_{-1}^{\alpha}(n)}{n^s} = \zeta(s)\zeta^{\alpha}(s+1)f_{\alpha}(s+1)$$

*with* $f_\alpha(s+1)$ *having an Euler product absolutely convergent in* $\sigma > -1/2$. *Further*, *if we write*

$$f_\alpha(s+1) = \sum_{n=1}^{\infty} \frac{b_f(n)}{n^{s+1}}$$

*then* $b_f(n) \ll_\varepsilon n^\varepsilon$.

Proof. By the multiplicativity of $\sigma_{-1}^{\alpha}(n)$, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\sigma_{-1}^{\alpha}(n)}{n^s} \\
&\quad = \prod_p \left(1 + \frac{(1+p^{-1})^\alpha}{p^s} + \frac{(1+p^{-1}+p^{-2})^\alpha}{p^{2s}} + \ldots\right) \\
&\quad = \zeta(s)\prod_p \left(1 + \frac{(1+p^{-1})^\alpha - 1}{p^s} + \frac{(1+p^{-1}+p^{-2})^\alpha - (1+p^{-1})^\alpha}{p^{2s}} + \ldots\right) \\
&\quad = \zeta(s)\zeta^\alpha(s+1)f_\alpha(s+1),
\end{aligned}$$

where

$$\begin{aligned}
f_\alpha(s+1) = {} & \prod_p \left(1 - \frac{\alpha}{p^{s+1}} + \frac{\binom{\alpha}{2}}{p^{2(s+1)}} - \ldots\right) \\
& \times \left(1 + \frac{(1+p^{-1})^\alpha - 1}{p^s} + \frac{(1+p^{-1}+p^{-2})^\alpha - (1+p^{-1})^\alpha}{p^{2s}} + \ldots\right) \\
= {} & \prod_p \left(1 - \frac{\alpha}{p^{s+1}} + \frac{\binom{\alpha}{2}}{p^{2(s+1)}} - \ldots\right) \\
& \times \left(1 + \frac{\alpha}{p^{s+1}} + \frac{c}{p^{s+2}} + \frac{d_2}{p^{2(s+1)}} + \frac{d_3}{p^{3(s+1)}} + \ldots\right)
\end{aligned}$$

with $d_r = p^r((1+p^{-1}+\ldots+p^{-r})^\alpha - (1+p^{-1}+\ldots+p^{-(r-1)})^\alpha) \ll_\alpha 1$, $c = p^2((1+p^{-1})^\alpha - 1 - \alpha/p) \ll_\alpha 1$ and $|\binom{\alpha}{r}| \ll r^\lambda$ for some $\lambda$ depending only on $\alpha$. Now it follows that the above Euler product for $f_\alpha(s+1)$ is absolutely convergent in $\sigma > -1/2$, and that $f_\alpha(s+1)$ has the representation $\sum_{n=1}^{\infty} b_f(n)/n^{s+1}$ with $b_f(n) \ll C^{\omega(n)} d^{\lambda+1}(n) \ll n^\varepsilon$. The proof of the lemma is complete.

Lemma 2. *We have*

$$\sum_{n=1}^{\infty} \frac{\left(\frac{\phi(n)}{n}\right)^{\alpha}}{n^s} = \zeta(s)\zeta^{-\alpha}(s+1)g_\alpha(s+1)$$

*with* $g_\alpha(s+1)$ *having an Euler product absolutely convergent in* $\sigma > -1/2$. *Further*, *if we write*

$$g_\alpha(s+1) = \sum_{n=1}^{\infty} \frac{b_g(n)}{n^{s+1}},$$

*then* $b_g(n) \ll_\varepsilon n^\varepsilon$.

Proof. By the multiplicativity of $\phi(n)$, we get

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\left(\frac{\phi(n)}{n}\right)^{\alpha}}{n^s} &= \prod_p \left(1 + \frac{(1-p^{-1})^\alpha}{p^s} + \frac{(1-p^{-1})^\alpha}{p^{2s}} + \ldots\right) \\ &= \prod_p \left(1 + \frac{(1-p^{-1})^\alpha}{p^s - 1}\right) = \zeta(s) \prod_p \left(1 + \frac{(1-p^{-1})^\alpha - 1}{p^s}\right) \\ &= \zeta(s)\zeta^{-\alpha}(s+1)g_\alpha(s+1),\end{aligned}$$

where

$$g_\alpha(s+1) = \prod_p \left(1 - \frac{1}{p^{s+1}}\right)^{-\alpha} \left(1 + \frac{(1-p^{-1})^\alpha - 1}{p^s}\right).$$

The proof can now be completed along the lines of the previous lemma's proof.

To conclude this subsection, we show that hypotheses (h1) and (h2) of Sections 3 and 4 are satisfied by the functions $v_f$ and $v_g$ associated with $\sigma_{-1}(n)$ and $\phi(n)/n$.

Lemma 3. *Let* $\alpha \in \mathbb{R}$. *Under the notation of Lemmata* 1 *and* 2 *above let*

$$\sum_{n=1}^{\infty} \frac{v_f(n)}{n^s} = \zeta^\alpha(s+1)f_\alpha(s+1) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{v_g(n)}{n^s} = \zeta^{-\alpha}(s+1)g_\alpha(s+1).$$

*Then*

$$\sum_{n \le x} (nv_h(n))^2 = O(x(\log x)^{\alpha^2 - 1}) \tag{1}$$

*and*

$$\sum_{n \le x} |v_h(n)| = O((\log x)^{|\alpha|}), \tag{2}$$

*where* $h$ *stands for either* $f$ *or* $g$.

Remarks. 1. From (1) we can derive only (using partial summation and the Cauchy–Schwarz inequality) $O((\log x)^{1+\alpha^2/2})$ instead of the right side of (2) (which is as (2) only when $|\alpha| = 1$).

2. For our purposes, any exponent $\beta$ instead of $\alpha^2 - 1$ in (2) is sufficient (see Theorem 2), and in some applications (as those of Section 1.2) such a weaker result can be proved by a simpler argument than below.

Proof of Lemma 3. It is easy to see with the help of Lemmata 1 and 2 that

$$\sum_{n\ge 1} \frac{(nv_h(n))^2}{n^s} = \zeta^{\alpha^2}(s)H_h(s),$$

where $H_h(s)$ is absolutely convergent in $\sigma > 1/2$. A calculation very similar to that of the proof of Lemma 2.3 above then yields

$$\sum_{n\le x} (nv_h(n))^2 = \frac{A_h}{\Gamma(\alpha^2)} x(\log x)^{\alpha^2-1}\left(1 + O\left(\frac{1}{\log x}\right)\right), \tag{3}$$

where $A_h$ is the constant term in the power series expansion for

$$(s-1)^{\alpha^2}\zeta^{\alpha^2}(s)H_h(s)/s \quad \text{at } s = 1.$$

This implies (1). For (2) it is sufficient to note that $|v_h(n)| \le w_h(n)$, where

$$\sum_{n\ge 1} \frac{w_h(n)}{n^s} := \zeta^{|\alpha|}(s+1)F_h(s+1),$$

and where (as can be seen from the proofs of Lemmata 1 and 2)

$$F_f(s+1) = \prod_p \left(1 - \frac{1}{p^{s+1}}\right)^{|\alpha|} \times \left(1 + \frac{|\alpha|}{p^{s+1}} + \frac{|(1+p^{-1})^\alpha - 1 - \alpha/p|}{p^s} + \frac{|d_2|}{p^{2(s+1)}} + \frac{|d_3|}{p^{3(s+1)}} + \ldots\right)$$

and

$$F_g(s+1) = \prod_p \left(1 - \frac{1}{p^{s+1}}\right)^{|\alpha|}\left(1 + \frac{|\alpha|}{p^{s+1}} + \frac{|\binom{\alpha}{2}|}{p^{s+2}} + \ldots\right).$$

Indeed, (2.3) of Lemma 2.3 holds with $v(n) = w_h(n)$ and $|\alpha|$ instead of $\alpha$, and this implies (1).

**5.2.** *Main results.* In virtue of Theorem 1 and Lemma 5.1 we can find a number $b$ with $0 < b < 1$ such that for every real number $\alpha$ we have

$$\sum_{n\le x} \left(\frac{\sigma(n)}{n}\right)^\alpha = \zeta^\alpha(2)f_\alpha(2)x + \sum_{r=0}^{[\alpha]} a_r(\log x)^{\alpha-r} + e_{f_\alpha}(x) + o(1), \tag{4}$$

where

$$e_{f_\alpha} := -\sum_{n\le y} v_{f_\alpha}\psi(x/n) \quad \text{and} \quad y := \exp(-(\log x)^b),$$

where $f_\alpha$ and $v_{f_\alpha}$ are defined by

$$\sum_{n=1}^{\infty} \frac{(\sigma(n)/n)^\alpha}{n^s} = \zeta(s)\zeta^\alpha(s+1)f_\alpha(s+1)$$

and

$$\sum_{n=1}^{\infty} \frac{v_{f_\alpha}(n)}{n^s} = \zeta^\alpha(s+1)f_\alpha(s+1),$$

and where the $a_r = a_r(\alpha)$ are certain real constants (the sum in which they appear being of course empty if $\alpha < 0$).

Similarly, with the help of Lemma 5.2, we have with an obvious notation

$$(5)\quad \sum_{n\le x}\left(\frac{\phi(n)}{n}\right)^\alpha = \zeta^{-\alpha}(2)g_\alpha(2)x + \sum_{r=0}^{[-\alpha]} b_r(\log x)^{-\alpha-r} + e_{g_\alpha}(x) + o(1).$$

We now establish the following estimates for the error terms $e_{f_\alpha}$ and $e_{g_\alpha}$ of these summatory functions.

THEOREM 4. *With the notation as just above we have, for each real number* $\alpha$,

$$(6)\qquad e_{h_\alpha} = O((\log x)^{2|\alpha|/3}(\log\log x)^{4|\alpha|/3}),$$

*where* $h$ *denotes either* $f$ *or* $g$.

THEOREM 5. *On the other hand, we have, also for each real number* $\alpha$,

$$(7)\qquad e_{h_\alpha} = \begin{cases} \Omega_\pm((\log\log x)^{|\alpha|}) & \text{if } h=f \text{ and } \alpha\ge 0,\\ & \text{or } h=g \text{ and } \alpha\le 0;\\ \Omega_\pm((\log\log x)^{|\alpha|/2}) & \text{if } h=f \text{ and } \alpha\le 0,\\ & \text{or } h=g \text{ and } \alpha\ge 0.\end{cases}$$

Comments. (1) For $\alpha = 1$ and $h = g$ Theorem 2 is Walfisz' result (1.13); for $\alpha = 1$ and $h = f$ though, it is not as good as estimate (1.12): Walfisz' proof of the latter exploits the monotonicity of $v_{f_1}(n) = 1/n$, and cannot be generalized to other values of $\alpha$. For positive values of $\alpha \ne 1$ and $h = g$ Theorem 2 improves on Il'yasov's (1.19) and Sivaramasarma's (1.20); for positive integral values of $\alpha$ it improves on Balakrishnan's (1.24). As for the other cases there are to our knowledge no $O$-estimates in the literature.

(2) We believe Theorem 5 is new, except when $\alpha = 1$ and $h = f$ and when $\alpha = \pm 1$ and $h = g$. In these three cases it is Pétermann's (1.17), (1.22) and Montgomery's (1.18).

Proof of Theorem 4. We consider the cases where the coefficients of the Dirichlet series expansion of $\zeta(s)\zeta^{\alpha}(s+1)f(s+1)$ are $\{(\sigma(n)/n)^{\alpha}\}$ and $\{(\phi(n)/n)^{\alpha}\}$. By the results in Subsection 5.1 above the corresponding sequences of coefficients $\{v_f(n)\}$ and $\{v_g(n)\}$ for $\zeta^{\alpha}(s+1)f_{\alpha}(s+1)$ and $\zeta^{-\alpha}(s+1)g_{\alpha}(s+1)$ both satisfy (h1) and (h2) (with of course $|\alpha|$ instead of $\alpha$ in (h1)). By Theorem 2 there thus remains to establish (h3). It is easy to see that $p^k v_h(p^k)$ is bounded for each $k$ and $h = f$ or $g$, and that

$$\begin{aligned} pv_g(p) &= p((1-p^{-1})^{\alpha}-1),\\ p^2 v_g(p^2) &= 0,\\ pv_f(p) &= p((1+p^{-1})^{\alpha}-1), \end{aligned}$$

are monotonic functions of $p$. As for

$$p^2 v_f(p^2) = p^2((1+p^{-1}+p^{-2})^{\alpha} - (1+p^{-1})^{\alpha}),$$

let us write $p = x^{-1}$ and $u = 1+x+x^2$. We have

$$\begin{aligned} p^2 v_f(p^2) =: m(x) &= \frac{u^{\alpha}-(u-x^2)^{\alpha}}{x^2} = \frac{(u-x^2)^{\alpha}}{x^2}\left(\left(1-\frac{x^2}{u}\right)^{-\alpha}-1\right)\\ &= \frac{(1+x)^{\alpha}}{x^2}\left(\alpha\frac{x^2}{u} + \sum_{n\ge 2}\binom{-\alpha}{n}(-1)^n\frac{x^{2n}}{u^n}\right)\\ &= (1+x)^{\alpha}\left(\alpha\frac{1}{u} + \sum_{n\ge 2}\binom{-\alpha}{n}(-1)^n\frac{x^{2n-2}}{u^n}\right), \end{aligned}$$

and we must ensure that $m$ is monotonic at least in some interval $(0,\varepsilon_{\alpha})$, where $\varepsilon_{\alpha} > 0$. This is immediate if $\alpha = 0$ or $\alpha = 1$. For other values of $\alpha$ this follows from

$$m'(x) = \frac{\alpha(1+x)^{\alpha-1}}{u^2}((\alpha-1)+O_{\alpha}(x)).$$

Proof of Theorem 5. We already ensured in Lemma 5.3 that (h1) holds when $v_h$ is one of the functions $v_f$ and $v_g$. And from the expression of $\zeta^{\alpha}(s+1)f_{\alpha}(s+1)$ and $\zeta^{-\alpha}(s+1)g_{\alpha}(s+1)$ in Lemmata 5.1 and 5.2, (h4) is clearly satisfied by both $v_h = v_f$ and $v_h = v_g$, with

$$\begin{aligned} * = + &\quad \text{when } h = f \text{ and } \alpha \ge 0 \quad \text{or} \quad h = g \text{ and } \alpha \le 0;\\ * = - &\quad \text{when } h = f \text{ and } \alpha \le 0 \quad \text{or} \quad h = g \text{ and } \alpha \ge 0. \end{aligned}$$

As regards (h5), since for every $p$, $\sum_{i\ge 0} v_h(p^i)/p^i$ is a factor in the Euler product for $\zeta^{\pm\alpha}(2)h_{\alpha}(2)$, it is sufficient to show that $h_{\alpha}(2) \ne 0$. In view of Theorem 1 this is a triviality when $* = +$ above (in which case (h5) is in fact not needed). Then, if for instance $\alpha > 0$ and $h = f$, an application of the Cauchy–Schwarz inequality yields

$$\sum_{n\le x}\left(\frac{n}{\sigma(n)}\right)^{-\frac{\alpha}{2}2} \gg \left(\sum_{n\le x}1\left(\sum_{n\le x}\left(\frac{\sigma(n)}{n}\right)^{-\frac{\alpha}{2}2}\right)^{-1/2}\right)^2 \gg x,$$

which (again in view of Theorem 1) shows that $f_\alpha(2)\neq 0$. The argument for the remaining case $\alpha<0$ and $h=g$ is exactly similar. Now, for $h=f$ or $g$ we have

$$1+|v_h(p)| = 1+\left|\left(1\pm\frac{1}{p}\right)^{\alpha}-1\right| = \left(1+\frac{1}{p}\right)^{|\alpha|}\left(1+O\left(\frac{1}{p^2}\right)\right),$$

whence, for some positive constant $C_0$,

$$\prod_{\substack{p\le m\\ p\in P}}(1+|v_h(p)|)\gg C_0(\log m)^{\varepsilon_*|\alpha|},$$

where $P=P(*)$ is as in Theorem 3 and $\varepsilon_+=1$, $\varepsilon_-=1/2$. Thus, by Theorem 3, if $A=A(m)$ and $x=x(m)$ are defined by equation (4.1) for some $a$ less than the $b$ of Theorem 1, there are numbers $X$ and $X'$ less than $(A+1)x$ and some positive constant $C$ with

$$E_h(X):=\sum_{n\le y(X)}v_h(n)\psi\left(\frac{X}{n}\right)\ge C(\log m)^{\varepsilon_*|\alpha|}$$

and

$$E_h(X')\le -C(\log m)^{\varepsilon_*|\alpha|}.$$

Since $\log m\gg\log\log((A+1)x)$, the proof is complete.

We conclude this section by establishing asymptotic expressions for the summatory functions of $\sigma^\beta$ and $\phi^\beta$, as well as $O$- and $\Omega$-estimates for their error terms.

To this purpose we let $a(n)$ be either $(\sigma(n)/n)^\beta$ or $(\phi(n)/n)^\beta$, and put

$$S(\beta,x):=\sum_{n\le x}n^\beta a(n).$$

We first note that there are three essentially distinct cases, excluding the trivial case $\beta=0$:

(I) $\beta>0$, where the error term is unbounded;

(II) $-1\le\beta<0$, where $S(\beta,x)$ is unbounded but where the error term is $o(1)$;

(III) $\beta<-1$, where $S(\beta,x)$ converges to $S_\beta<\infty$.

Case (I). We have, by Theorem 1 and with the notation introduced above,

$$\frac{1}{\beta}\sum_{n\le x}(x^\beta-n^\beta)a(n)=\sum_{n\le x}a(n)\int_n^x t^{\beta-1}\,dt=\int_1^x t^{\beta-1}S(0,t)\,dt$$
$$=\int_1^x t^{\beta-1}\Big(Ct+\sum_{r=0}^{[\alpha]}B_r(\log t)^{\alpha-r}+e_h(t)+o(1)\Big)\,dt,$$

where $h$, $\alpha$ and $C$ denote respectively either $f_\beta$, $\beta$ and $\zeta^\beta(2)f_\beta(2)$, or $g_\beta$, $-\beta$ and $\zeta^{-\beta}(2)g_\beta(2)$. Thus we have

$$\frac{1}{\beta}\sum_{n\le x}(x^\beta-n^\beta)a(n)$$
$$=\frac{C}{\beta+1}x^{\beta+1}+x^\beta\sum_{r=0}^{[\alpha]}B'_r(\log x)^{\alpha-r}+\int_1^x t^{\beta-1}e_h(t)\,dt+o(x^\beta). \tag{8}$$

And since we also have

$$\frac{x^\beta}{\beta}\sum_{n\le x}a(n)=\frac{C}{\beta}x^{\beta+1}+x^\beta\sum_{r=0}^{[\alpha]}B''_r(\log x)^{\alpha-r}+\frac{x^\beta}{\beta}e_h(x)+o(x^\beta), \tag{9}$$

we may write

$$S(\beta,x)=\frac{C}{\beta+1}x^{\beta+1}+x^\beta\sum_{r=0}^{[\alpha]}B'''_r(\log x)^{\alpha-r}+E_h(x)+o(x^\beta), \tag{10}$$

where

$$E_h(x)=x^\beta e_h(x)-\beta\int_1^x t^{\beta-1}e_h(t)\,dt. \tag{11}$$

But we have

$$\int_1^x t^{\beta-1}e_h(t)\,dt=\int_1^x t^{\beta-1}\sum_{n\le y(t)}v_n\psi(t/n)\,dt,$$

where $v_n$ is here either $v_{f_\beta}(n)$ or $v_{g_\beta}(n)$, and this is

$$\sum_{n\le y(x)}v_n\int_{w(n)}^{x}t^{\beta-1}\psi(t/n)\,dt=\sum_{n\le y(x)}v_n\int_{w(n)/n}^{x/n}n^\beta s^{\beta-1}\psi(s)\,ds$$
$$=\sum_{n\le y(x)}n^\beta v_n O((x/n)^{\beta-1}+1)=o(x^\beta),$$

where $w(n)$ denotes the inverse function of $y$ applied to $n$ if $n\ge y(1)$ and 1 otherwise, and where we use Lemma 5.3 for the last equality. Thus, from (11) and Theorems 4 and 5, (10) can be restated as

COROLLARY 1. *If* $\beta > 0$ *we have*

$$\sum_{n\le x}\sigma^{\beta}(n)=\frac{\zeta^{\beta}(2)f_{\beta}(2)}{\beta+1}x^{\beta+1}+x^{\beta}\sum_{r=0}^{[\beta]}a_r'(\log x)^{\beta-r}+E_{f_\beta}(x)+o(x^{\beta}), \tag{12}$$

*where the* $a_r' = a_r'(\beta)$ *are some real constants and*

$$E_{f_\beta}(x)=\begin{cases}O(x^{\beta}(\log x)^{2\beta/3}(\log\log x)^{4\beta/3}),\\ \Omega_{\pm}(x^{\beta}(\log\log x)^{\beta}).\end{cases} \tag{13}$$

*We also have*

$$\sum_{n\le x}\phi^{\beta}(n)=\frac{\zeta^{-\beta}(2)g_{\beta}(2)}{\beta+1}x^{\beta+1}+E_{g_\beta}(x)+o(x^{\beta}), \tag{14}$$

*with*

$$E_{g_\beta}(x)=\begin{cases}O(x^{\beta}(\log x)^{2\beta/3}(\log\log x)^{4\beta/3}),\\ \Omega_{\pm}(x^{\beta}(\log\log x)^{\beta/2}).\end{cases} \tag{15}$$

Case (II). Equation (9) holds unchanged, and equation (8) with an additional constant term on the right side as well as a principal term of the form $C\log x$ when $\beta=-1$. Thus we may write, instead of (10),

$$\begin{aligned} S(\beta,x)=&\left\{\begin{matrix}\dfrac{C}{\beta+1}x^{\beta+1} & \text{if } -1<\beta<0\\ C\log x & \text{if } \beta=-1\end{matrix}\right\}\\ &+x^{\beta}\sum_{r=0}^{[\alpha]}B_r'''(\log x)^{\alpha-r}+K+E_h'(x)+o(x^{\beta}),\end{aligned} \tag{16}$$

where $E_h'(x)$ satisfies (11). Since $\beta<0$ we have, by Theorem 2,

$$\int_1^x t^{\beta-1}e_h(t)\,dt=K'-\int_x^{\infty}t^{\beta-1}e_h(t)\,dt, \tag{17}$$

and the last integral is

$$\begin{aligned}&\int_x^{\infty}t^{\beta-1}\sum_{n\le y(t)}v_n\psi\left(\frac{t}{n}\right)dt\\ &=\sum_{n\le y(x)}v_nn^{\beta}\int_{x/n}^{\infty}s^{\beta-1}\psi(s)\,ds+\sum_{n>y(x)}v_nn^{\beta}\int_{w(n)/n}^{\infty}s^{\beta-1}\psi(s)\,ds\\ &=:I+II.\end{aligned} \tag{18}$$

For the first term we have, with the help of Lemma 5.3,

$$I = O\Big(\sum_{n\le y(x)} |v_n| n^{\beta}\Big(\frac{x}{n}\Big)^{\beta-1}\Big) = O(y(x)(\log x)^{|\beta|}x^{\beta-1}). \tag{19}$$

As for *II* it is, with the additional help of partial summation, $O$ of

$$\begin{aligned}\sum_{n>y}|v_n|nw(n)^{\beta-1} &= O\Big(\sum_{n>y} n(\log n)^{|\beta|}(w(n)^{\beta-1}-w(n+1)^{\beta-1})\Big)\\ &= O\Big(\sum_{n>y} n(\log n)^{|\beta|}w(n)^{\beta-2}w'(n)\Big)\\ &= O\Big(\int_y^\infty t(\log t)^{|\beta|}\,w(t)^{\beta-2}w'(t)\,dt\Big).\end{aligned}$$

We put $s = w(t)$ in the last integral and obtain

$$\begin{aligned} II &= O\Big(\int_x^\infty y(s)(\log y(s))^{|\beta|}s^{\beta-2}\,ds\Big) \\ &= (\log x)^{|\beta|}\,\exp(-(\log x)^b)\int_x^\infty s^{\beta-1}\,ds = o(x^{\beta}).\end{aligned} \tag{20}$$

Thus by (17)–(20) we have

$$\int_1^x t^{\beta-1}e_h(t)\,dt = K' + o(x^{\beta}), \tag{21}$$

and from (11) and Theorems 4 and 5 we may now restate (16) as

Corollary 2. *If* $-1\le\beta<0$ *we have*

$$\begin{aligned}\sum_{n\le x}\sigma^{\beta}(n) &\\ = \zeta^{\beta}(2)f_{\beta}(2)\times &\begin{Bmatrix} \dfrac{x^{\beta+1}}{\beta+1} & \textit{if } -1<\beta<0\\ \log x & \textit{if } \beta=-1\end{Bmatrix} + A + E_{f_\beta}(x) + o(x^{\beta}),\end{aligned} \tag{22}$$

*where* $A = A(\beta)$ *is a constant and* $E_{f_\beta}(x)$ *satisfies*

$$E_{f_\beta}(x) = \begin{cases} O(x^{\beta}(\log x)^{2|\beta|/3}(\log\log x)^{4|\beta|/3}),\\ \Omega_{\pm}(x^{\beta}(\log\log x)^{|\beta|/2}).\end{cases} \tag{23}$$

*We also have*

$$\begin{aligned}\sum_{n\le x}\phi^{\beta}(n) = \zeta^{-\beta}(2)g_{\beta}(2)\times &\begin{Bmatrix} \dfrac{x^{\beta+1}}{\beta+1} & (-1<\beta<0)\\ \log x & (\beta=-1)\end{Bmatrix}\\ &+ B + x^{\beta}\sum_{r=0}^{[-\beta]} b'_r(\log x)^{-\beta-r} + E_{g_\beta}(x) + o(x^{\beta}),\end{aligned} \tag{24}$$

*where* $b'_r = b'_r(\beta)$ *and* $B = B(\beta)$ *are constants and*

$$E_{g_\beta}(x) = \begin{cases} O(x^\beta (\log x)^{2|\beta|/3} (\log\log x)^{4|\beta|/3}), \\ \Omega_\pm(x^\beta (\log\log x)^{|\beta|}). \end{cases} \tag{25}$$

Case (III). The expansion of the left side of (8) has now a constant (depending on $\beta$) as principal term, and very similarly to Case (II) we obtain

Corollary 3. *If* $\beta < -1$ *then*

$$\sum_{n>x} \sigma^\beta(n) = -\frac{\zeta^\beta(2) f_\beta(2)}{\beta+1} x^{\beta+1} + E_{f_\beta}(x) + o(x^\beta), \tag{26}$$

*where* $E_{f_\beta}(x)$ *satisfies* (23), *and*

$$\sum_{n>x} \phi^\beta(n) = -\frac{\zeta^{-\beta}(2) g_\beta(2)}{\beta+1} x^{\beta+1} + x^\beta \sum_{r=0}^{[-\beta]} b'_r (\log x)^{-\beta-r} + E_{g_\beta}(x) + o(x^\beta),$$

*where* $E_{g_\beta}(x)$ *satisfies* (25).

**Added in proof.** We are grateful to Professor A. Schinzel for twisting our arms to make us read A. I. Saltykov's paper (*On Euler's function* (in Russian), Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1960 (6), 34–50), in which the better estimate than (1.13),

$$H(x) = O((\log x)^{2/3} (\log\log x)^{1+\varepsilon}), \tag{1.13'}$$

where $\varepsilon$ is an arbitrarily small positive number, is obtained. Saltykov's paper has been considered suspect, because it relies on a theorem proved by M. N. Korobov (*Estimates of trigonometrical sums and their applications* (in Russian), Uspekhi Mat. Nauk 13 (4) (1958), 185–192) in a paper in which he also makes an (as of today) unverified claim about the Riemann zeta-function (see [19], Notes on Chapter 5, p. 226). But in his work Saltykov only uses *proved* results of Korobov, and we agree with E. A. Bender, O. Patashnik and H. Rumsey Jr. (*Pizza slicing*, *Phi's*, *and the Riemann Hypothesis*, Amer. Math. Monthly 101 (1994), 307–317), when they state that Saltykov's estimate (1.13′) "is undisputed and is the best to date". We note in passing that in order to obtain (1.13) Walfisz also exploits Korobov's controversial paper (see [19], Paragraph 2.2).

Saltykov's proof follows very closely that of Walfisz—in an earlier version providing a weaker estimate than (1.13) (*Über die Wirksamkeit einiger Abschätzungen trigonometrischer Summen*, Acta Arith. 4 (1958), 108–180), and can be generalized in a way similar to that in our Section 3 to yield, instead of (3.1),

$$\sum_{n\le y} v_n \psi(x/n) = O(t^{2\alpha/3} u^{\alpha+\varepsilon}). \tag{3.1'}$$

We are preparing a complete proof of (3.1′), to appear in the (preprint series) "Publications internes de la Section de Mathématiques de l'Université de Genève", and which will be made available on request.

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