The mean square of the error term in a generalization of Dirichlet’s divisor problem

by

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1. Introduction. Let \( \sigma_a(n) \) denote the \( n \)th coefficient of the Dirichlet series \( \zeta(s)\zeta(s-a) \), where \( \zeta(s) \) is the Riemann zeta-function. Thus
\[
\sigma_a(n) = \sum_{d|n} d^a.
\]
We define
\[
D_a(y) = \sum_{n<y} \sigma_a(n) + \sigma_a(y)/2
\]
with the convention that \( \sigma_a(y) = 0 \) unless \( y \) is an integer. We also define
\[
\Delta_a(y) = D_a(y) - \zeta(1-a)y - \frac{\zeta(1+a)}{1+a} y^{1+a} + \frac{1}{2} \zeta(-a).
\]
In these definitions \( a \) may be any complex number. We prove

**Theorem.** Suppose \( x \geq 1 \). Then
\[
\int_1^x \Delta_a(y)^2 \, dy = \begin{cases}
  c_1 x^{3/2+a} + O(x) & \text{for } -1/2 < a < 0, \\
  c_2 x \log x + O(x) & \text{for } a = -1/2, \\
  O(x) & \text{for } -1 < a < -1/2,
\end{cases}
\]
where
\[
c_1 = (6 + 4a)^{-1} \pi^{-2} \zeta(3/2-a) \zeta(3/2+a) \zeta(3/2) \zeta(3)^{-1},
\]
\[
c_2 = \zeta(3/2)^2/(24\zeta(3))
\]
and the constants implied by the \( O \)-symbols may depend on \( a \).

This improves and generalizes a special case of a result of Kiuchi [3]. Kiuchi studied the situation in which \( \sigma_a(n) \) is multiplied by \( e^{2\pi i n h/k} \), where \( h \) and \( k \) are coprime integers. In the case \( k = 1 \) he proved that
\[
\int_1^x \Delta_a(y)^2 \, dy = c_1 x^{3/2+a} + O(x^{5/4+a/2+\varepsilon})
\]
for \(-1/2 < a < 0\) and any positive \( \varepsilon \).
It is also interesting to record the situation in the case $a = 0$. We have

$$
\Delta_0(y) = D_0(y) - y(\log y + 2\gamma - 1) - 1/4,
$$

where $\gamma$ is Euler’s constant. Tong [11] proved that

$$
\int_1^x \Delta_0(y)^2 dy = c_0 x^{3/2} + O(x \log^3 x),
$$

where $c_0 = \zeta(3/2)^4/(6\pi^2 \zeta(3))$. A simpler proof of this was later given by Meurman [6], and subsequently Preissmann [9] improved the error term to $O(x \log^4 x)$.

Our theorem is analogous to what has been proved by Matsumoto and Meurman [4, 5] for $E_{(1-a)/2}(T)$, the error term in the asymptotic formula for $\int_1^T |\zeta((1-a)/2 + it)|^2 dt$. However, in the special case $a = -1/2$ it is in fact sharper than what is suggested by the result just referred to.

The mean square estimates above show that the average order of $\Delta_a(y)$ is $O(y^{1/4+a/2})$ for $-1/2 < a \leq 0$, $O(\sqrt{\log y})$ for $a = -1/2$ and $O(1)$ for $-1 < a < -1/2$. They also show that $\Delta_a(y) = \Omega(y^{1/4+a/2})$ for $-1/2 < a \leq 0$ and $\Delta_{-1/2}(y) = \Omega(\sqrt{\log y})$. All this agrees with Pétérmann’s [8] conjecture (S) concerning individual values of $\Delta_a(y)$. As to the latter problem, a simple elementary argument starting with the definition of $\Delta_a(y)$ shows that $\Delta_a(y) \ll y^{(1+a)/2}$. Pétérmann [8] has a better result which is stated in terms of exponent pairs. Suffice it to say that it implies at least that $\Delta_a(y) \ll y^{(1+a)/3+\varepsilon}$ for any positive $\varepsilon$. However, the true order of magnitude of $\Delta_a(y)$ is as yet unknown. In the case $a = 0$ this problem is called the Dirichlet divisor problem.

It is not obvious in view of existing proofs in the case $a = 0$ how to prove our theorem. One of the difficulties is that the “Voronoï series” for $\Delta_a(y)$ may diverge for $a \leq -1/2$.

Our argument may be generalized to a wider set of real and complex values of $a$ including $a = 0$. In the case $a = 0$ it clearly gives Preissmann’s result mentioned above. But to keep it as simple as possible and referring also to the remarks in Section 4 we assume $-1 < a < 0$. Moreover, this assumption, being equivalent to $1/2 < (1 - a)/2 < 1$, is suggested by the analogy between $\Delta_a(x)$ and $E_{(1-a)/2}(T)$.

It seems difficult to improve the $O$-terms in our theorem. In fact, we believe that they are $\Omega(x)$. For $-1/2 < a < 0$ the $O$-term is $\Omega(x^{3/4+3a/2})$. This can be seen by following the proofs of Theorem 3 in [5] and Theorem 13.6 in [2].

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2. Proof of the Theorem. We begin by stating our main lemma. Its proof will be given in Sections 3–5.

Lemma 1. For $-1 < a < 0$, $y \geq 1$, $X \geq y$, $Z \geq 2y$ and $y$ not an integer we have

$$\Delta_a(y) = \Delta_a(y, X) + R_a(y, X, Z) + O(y^{-1/4+a/2}) + O(y^{-1/2}),$$

where

$$\Delta_a(y, X) = \frac{1}{\pi \sqrt{2}} y^{1/4+a/2} \int_{1}^{2} \sum_{n \leq u \leq X} \sigma_a(n)n^{-3/4-a/2} \cos(4\pi \sqrt{ny} - \pi/4) \, du$$

and

$$R_a(y, X, Z) = \frac{1}{2\pi} \sum_{n \leq Z} \sigma_a(n) \int_{1}^{2} \int_{u_X}^{\infty} t^{-1} \sin(4\pi (\sqrt{ny} - \sqrt{nt}) \sqrt{t}) \, dt \, du.$$

Remark. The expression $R_a(y, X, Z)$ accounts for the jumps of $\Delta_a(y)$ at integers. Therefore the easy estimate $R_a(y, X, Z) \ll y^2$—which is not good enough for our purpose—cannot be improved in general. But it can be improved if $y$ is not near an integer and $X$ is large. This way of argument is successful in the case $a = 0$ (see [6]) but not here. That is why we need the explicit expression for $R_a(y, X, Z)$ as given above.

Suppose that $x \geq 2$, $x/2 \leq y \leq x$, $Z = 2x$ and $y$ is not an integer. We apply Lemma 1 with two different values of $X$, viz. $4x$ and $x$. We abbreviate just for a moment $R_a(y, X, Z) = R_a(y, X)$. Then

$$\Delta_a(y)^2 = \Delta_a(y, 4x)^2 + 2\Delta_a(y, 4x)R_a(y, 4x) + R_a(y, 4x)^2$$

$$+ O(y^{-1/4}(|\Delta_a(y, 4x)| + |R_a(y, 4x)|) + y^{-1/2})$$

and

$$\Delta_a(y, 4x) = \Delta_a(y, x) + R_a(y, x) - R_a(y, 4x) + O(y^{-1/4}).$$

We combine these formulas to obtain

$$\Delta_a(y)^2 = \Delta_a(y, 4x)^2 + 2\Delta_a(y, x)R_a(y, 4x) + 2R_a(y, x)R_a(y, 4x)$$

$$- R_a(y, 4x)^2 + O(y^{-1/4}(|\Delta_a(y, 4x)| + |R_a(y, 4x)|) + y^{-1/2}).$$

Now we integrate for $y$ and use Cauchy’s inequality to obtain

$$\frac{x}{x/2} \int_{y/2}^{x} \Delta_a(y)^2 \, dy = I_1 + 2I_2 + O(\sqrt{T_3}I_3 + I_3 + x^{1/4}(\sqrt{I_1} + \sqrt{I_3}) + x^{1/2}),$$

where

$$I_1 = \int_{x/2}^{x} \Delta_a(y, 4x)^2 \, dy, \quad I_2 = \int_{x/2}^{x} \Delta_a(y, x)R_a(y, 4x, 2x) \, dy,$$
\[ I_3 = \int_{x/2}^{x} R_a(y, 4x, 2x)^2 dy \quad \text{and} \quad I'_3 = \int_{x/2}^{x} R_a(y, x, 2x)^2 dy. \]

Obviously it suffices to prove that

\[
I_1 = \begin{cases} 
  c_1(x^{3/2+a} - (x/2)^{3/2+a}) + O(x) & \text{for } -1/2 < a < 0, \\
  c_2(x \log x - (x/2) \log (x/2)) + O(x) & \text{for } a = -1/2, \\
  O(x) & \text{for } -1 < a < -1/2,
\end{cases}
\]

\[
I_2 \ll x, \quad I_3 \ll x \quad \text{and} \quad I'_3 \ll x.
\]

**Proof of (2.1).** We square out the expression for \( \Delta_a(y, 4x) \) given by Lemma 1 and get

\[
I_1 = I_{10} + I_{11},
\]

where

\[
I_{10} = \frac{1}{2\pi^2} \int_{x/2}^{x} y^{1/2+a} \sum_{n \leq 8x} b(n)^2 \cos^2(4\pi \sqrt{ny} - \pi/4) dy,
\]

\[
I_{11} = \frac{1}{2\pi^2} \int_{x/2}^{x} y^{1/2+a} \sum_{\substack{m,n \leq 8x \\ m \neq n}} b(m)b(n) \cos(4\pi \sqrt{my} - \pi/4) \cos(4\pi \sqrt{ny} - \pi/4) dy,
\]

\[
b(n) = \sigma_a(n)n^{-3/4-a/2} \int_{\max(1,n/(4x))}^{2} du.
\]

We first prove that \( I_{11} \ll x \) (which is acceptable in view of our claim (2.1)). This reduces to showing that \( J^\pm \ll x \), where

\[
J^\pm = \int_{x/2}^{x} y^{1/2+a} \sum_{\substack{m,n \leq 8x \\ m \neq n}} b(m)b(n)e^{4\pi i(\sqrt{m} \pm \sqrt{n})\sqrt{y}} dy.
\]

By the second mean value theorem there exist \( \xi_1 \) and \( \xi_2 \) between \( x/2 \) and \( x \) such that

\[
J^\pm \ll x^{1+a} \int_{\xi_1}^{\xi_2} y^{-1/2} \sum_{\substack{m,n \leq 8x \\ m \neq n}} b(m)b(n)e^{4\pi i(\sqrt{m} \pm \sqrt{n})\sqrt{y}} dy
\]

\[
\ll x^{1+a} \sum_{j=1}^{2} \sum_{\substack{m,n \leq 8x \\ m \neq n}} b(m)b(n)(\sqrt{m} \pm \sqrt{n})^{-1}e^{4\pi i(\sqrt{m} \pm \sqrt{n})\sqrt{\xi_j}}.
\]

Trivially \( J^+ \ll x \). Similarly we get trivially \( J^- \ll x \log x \), but this does not suffice. So, following Preissmann [9], we invoke a generalization of Hilbert’s
inequality, viz. the Montgomery–Vaughan inequality (see [2], (5.34)). We have
\[ \min_{m \neq n} |\sqrt{m} - \sqrt{n}| \gg n^{-1/2} \]
for any positive integer \( n \) and it follows that
\[ J^{-} \ll x^{1+a} \sum_{n \leq 8x} b(n)^2 n^{1/2} \ll x^{1+a} \sum_{n \leq 8x} \sigma_a(n)^2 n^{-a-1} \ll x \]
so that \( I_{11} \ll x \) as claimed.

Consider \( I_{10} \). Since \( \cos^2(4\pi \sqrt{ny} - \pi/4) = (1 + \sin(8\pi \sqrt{ny}))/2 \), and
\[ \sum_{n \leq 8x} b(n)^2 \int_{x/2}^{x} y^{1/2+a} \sin(8\pi \sqrt{ny}) \, dy \ll x^{1+a} \]
(see [10], Lemma 4.3), we have
\[ I_{10} = \frac{1}{(6 + 4a)\pi^2} (x^{3/2+a} - (x/2)^{3/2+a}) \sum_{n \leq 8x} b(n)^2 + O(x^{1+a}). \]
Here
\[ \sum_{n \leq 8x} b(n)^2 = \sum_{n \leq x} \sigma_a(n)^2 n^{-3/2-a} + O(x^{-1/2-a}), \]
which is \( O(x^{-1/2-a}) \) for \(-1 < a < -1/2 \) so that \( I_{10} \ll x \) in this case. For \(-1/2 < a < 0 \) we have (see [10], (1.3.3))
\[ \sum_{n \leq x} \sigma_a(n)^2 n^{-3/2-a} = \zeta(3/2-a)\zeta(3/2+a)\zeta(3/2)^2\zeta(3)^{-1} + O(x^{-1/2-a}) \]
so that \( I_{10} = c_1(x^{3/2+a} - (x/2)^{3/2+a}) + O(x) \) in this case. In the remaining case \( a = -1/2 \) we use Perron’s formula to obtain
\[ \sum_{n \leq x} \sigma_a(n)^2 n^{-1} = (3/2)^2 \zeta(2)\zeta(3)^{-1} \log x + O(1). \]
Since \( \zeta(2) = \pi^2/6 \) we conclude that \( I_{10} = c_2(x - x/2) \log x + O(x) \) in this case. This completes the proof of (2.1).

Proof of (2.2). By the second mean value theorem there exists \( \xi \) between \( x/2 \) and \( x \) such that
\[ I_2 = x^{3/4+a/2} \int_{\xi}^{x} y^{-3/4-a/2} \Delta_a(y, x)R_a(y, X, 2x) \, dy, \]
where \( X = 4x \). Lemma 1 then gives
\[ I_2 \ll x^{3/4+a/2} \sum_{m \leq 2x} \sigma_a(m) m^{-3/4-a/2} \sum_{n \leq 2x} \sigma_a(n) |J(m, n, X)|, \]
where

\[ J(m, n, X) = \int_{\xi}^{x} y^{-1/2} \cos(4\pi\sqrt{my} - \pi/4) \int_{1}^{\infty} t^{-1} \sin(4\pi\sqrt{y} - \sqrt{n}\sqrt{t}) dt \, du \, dy. \]

Then for the proof of (2.2) it clearly suffices to show that

\[ \sum_{n \leq 2x} \sigma_a(n) |J(m, n, X)| \ll 1. \]

For \( Y \geq X \) we have

\[ J(m, n, X) - J(m, n, Y) \ll x^{-1/2} \max_{y \in \{\xi, x\}} \min(1, |y - n|^{-1}) \]

uniformly in \( Y \), since

\[ \int_{uX}^{uY} t^{-1} e^{-4\pi i \sqrt{mt}} \int_{\xi}^{x} y^{-1/2} e^{4\pi i (\sqrt{t} \pm \sqrt{m})\sqrt{y}} dy \, dt \]

\[ \ll \max_{y \in \{\xi, x\}} \left| \int_{uX}^{uY} t^{-1} (\sqrt{t} \pm \sqrt{m})^{-1} e^{-4\pi i (\sqrt{t} \pm \sqrt{m})\sqrt{y}} dt \right| \]

\[ = \max_{y \in \{\xi, x\}} \left| \int_{uX}^{uY} t^{-1} (\sqrt{t} \pm \sqrt{m})^{-1} e^{4\pi i (\sqrt{y} - \sqrt{n})\sqrt{t}} dt \right| \]

\[ \ll \max_{y \in \{\xi, x\}} \min(X^{-1/2}, X^{-1} |\sqrt{y} - \sqrt{n}|^{-1}). \]

In the last step we applied Lemma 4.3 in [10], and made use of the fact that \( m \leq 2x = X/2 \). (At this point one can see why Lemma 1 was applied with two different values of \( X \).) On the other hand, \( \lim_{Y \to \infty} J(m, n, Y) = 0 \) by applying the same lemma to the innermost integral. Now (2.4) follows easily.

**Proof of (2.3).** We need the following lemma, the proof of which is a simple application of partial integration.

**Lemma 2.** For \( X \geq 1 \) and any real \( k \) we have

\[ \int_{1}^{2} \int_{uX}^{\infty} t^{-1} \sin(k\sqrt{t}) dt \, du \ll \min(1, X^{-1} k^{-2}). \]

Lemma 1, Lemma 2 and Cauchy’s inequality give
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\[ I_3 \ll \int_{x/2}^{x} \left( \sum_{n \leq 2x} \sigma_a(n) \min(1, (n - y)^{-2}) \right)^2 dy \]

\[ \ll \int_{x/2}^{x} \sum_{n \leq 2x} \sigma_a(n)^2 \min(1, (n - y)^{-2}) dy \]

\[ \ll \sum_{n \leq 2x} \sigma_a(n)^2 \int_{x/2}^{x} \min(1, (n - y)^{-2}) dy \]

\[ \ll \sum_{n \leq 2x} \sigma_a(n)^2 \ll x. \]

The integral \( I_3' \) is estimated similarly and (2.3) follows.

3. Analytic continuation. In the following sections we prove Lemma 1.

Let \( z \) be a complex variable and let \( p \) be a real variable, which will eventually tend to \( \infty \). Let \( w \) be a sufficiently many (three will suffice) times continuously differentiable function supported on the interval \([-2/3, 2/3]\) such that \( w(v) = 1 \) for \( v \in [-1/3, 1/3] \). It is clear that the function \( z \mapsto \Delta_z(y) \) is entire. Hence, defining

\[
(3.1) \quad \Delta_z,p(y) = p \int_{-\infty}^{\infty} w(v) e^{-\pi(pv)^2} (1 + v)^{1/2 - z} \Delta_z(y(1 + v)^2) dv,
\]

the function \( z \mapsto \Delta_z,p(y) \) is entire. We define

\[
B_z(t) = \sin(\pi z/2)J_{1+z}(4\pi \sqrt{t}) + \cos(\pi z/2)(Y_{1+z}(4\pi \sqrt{t}) + (2/\pi)K_{1+z}(4\pi \sqrt{t}))
\]

in the usual notation of Bessel functions. Oppenheim [7] has proved that

\[
\Delta_z(y) = -y^{(1+z)/2} \sum_{n=1}^{\infty} \sigma_z(n)n^{-(1+z)/2}B_z(ny)
\]

for \(-1/2 < z < 0\). The series here is boundedly convergent in any finite \( y \)-subinterval of \((0, \infty)\), as shown by Hafner [1]. Hence we may integrate term-by-term to obtain

\[
(3.2) \quad \Delta_z,p(y) = -py^{(1+z)/2} \sum_{n=1}^{\infty} \sigma_z(n)n^{-(1+z)/2}
\]

\[
\times \int_{-\infty}^{\infty} w(v) e^{-\pi(pv)^2} (1 + v)^{3/2} B_z(ny(1 + v)^2) dv.
\]

We now only know that (3.2) holds for real values of \( z \) satisfying \(-1/2 < z < 0\).
Consider the expression

\begin{equation}
py^{-1/4+z/2} \sum_{n=1}^{\infty} \sigma_z(n) n^{-5/4-z/2} \int_{-\infty}^{\infty} w(v) e^{-\pi(pv)^2} h_z(ny(1 + v)^2) \, dv,
\end{equation}

where

\[ h_z(t) = \frac{1}{\sqrt{2}} \left( \sqrt{t} \cos(4\pi \sqrt{t} - \pi/4) - \frac{4z^2 + 8z + 3}{32\pi} \sin(4\pi \sqrt{t} - \pi/4) \right). \]

By partial integration (this is where we need the function \( w \)) and the familiar formula

\[ \int_{-\infty}^{\infty} e^{Av-Bv^2} \, dv = \sqrt{\frac{\pi}{Be^{A^2/4B}}} \quad (\Re(B) > 0) \]

(see e.g. [2], (A.38)), we get

\begin{equation}
p \int_{-\infty}^{\infty} w(v) e^{-\pi(pv)^2 + 4\pi iv \sqrt{ny}} \, dv = e^{-4\pi ny/p^2} + O((|y|^{-3/2} e^{-p})
\end{equation}

and (using (3.4))

\begin{equation}
p \int_{-\infty}^{\infty} w(v) ve^{-\pi(pv)^2 + 4\pi iv \sqrt{ny}} \, dv
\end{equation}

\[ = 2i(ny)^{1/2} p^{-2} e^{-4\pi ny/p^2} + O((ny)^{-1} e^{-p}). \]

Let \( \mathcal{C} \) be a compact subset of \( \mathcal{D} = \{ z \mid -3/2 < \Re(z) < 3/2 \} \). It follows that the series in (3.3) is absolutely and uniformly convergent in \( \mathcal{C} \) and hence that the expression (3.3) defines a holomorphic function \( z \mapsto \Delta^*_z, p(y), \) say, in \( \mathcal{D}. \)

By (3.2) and (3.3) we get a series representing \( \Delta_z, p(y) - \Delta^*_z, p(y) \) for \(-1/2 < z < 0 \). It has holomorphic terms in \( \mathcal{D}. \) It is absolutely and uniformly convergent and \( O(|y|^{-3/4+z/2}) \) in \( \mathcal{C} \), since, by well-known asymptotic formulas for Bessel functions (see [12], Sec. 7.21, 7.23),

\[ t^{3/4} B_z(t) + h_z(t) \ll t^{-1/2} \]

uniformly in \( \mathcal{C}. \) Hence it is holomorphic and represents \( \Delta_z, p(y) - \Delta^*_z, p(y) \) in the whole \( \mathcal{D}, \) and we get \( \Delta_z, p(y) - \Delta^*_z, p(y) \ll |y|^{-3/4+z/2} \) for \( z \) in \( \mathcal{D}. \)

Finally, we evaluate \( \Delta^*_z, p(y) \) using (3.4) and (3.5) and conclude that

\begin{equation}
\Delta_a, p(y) = \Delta^{(1)}_{a, p}(y) + \Delta^{(2)}_{a, p}(y) + O(|y|^{-3/4+a/2}),
\end{equation}

where

\begin{equation}
\Delta^{(1)}_{a, p}(y) = \frac{1}{\pi \sqrt{2}} y^{1/4+a/2} \sum_{n=1}^{\infty} \sigma_a(n) n^{-3/4-a/2} e^{-4\pi ny/p^2}
\times \cos(4\pi \sqrt{ny} - \pi/4),
\end{equation}

\begin{equation}
\Delta^{(2)}_{a, p}(y) = \frac{1}{\pi \sqrt{2}} y^{1/4+a/2} \sum_{n=1}^{\infty} \sigma_a(n) n^{-3/4-a/2} e^{-4\pi ny/p^2}
\times \sin(4\pi \sqrt{ny} - \pi/4).
\end{equation}
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\[(3.8) \quad \Delta_{a,p}^{(2)}(y) = y^{-1/4+a/2} \sum_{n=1}^{\infty} \sigma_a(n)(c_3 - \sqrt{2} \pi^{-1} ny/p^2)n^{-5/4-a/2} \]
\[\times e^{-4\pi ny/p^2} \sin(4\pi \sqrt{ny} - \pi/4) \]
and \(c_3 = -(4a^2 + 8a + 3)/(32\pi^2 \sqrt{2}).\)

Remarks. The quantity \(c_3\) vanishes at \(a = -1/2\). The implied constant in (3.6) does not depend on \(p\) and the formula is valid for any \(a\) in \(D\). This range can be further extended by replacing \(-h_z(t)\) with a sharper approximation of \(t^{3/4} B_z(t)\).

4. Lemmata

Lemma 3. For \(X \geq 1, Y \geq X, V > 0, t \) fixed and any real \(k\) we have
\[\int \int_{uX}^{uY} t^{-1/4} e^{-t/V + ik\sqrt{t}} dt du \ll \begin{cases} X^{-1} \min(V, k^{-2}) & \text{for } l \geq 0, \\ X^{-1} \min(X, k^{-2}) & \text{for } l > 1. \end{cases} \]

Proof. Partial integration gives \(O(X^{-1}k^{-2})\) if \(k \neq 0\). The alternative estimates are trivial.

Lemma 4. For \(-3/2 < a < 3/2\) we have
\[\int_{0}^{y} \Delta_a(v) dv = c_4 + y^{3/4+a/2} \sum_{n=1}^{\infty} \sigma_a(n)n^{-5/4-a/2} g(ny) + O(y^{-3/4+a/2}),\]
where
\[g(t) = \sum_{\nu=0}^{2} e_\nu t^{-\nu/2} \cos(4\pi \sqrt{t} + \pi/4 + \pi \nu/2),\]
e_0 = 1/(2\pi^2 \sqrt{2}), e_1, e_2 and \(c_4\) may depend on \(a\) only and the series here is uniformly convergent on any finite closed subinterval of \((0, \infty)\).

Proof. The lemma is based on Theorem B and Lemma 2.1 of Hafner [1]. See also Section 2 of [5].

Lemma 5. For \(-1 < a < 1/2\) we have
\[\int_{0}^{y} \Delta_a(v) dv \ll y^{3/4+a/2} + y^{1/2} \log y.\]

Proof. The integral is \(O(y^{3/4+a/2})\) for \(-1/2 < a < 1/2\) by Lemma 4, whereas the case \(-1 < a \leq -1/2\) is covered by Lemma 2 of [5].

Remarks. The restriction \(-3/2 < a < 3/2\) in Lemma 4 is essential. Since the number \(r\) in Hafner’s Definition 1.1 is real, our \(a\) must be real. It is, however, possible to generalize Lemma 4 to complex values of \(a\). The
assumption $a > -1$ in Lemma 5 is not essential, but we have to accept it because it occurs in Lemma 2 of [5].

5. Transformation. The idea now is to truncate the series in (3.7) and (3.8), transform the remainder using Lemma 4 and then let $p \to \infty$ along a suitable sequence. The constants implied by the symbols $O$ and $\ll$ will be independent of $p$.

We define
\begin{equation}
(5.1) \quad f_p(t) = t^{-3/4-a/2} e^{-4\pi ty/p^2} \cos(4\pi \sqrt{ty} - \pi/4).
\end{equation}

Let $1 \leq u \leq 2$. We have
\begin{equation}
\sum_{n > uX} \sigma_a(n)f_p(n) = - \int_{uX}^{\infty} f'_p(t)(D_a(t) - D_a(uX))
\end{equation}
\begin{align*}
&= - \int_{uX}^{\infty} f'_p(t)\left[\zeta(1-a)v + \frac{\zeta(1+a)}{1+a}v^{1+a}\right]_{v=uX}^t dt \\
&\quad - \int_{uX}^{\infty} f'_p(t)(\Delta_a(t) - \Delta_a(uX)) dt \\
&= \int_{uX}^{\infty} f_p(t)(\zeta(1-a) + \zeta(1+a)t^a) dt \\
&\quad - f_p(uX)\Delta_a(uX) + \int_{uX}^{\infty} f''_p(t) \int_{uX}^t \Delta_a(v) dv dt.
\end{align*}

Hence
\begin{equation}
(5.2) \quad \int_1^{2} \sum_{n > uX} \sigma_a(n)f_p(n) du = S_1(p) + S_2(p) + \lim_{Y \to \infty} S_3(p, Y),
\end{equation}
where
\begin{align*}
S_1(p) &= - \int_1^{2} f_p(uX)\Delta_a(uX) du, \\
S_2(p) &= \int_1^{2} \int_{uX}^{\infty} f_p(t)(\zeta(1-a) + \zeta(1+a)t^a) dt du, \\
S_3(p, Y) &= \int_1^{2} \int_{uX}^{uY} f''_p(t) \int_{uX}^t \Delta_a(v) dv dt du.
\end{align*}

We claim that
\begin{equation}
(5.3) \quad S_1(p) \ll y^{-1/2}, \quad S_2(p) \ll y^{-1/2}.
\end{equation}
Concerning \( S_2(p) \) this is clear, since Lemma 3 gives \( S_2(p) \ll y^{-1}X^{-3/4-a/2} \).
Consider then \( S_1(p) \). We have

\[
S_1(p) = -X^{-1} \left[ f_p(t) \int_0^t \Delta_a(v) \, dv \right]_{t=X}^{2X} + X^{-1} \int_X^{2X} f_p'(t) \int_0^t \Delta_a(v) \, dv \, dt
\]

\[
= X^{-1} S_{11} + X^{-1} S_{12},
\]
say. Lemma 5 gives \( S_{11} \ll 1 + X^{-1/4-a/2} \log X \), which is acceptable. Lemma 4 gives

\[
S_{12} \ll \sum_{n=1}^{\infty} n^{-3/4} \left[ \int_X^{2X} f_p'(t) t^{3/4+a/2} g(nt) \, dt \right]^2 + \left[ \int_X^{2X} |f_p'(t)| t^{-3/4+a/2} \, dt \right]^2 = S_{121} + S_{122} + S_{123},
\]
say. By (5.1) we have

\[
(5.4) \quad f_p'(t) = t^{-3/4-a/2} (-2\pi y^{1/2} t^{-1/2} \sin(4\pi \sqrt{ty} - \pi/4) + (c_5 y^{-2} + c_6 t^{-1}) \cos(4\pi \sqrt{ty} - \pi/4)) e^{-4\pi ty/p^2},
\]

where \( c_5 = -4\pi \) and \( c_6 = -3/4 - a/2 \), so that

\[
\int_X^{2X} f_p'(t) t^{3/4+a/2} g(nt) \, dt \ll y^{1/2} \min(X^{1/2}, |n - \sqrt{y}|^{-1})
\]

either trivially or by Lemma 4.3 of [10]. Hence \( S_{121} \ll y^{1/4} \log y + X^{1/2} y^{-1/4} \).
Finally, it is plain that \( S_{122} \ll y^{1/2} X^{-1} \), \( S_{123} \ll X^{-3/4-a/2} \) and (5.3) has been proved.

Consider \( S_3(p,Y) \). We apply Lemma 4 and integrate term-by-term to get

\[
S_3(p,Y) = \sum_{n=1}^{\infty} \sigma_a(n) n^{-5/4-a/2} \int_1^Y \int_{uX}^{uY} f_p''(t) \left[ y^{3/4+a/2} g(ny) \right]_{y=uX}^{t} \, dt \, du
\]

\[
+ O \left( X^{-3/4+a/2} \int_X^{2Y} |f_p''(t)| \, dt \right).
\]

The \( O \)-term here is \( O(yX^{-3/2}) \), since (5.4) implies that \( f_p''(t) \ll yt^{-7/4-a/2} \).
Then we integrate by parts and note that the integrated term is

\[
\int_1^2 \int_{uX}^{uY} f_p''(uY) \left( \int_{uX}^{uY} \Delta_a(v) \, dv + O(X^{-3/4+a/2}) \right) \, du \ll Y^2 e^{-Y y/p^2}.
\]
Hence
\[
S_3(p, Y) = - \sum_{n=1}^{\infty} \sigma_a(n)n^{-5/4-a/2} \int_{1}^{2} \int_{uX}^{uY} f_p'(t)(t^{3/4+a/2}g(nt))' \, dt \, du \\
+ O(yX^{-3/2}) + O_y(Y^2e^{-Yy/p^2}).
\]

We have
\[
(t^{3/4+a/2}g(nt))' = t^{3/4+a/2}((\pi\sqrt{2})^{-1}n^{1/2}t^{-1/2}\cos(4\pi\sqrt{nt} - \pi/4) \\
+ c_7 t^{-1}\sin(4\pi\sqrt{nt} - \pi/4) + O(n^{-1/2}t^{-3/2})),
\]
where \(c_7\) may depend on \(a\) only. Hence, by (5.4), Lemma 3 and using the formula \(\sin \alpha \cos \beta = (\sin(\alpha + \beta) + \sin(\alpha - \beta))/2\), we get (assuming that \(p^2 > Xy\))
\[
\int_{1}^{2} \int_{uX}^{uY} f_p'(t)(t^{3/4+a/2}g(nt))' \, dt \, du \\
= -(1/\sqrt{2})(ny)^{1/2}I(n, p, Y) \\
+ O((y + n)^{1/2}X^{-3/2}\min(X, (\sqrt{n} - \sqrt{y})^{-2}) \\
+ O(y^{1/2}X^{-1}n^{-1/2}),
\]
where
\[
I(n, p, Y) = \int_{1}^{2} \int_{uX}^{uY} t^{-1}e^{-4\pi y/\sqrt{t}} \sin(4\pi(\sqrt{y} - \sqrt{n})\sqrt{t}) \, dt \, du.
\]

For \(n > Z\) we have \(I(n, p, Y) \ll (Xn)^{-1}\) by Lemma 3, since \(Z \geq 2y\) by assumption. Hence
\[
S_3(p, Y) = \frac{1}{\sqrt{2}}y^{1/2} \sum_{n \leq Z} \sigma_a(n)n^{-3/4-a/2}I(n, p, Y) \\
+ O_y(Y^2e^{-Yy/p^2}) + O(y^{1/2}X^{-1}) + O(y^{-3/4-a/2+\varepsilon}X^{-1/2})
\]
for any \(\varepsilon > 0.\)

We combine (3.7), (5.2), (5.3) and (5.5). This gives
\[
\Delta_{a, p}^{(1)}(Y) = \frac{1}{\pi\sqrt{2}}y^{1/4+a/2} \int_{1}^{2} \sum_{n \leq uX} \sigma_a(n)f_p(n) \, du \\
+ \frac{1}{2\pi}y^{3/4+a/2} \sum_{n \leq Z} \sigma_a(n)n^{-3/4-a/2} \lim_{Y \to \infty} I(n, p, Y) \\
+ O(y^{-1/4+a/2}).
\]
Concerning $\Delta^{(2)}_{a,p}(y)$, as defined by (3.8), we argue similarly with $X$ replaced by $y$ and estimate trivially the contribution of the terms with $a \ll y$. Here it is to be noted that $c_3 = 0$ at $a = -1/2$. The result is that

$$\Delta^{(2)}_{a,p}(y) \ll y^{-1/4+a/2} + y^{-1/2}. \tag{5.7}$$

Next, we show that

$$\lim_{p \to \infty} \Delta_{a,p}(y) = \Delta_a(y) \tag{5.8}$$

unless $y$ is an integer. First of all we have

$$p \int_{-\infty}^{\infty} w(v) e^{-\pi(pv)^2} (1 + v)^{1/2-a} \; dv = 1 + O(p^{-1}).$$

It follows that (see (3.1))

$$\Delta_{a,p}(y) - \Delta_a(y)$$

$$= p \int_{-\infty}^{\infty} w(v) e^{-\pi(pv)^2} (1 + v)^{1/2-a} (\Delta_a(y(1 + v)^2) - \Delta_a(y)) \; dv$$

$$+ O(|\Delta_a(y)/p)$$

$$\ll p \int_{-2/3}^{2/3} e^{-\pi(pv)^2} |\Delta_a(y(1 + v)^2) - \Delta_a(y)| \; dv + yp^{-1}$$

$$\ll p \int_{0}^{2/3} e^{-\pi(pv)^2} \left( yv + \sum_{|n-y| \leq 2yv} \sigma_a(n) \right) \; dv + yp^{-1}$$

$$\ll \int_{0}^{2p/3} e^{-\pi v^2} \sum_{|n-y| \leq 2yv/p} \sigma_a(n) \; dv + yp^{-1}.$$

Clearly this tends to zero as $p \to \infty$ unless $y$ is an integer, as claimed.

We combine (3.6), (5.6)–(5.8) and let $p \to \infty$. This gives

$$\Delta_a(y) = \Delta_a(y, X) + \frac{1}{2\pi} y^{3/4+a/2} \sum_{n \leq Z} \sigma_a(n) n^{-3/4-a/2} \lim_{p \to \infty} \lim_{Y \to \infty} I(n, p, Y)$$

$$+ O(y^{-1/4+a/2}) + O(y^{-1/2})$$

unless $y$ is an integer. It is easy to show that

$$\lim_{p \to \infty} \lim_{Y \to \infty} I(n, p, Y) = \int_{1}^{2} \int_{\sqrt{y}}^{\infty} t^{-1} \sin(4\pi(\sqrt{y} - \sqrt{n})\sqrt{t}) \; dt \; du.$$
Finally, we replace $n^{-3/4-a/2}$ in the sum by $y^{-3/4-a/2}$. By Lemma 2, this produces a term $O(y^{-1/4+a/2})$ to the whole expression. The proof of Lemma 1 is thus complete.

References