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The mean square of the error term in a generalization of Dirichlet's divisor problem

by

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1. Introduction. Let $\sigma_a(n)$ denote the nth coefficient of the Dirichlet series $\zeta(s)\zeta(s-a)$, where $\zeta(s)$ is the Riemann zeta-function. Thus

$$\sigma_a(n)=\sum_{d\mid n} d^a.$$

We define

$$D_a(y)=\sum_{n<y}\sigma_a(n)+\sigma_a(y)/2$$

with the convention that $\sigma_a(y)=0$ unless y is an integer. We also define

$$\Delta_a(y)=D_a(y)-\zeta(1-a)y-\frac{\zeta(1+a)}{1+a}y^{1+a}+\frac{1}{2}\zeta(-a).$$

In these definitions a may be any complex number. We prove

Theorem. *Suppose $x\ge 1$. Then*

$$\int_1^x \Delta_a(y)^2\,dy=\begin{cases} c_1x^{3/2+a}+O(x) & \text{for } -1/2<a<0,\\ c_2x\log x+O(x) & \text{for } a=-1/2,\\ O(x) & \text{for } -1<a<-1/2,\end{cases}$$

where

$$c_1=(6+4a)^{-1}\pi^{-2}\zeta(3/2-a)\zeta(3/2+a)\zeta(3/2)^2\zeta(3)^{-1},$$
$$c_2=\zeta(3/2)^2/(24\zeta(3))$$

and the constants implied by the O-symbols may depend on a.

This improves and generalizes a special case of a result of Kiuchi [3]. Kiuchi studied the situation in which $\sigma_a(n)$ is multiplied by $e^{2\pi inh/k}$, where h and k are coprime integers. In the case $k=1$ he proved that

$$\int_1^x \Delta_a(y)^2\,dy=c_1x^{3/2+a}+O(x^{5/4+a/2+\varepsilon})$$

for $-1/2<a<0$ and any positive ε.

It is also interesting to record the situation in the case $a = 0$. We have

$$\Delta_0(y) = D_0(y) - y(\log y + 2\gamma - 1) - 1/4,$$

where γ is Euler's constant. Tong [11] proved that

$$\int_1^x \Delta_0(y)^2\,dy = c_0 x^{3/2} + O(x\log^5 x),$$

where $c_0 = \zeta(3/2)^4/(6\pi^2\zeta(3))$. A simpler proof of this was later given by Meurman [6], and subsequently Preissmann [9] improved the error term to $O(x\log^4 x)$.

Our theorem is analogous to what has been proved by Matsumoto and Meurman [4, 5] for $E_{(1-a)/2}(T)$, the error term in the asymptotic formula for $\int_1^T |\zeta((1-a)/2+it)|^2\,dt$. However, in the special case $a = -1/2$ it is in fact sharper than what is suggested by the result just referred to.

The mean square estimates above show that the average order of $\Delta_a(y)$ is $O(y^{1/4+a/2})$ for $-1/2 < a \le 0$, $O(\sqrt{\log y})$ for $a = -1/2$ and $O(1)$ for $-1 < a < -1/2$. They also show that $\Delta_a(y) = \Omega(y^{1/4+a/2})$ for $-1/2 < a \le 0$ and $\Delta_{-1/2}(y) = \Omega(\sqrt{\log y})$. All this agrees with Pétermann's [8] conjecture (S) concerning individual values of $\Delta_a(y)$. As to the latter problem, a simple elementary argument starting with the definition of $\Delta_a(y)$ shows that $\Delta_a(y) \ll y^{(1+a)/2}$. Pétermann [8] has a better result which is stated in terms of exponent pairs. Suffice it to say that it implies at least that $\Delta_a(y) \ll y^{(1+a)/3+\varepsilon}$ for any positive ε. However, the true order of magnitude of $\Delta_a(y)$ is as yet unknown. In the case $a = 0$ this problem is called the *Dirichlet divisor problem*.

It is not obvious in view of existing proofs in the case $a = 0$ how to prove our theorem. One of the difficulties is that the "Voronoï series" for $\Delta_a(y)$ may diverge for $a \le -1/2$.

Our argument may be generalized to a wider set of real and complex values of a including $a = 0$. In the case $a = 0$ it clearly gives Preissmann's result mentioned above. But to keep it as simple as possible and referring also to the remarks in Section 4 we assume $-1 < a < 0$. Moreover, this assumption, being equivalent to $1/2 < (1-a)/2 < 1$, is suggested by the analogy between $\Delta_a(x)$ and $E_{(1-a)/2}(T)$.

It seems difficult to improve the O-terms in our theorem. In fact, we believe that they are $\Omega(x)$. For $-1/2 < a < 0$ the O-term is $\Omega(x^{3/4+3a/2})$. This can be seen by following the proofs of Theorem 3 in [5] and Theorem 13.6 in [2].

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2. Proof of the Theorem. We begin by stating our main lemma. Its proof will be given in Sections 3–5.

LEMMA 1. *For $-1 < a < 0$, $y \ge 1$, $X \ge y$, $Z \ge 2y$ and y not an integer we have*

$$\Delta_a(y) = \Delta_a(y, X) + R_a(y, X, Z) + O(y^{-1/4+a/2}) + O(y^{-1/2}),$$

where

$$\Delta_a(y, X) = \frac{1}{\pi\sqrt{2}} y^{1/4+a/2} \int_1^2 \sum_{n\le uX} \sigma_a(n) n^{-3/4-a/2} \cos(4\pi\sqrt{ny} - \pi/4)\, du$$

and

$$R_a(y, X, Z) = \frac{1}{2\pi} \sum_{n\le Z} \sigma_a(n) \int_1^2 \int_{uX}^{\infty} t^{-1} \sin(4\pi(\sqrt{y} - \sqrt{n})\sqrt{t})\, dt\, du.$$

Remark. The expression $R_a(y, X, Z)$ accounts for the jumps of $\Delta_a(y)$ at integers. Therefore the easy estimate $R_a(y, X, Z) \ll y^{\varepsilon}$—which is not good enough for our purpose—cannot be improved in general. But it can be improved if y is not near an integer and X is large. This way of argument is successful in the case $a = 0$ (see [6]) but not here. That is why we need the explicit expression for $R_a(y, X, Z)$ as given above.

Suppose that $x \ge 2$, $x/2 \le y \le x$, $Z = 2x$ and y is not an integer. We apply Lemma 1 with two different values of X, viz. $4x$ and x. We abbreviate just for a moment $R_a(y, X, Z) = R_a(y, X)$. Then

$$\begin{aligned}\Delta_a(y)^2 &= \Delta_a(y, 4x)^2 + 2\Delta_a(y, 4x)R_a(y, 4x) + R_a(y, 4x)^2 \\ &\quad + O(y^{-1/4}(|\Delta_a(y, 4x)| + |R_a(y, 4x)|) + y^{-1/2})\end{aligned}$$

and

$$\Delta_a(y, 4x) = \Delta_a(y, x) + R_a(y, x) - R_a(y, 4x) + O(y^{-1/4}).$$

We combine these formulas to obtain

$$\begin{aligned}\Delta_a(y)^2 &= \Delta_a(y, 4x)^2 + 2\Delta_a(y, x)R_a(y, 4x) + 2R_a(y, x)R_a(y, 4x) \\ &\quad - R_a(y, 4x)^2 + O(y^{-1/4}(|\Delta_a(y, 4x)| + |R_a(y, 4x)|) + y^{-1/2}).\end{aligned}$$

Now we integrate for y and use Cauchy's inequality to obtain

$$\int_{x/2}^{x} \Delta_a(y)^2\, dy = I_1 + 2I_2 + O(\sqrt{I_3 I_3'} + I_3 + x^{1/4}(\sqrt{I_1} + \sqrt{I_3}) + x^{1/2}),$$

where

$$I_1 = \int_{x/2}^{x} \Delta_a(y, 4x)^2\, dy, \qquad I_2 = \int_{x/2}^{x} \Delta_a(y, x)R_a(y, 4x, 2x)\, dy,$$

$$I_3 = \int\limits_{x/2}^{x} R_a(y,4x,2x)^2\,dy \quad\text{and}\quad I_3' = \int\limits_{x/2}^{x} R_a(y,x,2x)^2\,dy.$$

Obviously it suffices to prove that

$$I_1 = \begin{cases} c_1(x^{3/2+a} - (x/2)^{3/2+a}) + O(x) & \text{for } -1/2 < a < 0,\\ c_2(x\log x - (x/2)\log(x/2)) + O(x) & \text{for } a = -1/2,\\ O(x) & \text{for } -1 < a < -1/2,\end{cases} \tag{2.1}$$

$$I_2 \ll x, \tag{2.2}$$

$$I_3 \ll x \quad\text{and}\quad I_3' \ll x. \tag{2.3}$$

Proof of (2.1). We square out the expression for $\Delta_a(y,4x)$ given by Lemma 1 and get $I_1 = I_{10} + I_{11}$, where

$$I_{10} = \frac{1}{2\pi^2}\int\limits_{x/2}^{x} y^{1/2+a}\sum_{n\le 8x} b(n)^2\cos^2(4\pi\sqrt{ny}-\pi/4)\,dy,$$

$$I_{11} = \frac{1}{2\pi^2}\int\limits_{x/2}^{x} y^{1/2+a}\sum_{\substack{m,n\le 8x\\ m\ne n}} b(m)b(n)\cos(4\pi\sqrt{my}-\pi/4)\cos(4\pi\sqrt{ny}-\pi/4)\,dy,$$

$$b(n) = \sigma_a(n)n^{-3/4-a/2}\int\limits_{\max(1,n/(4x))}^{2}\qquad du.$$

We first prove that $I_{11} \ll x$ (which is acceptable in view of our claim (2.1)). This reduces to showing that $J^\pm \ll x$, where

$$J^\pm = \int\limits_{x/2}^{x} y^{1/2+a}\sum_{\substack{m,n\le 8x\\ m\ne n}} b(m)b(n)e^{4\pi i(\sqrt{m}\pm\sqrt{n})\sqrt{y}}\,dy.$$

By the second mean value theorem there exist ξ_1 and ξ_2 between $x/2$ and x such that

$$\begin{aligned} J^\pm &\ll x^{1+a}\Big|\int\limits_{\xi_1}^{\xi_2} y^{-1/2}\sum_{\substack{m,n\le 8x\\ m\ne n}} b(m)b(n)e^{4\pi i(\sqrt{m}\pm\sqrt{n})\sqrt{y}}\,dy\Big|\\ &\ll x^{1+a}\sum_{j=1}^{2}\Big|\sum_{\substack{m,n\le 8x\\ m\ne n}} b(m)b(n)(\sqrt{m}\pm\sqrt{n})^{-1}e^{4\pi i(\sqrt{m}\pm\sqrt{n})\sqrt{\xi_j}}\Big|.\end{aligned}$$

Trivially $J^+ \ll x$. Similarly we get trivially $J^- \ll x\log x$, but this does not suffice. So, following Preissmann [9], we invoke a generalization of Hilbert's inequality, viz. the Montgomery–Vaughan inequality (see [2], (5.34)). We have

$$\min_{m\neq n} |\sqrt{m}-\sqrt{n}| \gg n^{-1/2}$$

for any positive integer n and it follows that

$$J^{-} \ll x^{1+a}\sum_{n\le 8x} b(n)^2 n^{1/2} \ll x^{1+a}\sum_{n\le 8x}\sigma_a(n)^2 n^{-a-1} \ll x$$

so that $I_{11} \ll x$ as claimed.

Consider I_{10}. Since $\cos^2(4\pi\sqrt{ny}-\pi/4) = (1+\sin(8\pi\sqrt{ny}))/2$, and

$$\sum_{n\le 8x} b(n)^2 \int\limits_{x/2}^{x} y^{1/2+a}\sin(8\pi\sqrt{ny})\,dy \ll x^{1+a}$$

(see [10], Lemma 4.3), we have

$$I_{10} = \frac{1}{(6+4a)\pi^2}(x^{3/2+a}-(x/2)^{3/2+a})\sum_{n\le 8x} b(n)^2 + O(x^{1+a}).$$

Here

$$\sum_{n\le 8x} b(n)^2 = \sum_{n\le x}\sigma_a(n)^2 n^{-3/2-a} + O(x^{-1/2-a}),$$

which is $O(x^{-1/2-a})$ for $-1<a<-1/2$ so that $I_{10}\ll x$ in this case. For $-1/2<a<0$ we have (see [10], (1.3.3))

$$\sum_{n\le x}\sigma_a(n)^2 n^{-3/2-a} = \zeta(3/2-a)\zeta(3/2+a)\zeta(3/2)^2\zeta(3)^{-1} + O(x^{-1/2-a})$$

so that $I_{10} = c_1(x^{3/2+a}-(x/2)^{3/2+a}) + O(x)$ in this case. In the remaining case $a=-1/2$ we use Perron's formula to obtain

$$\sum_{n\le x}\sigma_a(n)^2 n^{-1} = \zeta(3/2)^2\zeta(2)\zeta(3)^{-1}\log x + O(1).$$

Since $\zeta(2)=\pi^2/6$ we conclude that $I_{10} = c_2(x-x/2)\log x + O(x)$ in this case. This completes the proof of (2.1).

Proof of (2.2). By the second mean value theorem there exists ξ between $x/2$ and x such that

$$I_2 = x^{3/4+a/2}\int\limits_{\xi}^{x} y^{-3/4-a/2}\Delta_a(y,x)R_a(y,X,2x)\,dy,$$

where $X=4x$. Lemma 1 then gives

$$I_2 \ll x^{3/4+a/2}\sum_{m\le 2x}\sigma_a(m)m^{-3/4-a/2}\sum_{n\le 2x}\sigma_a(n)|J(m,n,X)|,$$

where

$$J(m,n,X) = \int_{\xi}^{x} y^{-1/2}\cos(4\pi\sqrt{my}-\pi/4)\int_{1}^{2}\int_{uX}^{\infty} t^{-1}\sin(4\pi(\sqrt{y}-\sqrt{n})\sqrt{t})\,dt\,du\,dy.$$

Then for the proof of (2.2) it clearly suffices to show that

$$\sum_{n\le 2x}\sigma_a(n)|J(m,n,X)|\ll 1. \tag{2.4}$$

For $Y\ge X$ we have

$$J(m,n,X)-J(m,n,Y)\ll x^{-1/2}\max_{y\in\{\xi,x\}}\min(1,|y-n|^{-1})$$

uniformly in Y, since

$$\begin{aligned}\int_{uX}^{uY} t^{-1}e^{-4\pi i\sqrt{nt}}\int_{\xi}^{x} y^{-1/2}e^{4\pi i(\sqrt{t}\pm\sqrt{m})\sqrt{y}}\,dy\,dt \\ &\ll \max_{y\in\{\xi,x\}}\Big|\int_{uX}^{uY} t^{-1}(\sqrt{t}\pm\sqrt{m})^{-1}e^{-4\pi i(\sqrt{nt}-(\sqrt{t}\pm\sqrt{m})\sqrt{y})}\,dt\Big| \\ &= \max_{y\in\{\xi,x\}}\Big|\int_{uX}^{uY} t^{-1}(\sqrt{t}\pm\sqrt{m})^{-1}e^{4\pi i(\sqrt{y}-\sqrt{n})\sqrt{t}}\,dt\Big| \\ &\ll \max_{y\in\{\xi,x\}}\min(X^{-1/2},X^{-1}|\sqrt{y}-\sqrt{n}|^{-1}).\end{aligned}$$

In the last step we applied Lemma 4.3 in [10], and made use of the fact that $m\le 2x=X/2$. (At this point one can see why Lemma 1 was applied with two different values of X.) On the other hand, $\lim_{Y\to\infty}J(m,n,Y)=0$ by applying the same lemma to the innermost integral. Now (2.4) follows easily.

Proof of (2.3). We need the following lemma, the proof of which is a simple application of partial integration.

Lemma 2. *For $X\ge 1$ and any real k we have*

$$\int_{1}^{2}\int_{uX}^{\infty} t^{-1}\sin(k\sqrt{t})\,dt\,du\ll\min(1,X^{-1}k^{-2}).$$

Lemma 1, Lemma 2 and Cauchy's inequality give

$$\begin{aligned} I_3 &\ll \int\limits_{x/2}^{x} \Big(\sum_{n \le 2x} \sigma_a(n) \min(1, (n-y)^{-2}) \Big)^2 \, dy \\ &\ll \int\limits_{x/2}^{x} \sum_{n \le 2x} \sigma_a(n)^2 \min(1, (n-y)^{-2}) \, dy \\ &\ll \sum_{n \le 2x} \sigma_a(n)^2 \int\limits_{x/2}^{x} \min(1, (n-y)^{-2}) \, dy \\ &\ll \sum_{n \le 2x} \sigma_a(n)^2 \ll x. \end{aligned}$$

The integral I_3' is estimated similarly and (2.3) follows.

3. Analytic continuation. In the following sections we prove Lemma 1.

Let z be a complex variable and let p be a real variable, which will eventually tend to ∞. Let w be a sufficiently many (three will suffice) times continuously differentiable function supported on the interval $[-2/3, 2/3]$ such that $w(v) = 1$ for $v \in [-1/3, 1/3]$. It is clear that the function $z \mapsto \Delta_z(y)$ is entire. Hence, defining

$$\Delta_{z,p}(y) = p \int\limits_{-\infty}^{\infty} w(v) e^{-\pi(pv)^2} (1+v)^{1/2-z} \Delta_z(y(1+v)^2) \, dv, \tag{3.1}$$

the function $z \mapsto \Delta_{z,p}(y)$ is entire. We define

$$B_z(t) = \sin(\pi z/2) J_{1+z}(4\pi\sqrt{t}) + \cos(\pi z/2)(Y_{1+z}(4\pi\sqrt{t}) + (2/\pi) K_{1+z}(4\pi\sqrt{t}))$$

in the usual notation of Bessel functions. Oppenheim [7] has proved that

$$\Delta_z(y) = -y^{(1+z)/2} \sum_{n=1}^{\infty} \sigma_z(n) n^{-(1+z)/2} B_z(ny)$$

for $-1/2 < z < 0$. The series here is boundedly convergent in any finite y-subinterval of $(0, \infty)$, as shown by Hafner [1]. Hence we may integrate term-by-term to obtain

$$\begin{aligned} \Delta_{z,p}(y) = {} & -py^{(1+z)/2} \sum_{n=1}^{\infty} \sigma_z(n) n^{-(1+z)/2} \\ & \times \int\limits_{-\infty}^{\infty} w(v) e^{-\pi(pv)^2} (1+v)^{3/2} B_z(ny(1+v)^2) \, dv. \end{aligned} \tag{3.2}$$

We now only know that (3.2) holds for real values of z satisfying $-1/2 < z < 0$.

Consider the expression

$$(3.3)\quad py^{-1/4+z/2}\sum_{n=1}^{\infty}\sigma_z(n)n^{-5/4-z/2}\int_{-\infty}^{\infty} w(v)e^{-\pi(pv)^2}h_z(ny(1+v)^2)\,dv,$$

where

$$h_z(t)=\frac{1}{\pi\sqrt{2}}\left(\sqrt{t}\cos(4\pi\sqrt{t}-\pi/4)-\frac{4z^2+8z+3}{32\pi}\sin(4\pi\sqrt{t}-\pi/4)\right).$$

By partial integration (this is where we need the function w) and the familiar formula

$$\int_{-\infty}^{\infty} e^{Av-Bv^2}\,dv=\sqrt{\pi/B}e^{A^2/(4B)}\qquad(\Re(B)>0)$$

(see e.g. [2], (A.38)), we get

$$(3.4)\qquad p\int_{-\infty}^{\infty} w(v)e^{-\pi(pv)^2+4\pi iv\sqrt{ny}}\,dv=e^{-4\pi ny/p^2}+O((ny)^{-3/2}e^{-p})$$

and (using (3.4))

$$(3.5)\qquad p\int_{-\infty}^{\infty} w(v)ve^{-\pi(pv)^2+4\pi iv\sqrt{ny}}\,dv = 2i(ny)^{1/2}p^{-2}e^{-4\pi ny/p^2}+O((ny)^{-1}e^{-p}).$$

Let $\mathcal{C}$ be a compact subset of $\mathcal{D}=\{z\mid -3/2<\Re(z)<3/2\}$. It follows that the series in (3.3) is absolutely and uniformly convergent in $\mathcal{C}$ and hence that the expression (3.3) defines a holomorphic function $z\mapsto\Delta^*_{z,p}(y)$, say, in $\mathcal{D}$.

By (3.2) and (3.3) we get a series representing $\Delta_{z,p}(y)-\Delta^*_{z,p}(y)$ for $-1/2<z<0$. It has holomorphic terms in $\mathcal{D}$. It is absolutely and uniformly convergent and $O(|y^{-3/4+z/2}|)$ in $\mathcal{C}$, since, by well-known asymptotic formulas for Bessel functions (see [12], Sec. 7.21, 7.23),

$$t^{3/4}B_z(t)+h_z(t)\ll t^{-1/2}$$

uniformly in $\mathcal{C}$. Hence it is holomorphic and represents $\Delta_{z,p}(y)-\Delta^*_{z,p}(y)$ in the whole $\mathcal{D}$, and we get $\Delta_{z,p}(y)-\Delta^*_{z,p}(y)\ll|y^{-3/4+z/2}|$ for z in $\mathcal{D}$.

Finally, we evaluate $\Delta^*_{a,p}(y)$ using (3.4) and (3.5) and conclude that

$$(3.6)\qquad \Delta_{a,p}(y)=\Delta^{(1)}_{a,p}(y)+\Delta^{(2)}_{a,p}(y)+O(|y^{-3/4+a/2}|),$$

where

$$(3.7)\qquad \Delta^{(1)}_{a,p}(y)=\frac{1}{\pi\sqrt{2}}y^{1/4+a/2}\sum_{n=1}^{\infty}\sigma_a(n)n^{-3/4-a/2}e^{-4\pi ny/p^2}\times\cos(4\pi\sqrt{ny}-\pi/4),$$

(3.8) $$\Delta_{a,p}^{(2)}(y) = y^{-1/4+a/2}\sum_{n=1}^{\infty}\sigma_a(n)(c_3 - \sqrt{2}\pi^{-1}ny/p^2)n^{-5/4-a/2} \times e^{-4\pi ny/p^2}\sin(4\pi\sqrt{ny}-\pi/4)$$

and $c_3 = -(4a^2+8a+3)/(32\pi^2\sqrt{2})$.

Remarks. The quantity c_3 vanishes at $a=-1/2$. The implied constant in (3.6) does not depend on p and the formula is valid for any a in $\mathcal{D}$. This range can be further extended by replacing $-h_z(t)$ with a sharper approximation of $t^{3/4}B_z(t)$.

4. Lemmata

Lemma 3. *For $X\ge 1$, $Y\ge X$, $V>0$, l fixed and any real k we have*

$$\int_1^2\int_{uX}^{uY} t^{-l}e^{-t/V+ik\sqrt{t}}\,dt\,du \ll \begin{cases} X^{-l}\min(V,k^{-2}) & \text{for } l\ge 0,\\ X^{-l}\min(X,k^{-2}) & \text{for } l>1.\end{cases}$$

Proof. Partial integration gives $O(X^{-l}k^{-2})$ if $k\ne 0$. The alternative estimates are trivial.

Lemma 4. *For $-3/2<a<3/2$ we have*

$$\int_0^y \Delta_a(v)\,dv = c_4 + y^{3/4+a/2}\sum_{n=1}^{\infty}\sigma_a(n)n^{-5/4-a/2}g(ny) + O(y^{-3/4+a/2}),$$

where

$$g(t)=\sum_{\nu=0}^{2}e_\nu t^{-\nu/2}\cos(4\pi\sqrt{t}+\pi/4+\pi\nu/2),$$

$e_0 = 1/(2\pi^2\sqrt{2})$, e_1, e_2 *and* c_4 *may depend on a only and the series here is uniformly convergent on any finite closed subinterval of* $(0,\infty)$.

Proof. The lemma is based on Theorem B and Lemma 2.1 of Hafner [1]. See also Section 2 of [5].

Lemma 5. *For $-1<a<1/2$ we have*

$$\int_0^y \Delta_a(v)\,dv \ll y^{3/4+a/2}+y^{1/2}\log y.$$

Proof. The integral is $O(y^{3/4+a/2})$ for $-1/2<a<1/2$ by Lemma 4, whereas the case $-1<a\le -1/2$ is covered by Lemma 2 of [5].

Remarks. The restriction $-3/2<a<3/2$ in Lemma 4 is essential. Since the number r in Hafner's Definition 1.1 is real, our a must be real. It is, however, possible to generalize Lemma 4 to complex values of a. The assumption $a > -1$ in Lemma 5 is not essential, but we have to accept it because it occurs in Lemma 2 of [5].

5. Transformation. The idea now is to truncate the series in (3.7) and (3.8), transform the remainder using Lemma 4 and then let $p \to \infty$ along a suitable sequence. The constants implied by the symbols O and $\ll$ will be independent of p.

We define

$$f_p(t) = t^{-3/4-a/2} e^{-4\pi ty/p^2} \cos(4\pi\sqrt{ty} - \pi/4). \tag{5.1}$$

Let $1 \le u \le 2$. We have

$$\begin{aligned}
\sum_{n>uX} \sigma_a(n) f_p(n) &= -\int_{uX}^{\infty} f_p'(t)(D_a(t) - D_a(uX))\, dt \\
&= -\int_{uX}^{\infty} f_p'(t)\Big[\zeta(1-a)v + \frac{\zeta(1+a)}{1+a} v^{1+a}\Big]_{v=uX}^{t} dt \\
&\quad - \int_{uX}^{\infty} f_p'(t)(\Delta_a(t) - \Delta_a(uX))\, dt \\
&= \int_{uX}^{\infty} f_p(t)(\zeta(1-a) + \zeta(1+a)t^a)\, dt \\
&\quad - f_p(uX)\Delta_a(uX) + \int_{uX}^{\infty} f_p''(t) \int_{uX}^{t} \Delta_a(v)\, dv\, dt.
\end{aligned}$$

Hence

$$\int_1^2 \sum_{n>uX} \sigma_a(n) f_p(n)\, du = S_1(p) + S_2(p) + \lim_{Y\to\infty} S_3(p, Y), \tag{5.2}$$

where

$$\begin{aligned}
S_1(p) &= -\int_1^2 f_p(uX)\Delta_a(uX)\, du, \\
S_2(p) &= \int_1^2 \int_{uX}^{\infty} f_p(t)(\zeta(1-a) + \zeta(1+a)t^a)\, dt\, du, \\
S_3(p, Y) &= \int_1^2 \int_{uX}^{uY} f_p''(t) \int_{uX}^{t} \Delta_a(v)\, dv\, dt\, du.
\end{aligned}$$

We claim that

$$S_1(p) \ll y^{-1/2}, \qquad S_2(p) \ll y^{-1/2}. \tag{5.3}$$

Concerning $S_2(p)$ this is clear, since Lemma 3 gives $S_2(p) \ll y^{-1}X^{-3/4-a/2}$. Consider then $S_1(p)$. We have

$$\begin{aligned} S_1(p) &= -X^{-1}\Big[f_p(t)\int\limits_0^t \Delta_a(v)\,dv\Big]_{t=X}^{2X} + X^{-1}\int\limits_X^{2X} f_p'(t)\int\limits_0^t \Delta_a(v)\,dv\,dt \\ &= X^{-1}S_{11} + X^{-1}S_{12}, \end{aligned}$$

say. Lemma 5 gives $S_{11} \ll 1 + X^{-1/4-a/2}\log X$, which is acceptable. Lemma 4 gives

$$\begin{aligned} S_{12} &\ll \sum_{n=1}^{\infty} n^{-3/4}\Big|\int\limits_X^{2X} f_p'(t)t^{3/4+a/2}g(nt)\,dt\Big| + \int\limits_X^{2X} |f_p'(t)|t^{-3/4+a/2}\,dt \\ &\quad + \Big|\int\limits_X^{2X} f_p'(t)\,dt\Big| \\ &= S_{121} + S_{122} + S_{123}, \end{aligned}$$

say. By (5.1) we have

$$\begin{aligned} f_p'(t) = t^{-3/4-a/2}(&-2\pi y^{1/2}t^{-1/2}\sin(4\pi\sqrt{ty} - \pi/4) \\ &+ (c_5 yp^{-2} + c_6 t^{-1})\cos(4\pi\sqrt{ty} - \pi/4))e^{-4\pi ty/p^2}, \end{aligned} \tag{5.4}$$

where $c_5 = -4\pi$ and $c_6 = -3/4 - a/2$, so that

$$\int\limits_X^{2X} f_p'(t)t^{3/4+a/2}g(nt)\,dt \ll y^{1/2}\min(X^{1/2}, |\sqrt{n} - \sqrt{y}|^{-1})$$

either trivially or by Lemma 4.3 of [10]. Hence $S_{121} \ll y^{1/4}\log y + X^{1/2}y^{-1/4}$. Finally, it is plain that $S_{122} \ll y^{1/2}X^{-1}$, $S_{123} \ll X^{-3/4-a/2}$ and (5.3) has been proved.

Consider $S_3(p, Y)$. We apply Lemma 4 and integrate term-by-term to get

$$\begin{aligned} S_3(p, Y) &= \sum_{n=1}^{\infty} \sigma_a(n)n^{-5/4-a/2}\int\limits_1^2 \int\limits_{uX}^{uY} f_p''(t)[y^{3/4+a/2}g(ny)]_{y=uX}^t\,dt\,du \\ &\quad + O\Big(X^{-3/4+a/2}\int\limits_X^{2Y} |f_p''(t)|\,dt\Big). \end{aligned}$$

The O-term here is $O(yX^{-3/2})$, since (5.4) implies that $f_p''(t) \ll yt^{-7/4-a/2}$. Then we integrate by parts and note that the integrated term is

$$\int\limits_1^2 f_p'(uY)\Big(\int\limits_{uX}^{uY} \Delta_a(v)\,dv + O(X^{-3/4+a/2})\Big)\,du \ll_y Y^2 e^{-Yy/p^2}.$$

Hence

$$S_3(p,Y) = -\sum_{n=1}^{\infty}\sigma_a(n)n^{-5/4-a/2}\int\limits_1^2\int\limits_{uX}^{uY} f_p'(t)(t^{3/4+a/2}g(nt))'\,dt\,du + O(yX^{-3/2}) + O_y(Y^2e^{-Yy/p^2}).$$

We have

$$(t^{3/4+a/2}g(nt))' = t^{3/4+a/2}((\pi\sqrt{2})^{-1}n^{1/2}t^{-1/2}\cos(4\pi\sqrt{nt}-\pi/4) + c_7t^{-1}\sin(4\pi\sqrt{nt}-\pi/4) + O(n^{-1/2}t^{-3/2})),$$

where c_7 may depend on a only. Hence, by (5.4), Lemma 3 and using the formula $\sin\alpha\cos\beta = (\sin(\alpha+\beta)+\sin(\alpha-\beta))/2$, we get (assuming that $p^2 > Xy$)

$$\int\limits_1^2\int\limits_{uX}^{uY} f_p'(t)(t^{3/4+a/2}g(nt))'\,dt\,du = -(1/\sqrt{2})(ny)^{1/2}I(n,p,Y) + O((y+n)^{1/2}X^{-3/2}\min(X,(\sqrt{n}-\sqrt{y})^{-2})) + O(y^{1/2}X^{-1}n^{-1/2}),$$

where

$$I(n,p,Y) = \int\limits_1^2\int\limits_{uX}^{uY} t^{-1}e^{-4\pi yt/p^2}\sin(4\pi(\sqrt{y}-\sqrt{n})\sqrt{t})\,dt\,du.$$

For $n > Z$ we have $I(n,p,Y) \ll (Xn)^{-1}$ by Lemma 3, since $Z \ge 2y$ by assumption. Hence

$$S_3(p,Y) = \frac{1}{\sqrt{2}}y^{1/2}\sum_{n\le Z}\sigma_a(n)n^{-3/4-a/2}I(n,p,Y) + O_y(Y^2e^{-Yy/p^2}) + O(y^{1/2}X^{-1}) + O(y^{-3/4-a/2+\varepsilon}X^{-1/2}) \tag{5.5}$$

for any $\varepsilon > 0$.

We combine (3.7), (5.2), (5.3) and (5.5). This gives

$$\Delta_{a,p}^{(1)}(y) = \frac{1}{\pi\sqrt{2}}y^{1/4+a/2}\int\limits_1^2\sum_{n\le uX}\sigma_a(n)f_p(n)\,du + \frac{1}{2\pi}y^{3/4+a/2}\sum_{n\le Z}\sigma_a(n)n^{-3/4-a/2}\lim_{Y\to\infty}I(n,p,Y) + O(y^{-1/4+a/2}). \tag{5.6}$$

Concerning $\Delta^{(2)}_{a,p}(y)$, as defined by (3.8), we argue similarly with X replaced by y and estimate trivially the contribution of the terms with $n \ll y$. Here it is to be noted that $c_3 = 0$ at $a = -1/2$. The result is that

$$\Delta^{(2)}_{a,p}(y) \ll y^{-1/4+a/2} + y^{-1/2}. \tag{5.7}$$

Next, we show that

$$\lim_{p\to\infty} \Delta_{a,p}(y) = \Delta_a(y) \tag{5.8}$$

unless y is an integer. First of all we have

$$p \int_{-\infty}^{\infty} w(v) e^{-\pi(pv)^2} (1+v)^{1/2-a}\, dv = 1 + O(p^{-1}).$$

It follows that (see (3.1))

$$\begin{aligned}
&\Delta_{a,p}(y) - \Delta_a(y) \\
&\quad = p \int_{-\infty}^{\infty} w(v) e^{-\pi(pv)^2} (1+v)^{1/2-a} (\Delta_a(y(1+v)^2) - \Delta_a(y))\, dv \\
&\qquad + O(|\Delta_a(y)|/p) \\
&\quad \ll p \int_{-2/3}^{2/3} e^{-\pi(pv)^2} |\Delta_a(y(1+v)^2) - \Delta_a(y)|\, dv + yp^{-1} \\
&\quad \ll p \int_{0}^{2/3} e^{-\pi(pv)^2} \Big(yv + \sum_{|n-y|\le 2yv} \sigma_a(n) \Big)\, dv + yp^{-1} \\
&\quad \ll \int_{0}^{2p/3} e^{-\pi v^2} \sum_{|n-y|\le 2yv/p} \sigma_a(n)\, dv + yp^{-1}.
\end{aligned}$$

Clearly this tends to zero as $p \to \infty$ unless y is an integer, as claimed.

We combine (3.6), (5.6)–(5.8) and let $p \to \infty$. This gives

$$\begin{aligned}
\Delta_a(y) = \Delta_a(y, X) &+ \frac{1}{2\pi} y^{3/4+a/2} \sum_{n\le Z} \sigma_a(n) n^{-3/4-a/2} \lim_{p\to\infty} \lim_{Y\to\infty} I(n,p,Y) \\
&+ O(y^{-1/4+a/2}) + O(y^{-1/2})
\end{aligned}$$

unless y is an integer. It is easy to show that

$$\lim_{p\to\infty} \lim_{Y\to\infty} I(n,p,Y) = \int_1^2 \int_{uX}^{\infty} t^{-1} \sin(4\pi(\sqrt{y} - \sqrt{n})\sqrt{t})\, dt\, du.$$

Finally, we replace $n^{-3/4-a/2}$ in the sum by $y^{-3/4-a/2}$. By Lemma 2, this produces a term $O(y^{-1/4+a/2})$ to the whole expression. The proof of Lemma 1 is thus complete.

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